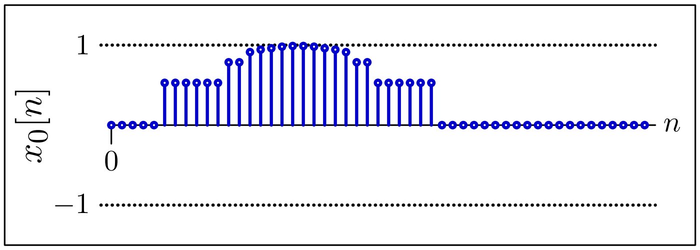
Dome Sweet Dome

Ben Bitdiddle created a signal $x_0[n]$ representing the MIT dome, but he only saved the DTFS coefficients $X_0[k]$ (and not the original signal). However, he knew that one period of the original signal (which is periodic in $N = 51$) looked like this:

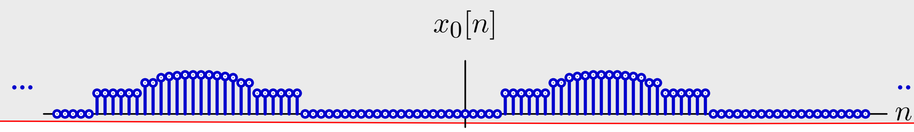


Ben tried several different methods of recovering the original image based on $X_0[k]$, by applying the DTFS synthesis equation to the following sets of coefficients.

For each set of Fourier coefficients described below (X_A through X_I), determine the corresponding signal from the 24 options shown on the next page (x_1 through x_{24}).

Assume that all 24 of those signals are purely real and are periodic in $N = 51$. If the required signal would be complex-valued, record your answer as "must be complex." Otherwise, write the name of the signal from the following page.

The original signal is periodic in $N = 51$ as shown below.



a. $X_A[k] = \mathrm{Re}\left(X_0[k]\right)$

$$X_A[k] = \mathrm{Re}\left(X_0[k]\right) = \frac{1}{2}X_0[k] + \frac{1}{2}X_0^*[k] \quad \text{(property of complex numbers)}$$

Now find the effect of conjugating $X[k]$.

$$X[k] = \frac{1}{N}\sum x[n]e^{-\frac{j2\pi kn}{N}} \quad \text{(Fourier analysis equation)}$$

$$X^*[k] = \frac{1}{N}\sum x^*[n]e^{\frac{j2\pi kn}{N}} \quad \text{(conjugate both sides)}$$

$$X^*[k] = \frac{1}{N}\sum x^*[-n]e^{-\frac{j2\pi kn}{N}} \quad (n \to -n)$$

$$x^*[-n] \overset{\text{FT}}{\Longrightarrow} X^*[k] \quad \text{(Fourier analysis equation)}$$

Then

$$x_A[n] = \frac{1}{2}x_0[n] + \frac{1}{2}x_0^*[-n] = \frac{1}{2}x_0[n] + \frac{1}{2}x_0[-n]$$

since $x_0[n]$ is real-valued. The flipped signal $x_0[-n]$ looks a lot like $x_0[n]$ (since that function is symmetric about $n = 18.5$) but it is shifted by 15 samples. Thus when $x_0[n]$ is added to $x_0[-n]$, part of the dome from $x_0[n]$ overlaps part of the dome from $x_0[-n]$. The result looks like $x_{16}[n]$.

We can think about symmetry properties as a way to check this answer. The sum of $x_0[n]$ and $x_0[-n]$ (which is a flipped version about $n = 0$) will be an even function of n. Since $x_0[n]$ is also periodic in $n = 51$, the result of adding $x_0[n]$ to $x_0[-n]$ is also symmetric about $n = 25.5$. There are only four signals with this symmetry: x_9, x_{11}, x_{16}, and x_{22}. (Notice that x_{14} is not quite right since there are only four leading values of zero.) However, the signal is clearly not zero, eliminating x_{11}. Also x_9 is upside down and x_{22} is upside-down plus a constant. Thus the answer must be x_{16}.

b. $X_B[k] = \text{Im}\left(X_0[k]\right)$

$$X_B[k] = \text{Im}\left(X_0[k]\right) = \frac{1}{2j}X_0[k] - \frac{1}{2j}X_0^*[k] \quad \text{(property of complex numbers)}$$

Then

$$x_B[n] = \frac{1}{2j}x_0[n] - \frac{1}{2j}x_0^*[-n] = \frac{1}{2j}x_0[n] - \frac{1}{2j}x_0[-n]$$

Since $x_0[n]$ is real-valued, $x_B[n]$ must be complex-valued.

None of the possible answers are complex valued, so the answer is "COMPLEX".

c. $X_C[k] = j\text{Im}\left(X_0[k]\right)$

$$X_C[k] = j\,\text{Im}\left(X_0[k]\right) = \frac{1}{2}X_0[k] - \frac{1}{2}X_0^*[k] \quad \text{(property of complex numbers)}$$

Thus

$$x_C[n] = \frac{1}{2}x_0[n] - \frac{1}{2}x_0^*[-n] = \frac{1}{2}x_0[n] - \frac{1}{2}x_0[-n]$$

since $x_0[n]$ is real-valued. When $x_0[-n]$ is subtracted from $x_0[n]$, the result is an odd function of n. Since $x_0[n]$ is also periodic in $N = 51$, the result is also antisymmetric about $n = 25.5$. The result looks like $x_8[n]$.

x_{11} has the right symmetry properties, but our answer is clearly not zero. Also x_{13} clearly has the wrong shape. x_{21} is the negative of the right answer, i.e.,$x[-n] - x[n]$. So the answer must be x_8.

d. $X_D[k] = \begin{cases} 0 & \text{if } k = 0 \\ X_0[k] & \text{otherwise} \end{cases}$

By setting $k = 0$ in the analysis equation,

$$X_0[k] = \frac{1}{N}\sum x_0[n]e^{\frac{-j2\pi kn}{N}}$$

we can see $X_0[0]$ is the average value of $x_0[n]$. Let $\bar{x}$ represent the average value of $x_0[n]$. Then by linearity

$$x_0[n] - \bar{x} \overset{\text{FT}}{\Longrightarrow} X_0[k] - X_0[0]$$

Setting $X_0[0]$ to zero is thus equivalent to subtracting the average value of $x_0[n]$ from $x[n]$ for all n.

Two signals $x_6[n]$ and $x_{19}[n]$ are simple vertical shifts of $x_0[n]$. Since $x_{19}[n]$ is shifted in the wrong direction, the answer must be $x_6[n]$.

e. $X_E[k] = \begin{cases} 0 & \text{if } k = 25 \\ X_0[k] & \text{otherwise} \end{cases}$

Setting the twenty-fifth component of the Fourier series to zero is equivalent to subtracting a complex exponential with frequency of $\frac{2\pi 25}{51}$ from $x_0[n]$.

So our new signal would be $x_E[n] = x_0[n] - X_0[25]e^{j2\pi(25/51)n}$. Unless $X_0[25] = 0$, this extra term will be complex-valued.

f. $X_F[k] = X_0[k] + 1/51$

By linearity, adding a constant to $X_0[k]$ adds a signal $y[n]$ to $x_0[n]$ where $y[n]$ is the signal whose Fourier series $Y[k]$ is $1/51$ for all k:

$$y[n] = \sum \frac{1}{51} e^{\frac{j\omega kn}{51}}$$

By orthogonality, $y[n]$ must be $\delta[n]$ since the above sum goes to zero except at $n = 0$.

Thus the solution is $x_{20}[n]$.

g. $X_G[k] = e^{j\pi} X_0[k]$

The multiplier $e^{j\pi}$ is equal to -1. Therefore the new signal is flipped about the horizontal axis. The solution must be $x_{23}[n]$.

h. $X_H[k] = \begin{cases} X_0[0] & \text{if } k = 0 \\ e^{j\pi} X_0[k] & \text{otherwise} \end{cases}$

The multiplier here is the same as in Part 7.

However, the DC term is still that of the original signal (which is positive). The resulting effect is that $x_H[n] = 2X_0[0] - x_0[n]$ (i.e., it is reflected about the horizontal axis, and then shifted to account for the change in DC value).

The solution is $x_{15}[n]$.

i. $X_I[k] = |X_0[k]| e^{j(-\angle X_0[k])}$

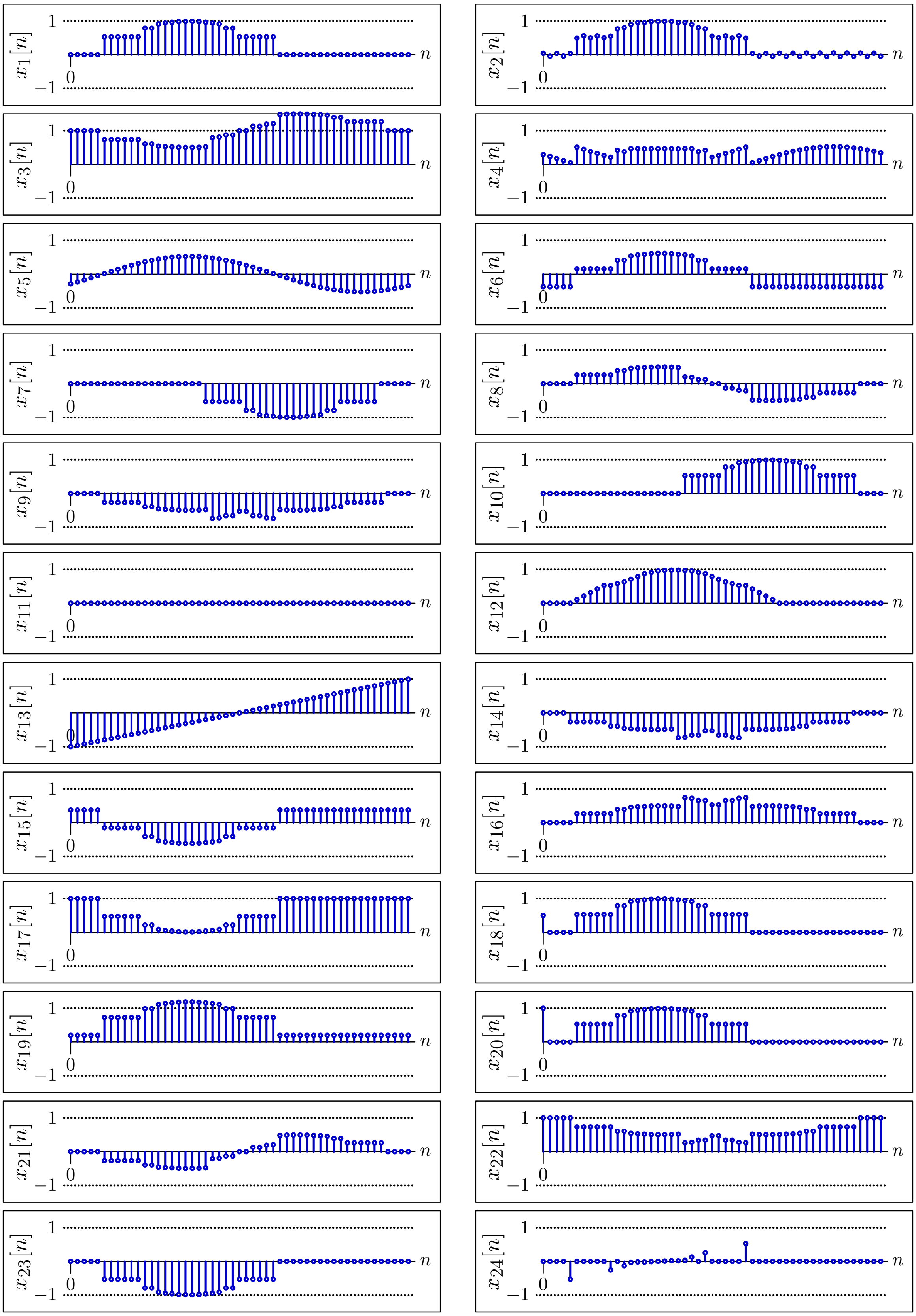
Negating the angle of a complex number while holding the magnitude constant has the same effect as taking the complex conjugate of the original number. This follows from thinking about the definition of magnitude and angle of a complex number a:

$$a = |a| e^{j\angle a}$$
$$a^* = |a| e^{-j\angle a}$$

Thus $X_I[k] = X_0^*[k]$.

Conjugating the Fourier series has the effect of conjugating the time function and then flipping it about $n = 0$. Since $x_0[n]$ is real-valued, the result is just a time flip, and the answer is x_{10}.

Ben's Graphs:


$x_1[n]$
$x_2[n]$
$x_3[n]$
$x_4[n]$
$x_5[n]$
$x_6[n]$
$x_7[n]$
$x_8[n]$
$x_9[n]$
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$x_{21}[n]$
$x_{22}[n]$
$x_{23}[n]$
$x_{24}[n]$
1
0
−1
n