6.300: Signal Processing

Sampling and Aliasing

Sampling: $x[n] \triangleq x(n\Delta)$, where $\Delta = \frac{1}{f_s}$ is the sampling interval. Brackets matter! Unless $\Delta = 1$, $x[n] \neq x(n)$.

Aliasing: Sometimes, $x_1[n] = x_2[n]$ even if $x_1(t) \neq x_2(t)$. Different continuous-time signals may yield the same set of discrete-time samples! (Generally undesirable.)

Sampling theorem: Suppose that f_{max} is the highest non-zero frequency in a CT signal. A sampling rate $f_s \geq 2f_{\text{max}}$ will prevent aliasing in frequencies $|f| \leq f_{\text{max}}$.

February 12, 2026

Agenda for Recitation

- Dimensional analysis
- Sampling
- Aliasing and the sampling theorem

What questions do you have from lecture?

Dimensional Analysis

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We can often get the answer just by "mashing" quantities together to get the right dimensions in the end.

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> If a car is traveling at 60 miles per hour, how far does the car travel in 2.5 hours?

"Mash" quantities together to get the right dimensions.

$$\underbrace{\left(60\,\frac{\text{miles}}{\text{hour}}\right)}_{\text{rate}} \times \underbrace{(2.5\ \text{hours})}_{\text{time}} = \underbrace{(150\ \text{miles})}_{\text{distance}}$$

Dimensional Analysis

Consider a **pendulum**. The rod has length L and the bob has mass M. Let G denote the acceleration due to gravity. What is the period T of the oscillations?

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Consider a **pendulum**. The rod has length L and the bob has mass M. Let G denote the acceleration due to gravity. What is the period T of the oscillations?

Figure out the dimensions of the quantities we're given.

- L length (e.g., meters)
- M mass (e.g., kilograms)
- G length per time^2 (e.g., meters per second^2)
- T time (e.g., seconds)

"Mash" all these together to get the right dimensions.

$$\text{time} = \sqrt{\frac{\text{length}}{\text{length per time}^2}}$$

So, $T \propto \sqrt{L/G}$, i.e., $T = C\sqrt{L/G}$ for some constant C.

Dimensional Analysis

Sinusoids in Continuous Time (CT)

$$\cos(\omega_0 t) = \cos(2\pi f_0 t) = \cos\left(\tfrac{2\pi}{T} t\right)$$

Sinusoids in Discrete Time (DT)

$$\cos(\Omega_0 n) = \cos(2\pi F_0 n) = \cos\left(\tfrac{2\pi}{N} n\right)$$

Determine units (e.g., seconds) for each term below.

- T period (CT)
- f_0 cyclical frequency (CT)
- ω_0 angular frequency (CT)
- N period (DT)
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Determine units (e.g., seconds) for each term below.

- T period (CT) — seconds
- f_0 cyclical freq. (CT) — cycles per second (Hz)
- ω_0 angular freq. (CT) — radians per second
- N period (DT) — samples
- F_0 cyclical freq. (DT) — cycles per sample
- Ω_0 angular freq. (DT) — radians per sample

Dimensional Analysis

Suppose that we record 5 seconds of audio using a sampling rate $f_s = 20$ kHz. How many samples long is the resulting discrete-time (DT) signal?

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Suppose that we record 5 seconds of audio using a sampling rate $f_s = 20$ kHz. How many samples long is the resulting discrete-time (DT) signal?

Always keep track of the dimensions!

$$\underbrace{(5 \text{ seconds})}_{\text{time}} \times \underbrace{\left(20 \times 10^3 \, \frac{\text{samples}}{\text{second}}\right)}_{\text{samples per unit time}} = 10^5 \text{ samples}$$

The sampling rate f_s is the "currency exchange rate" between seconds T and samples N.

Dimensional Analysis

“Middle C” is approximately 261.63 hertz (Hz). Let's generate a “middle C” tone that lasts 5 seconds using a sampling rate of 44,100 samples per second (Hz).

```
import math
from lib6300.audio import wav_write
tone = [cos(EXPR1 * n) for n in range(0, EXPR2)]
wav_write(tone, 44100, 'audio.wav')
```

Determine values for `EXPR1` and `EXPR2`.

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Determine values for `EXPR1` and `EXPR2`.

`EXPR1 = 2 * math.pi * 261.63 / 44100` is the frequency, Ω_0.

`EXPR2 = int(5 * 44100)` is the number of samples.

Sampling

A **continuous-time (CT) signal** is a **function**.
Time is indexed by t, a real number.

A **discrete-time (DT) signal** is a **sequence**.
Time is indexed by n, an integer.

Sampling a CT signal yields a DT signal: $t \to n\Delta = n/f_s$.

$$x[n] \triangleq x(n\Delta) = x\left(\frac{n}{f_s}\right)$$

- Δ (seconds per sample) is the **sampling interval**.
- f_s (samples per second) is the **sampling rate**.

Brackets matter! Unless $\Delta = f_s = 1$, $x[n] \neq x(n)$.

Sampling

Sketch $x(t) = \cos(2\pi t)$ for $0 \leq t \leq 3$.

Next, sketch $x[n]$ for each choice of Δ given below.

- $\Delta = 0.25$ seconds per sample
- $\Delta = 0.5$ seconds per sample
- $\Delta = 0.75$ seconds per sample
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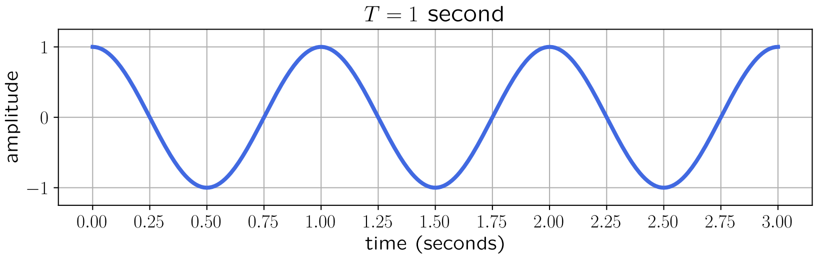

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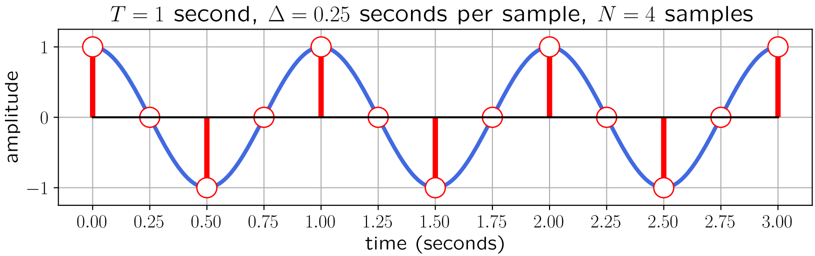

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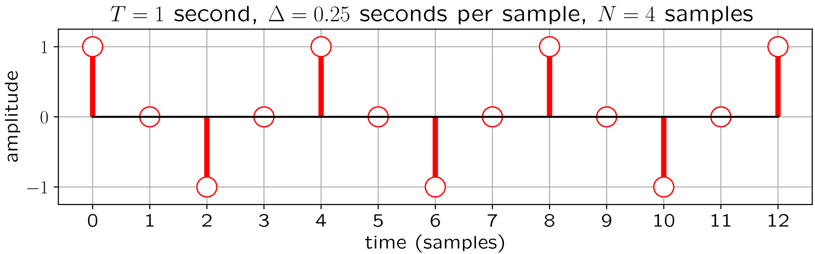

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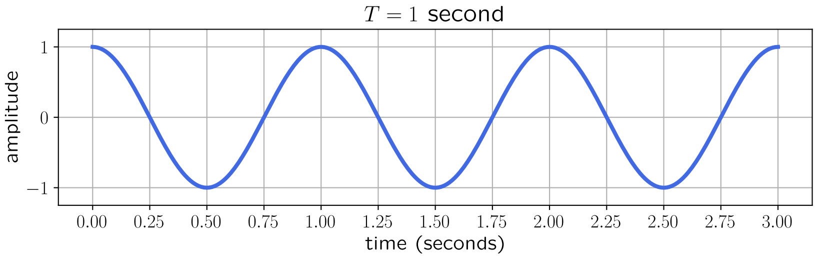

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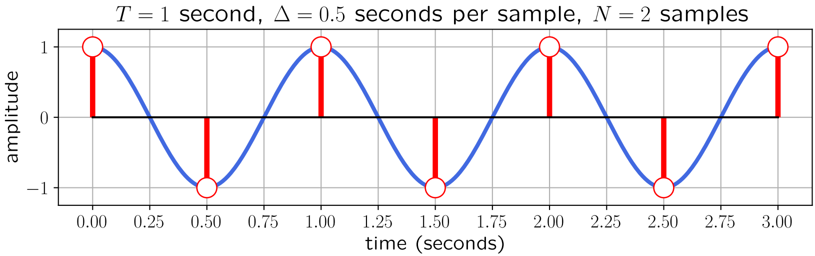

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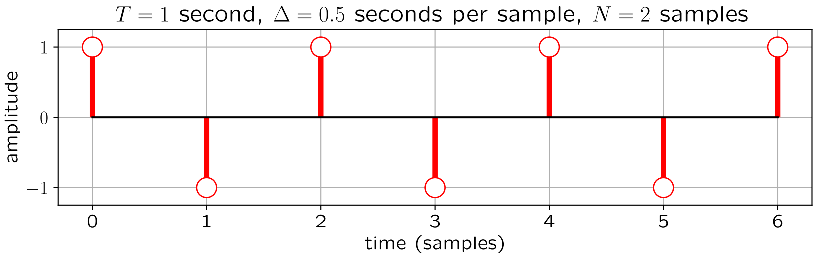

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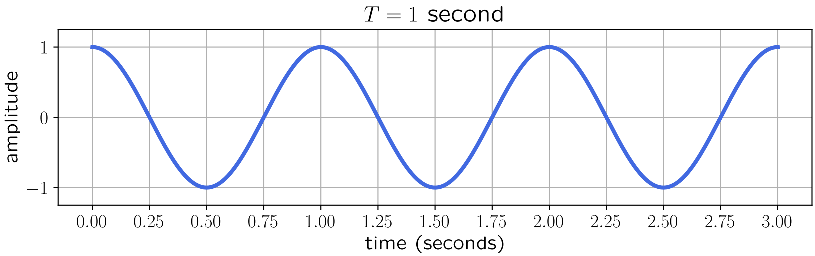

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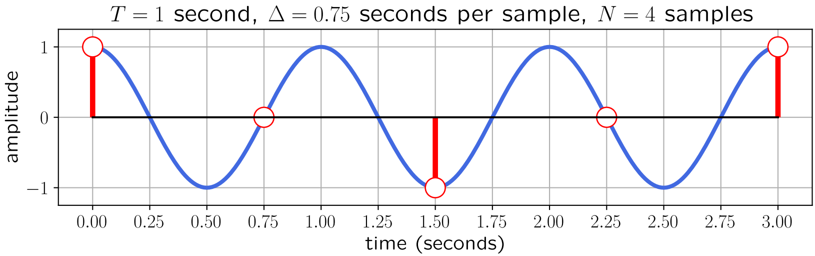

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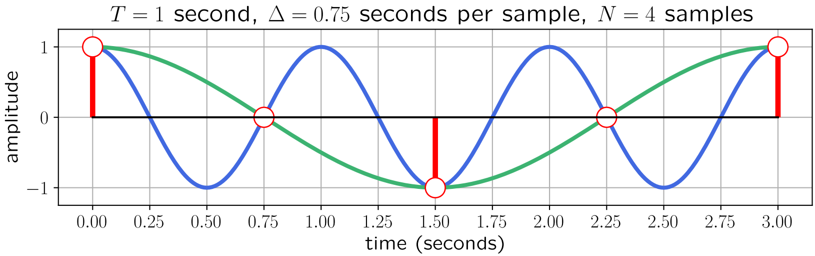

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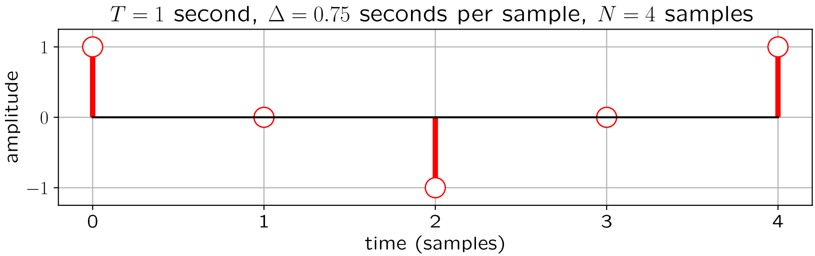

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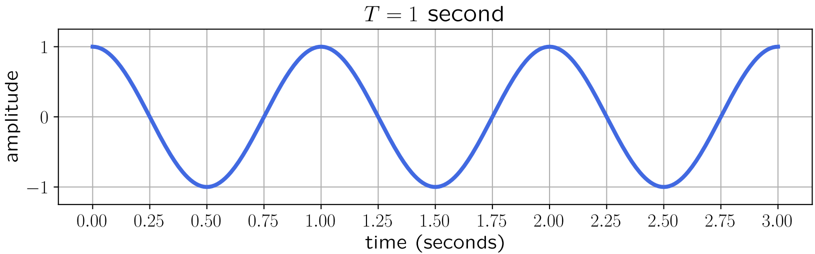

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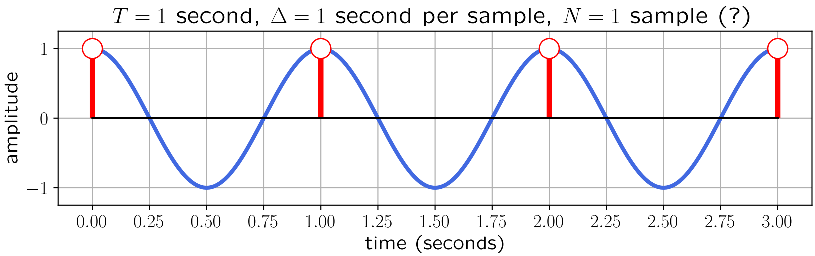

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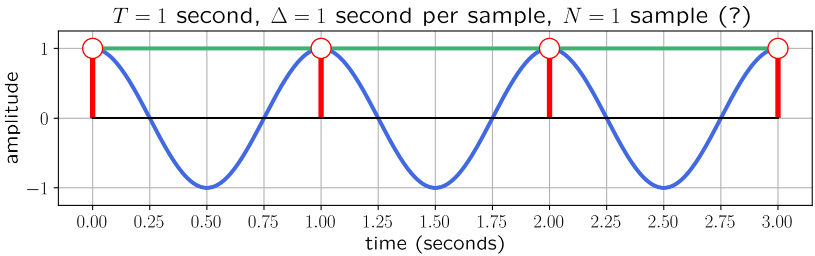

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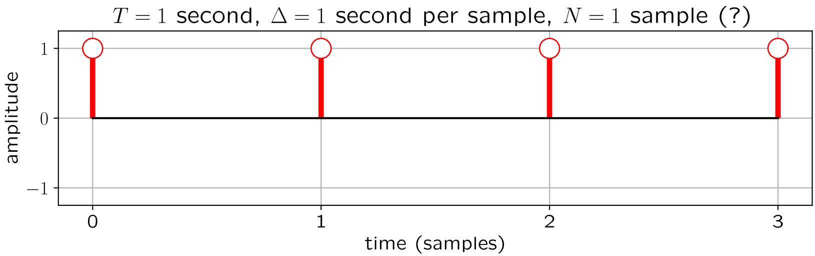

Sampling

What is the fundamental period (T) of $x(t)$?

$$x(t) = \cos\left(\frac{2\pi}{7}t\right)$$

Suppose we sample $x(t)$ using sampling interval $\Delta = 3$ seconds per sample. What is the fundamental period (N) of $x[n]$, the resulting discrete-time signal?

Sampling

What is the fundamental period (T) of $x(t)$?

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Suppose we sample $x(t)$ using sampling interval $\Delta = 3$ seconds per sample. What is the fundamental period (N) of $x[n]$, the resulting discrete-time signal?

$T = 7$ seconds is the fundamental period of $x(t)$.

The fundamental period of $x[n]$ **must be an integer!**

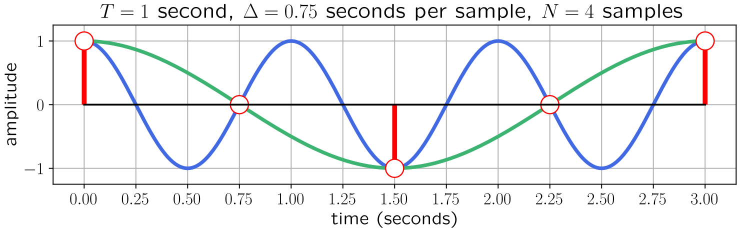
$$N \text{ (samples)} = T \text{ (seconds)} \times \Delta^{-1} \text{ (seconds per sample)}^{-1}$$
$$= 7 \times \frac{1}{3} \implies 7 \text{ is the least integer multiple}$$

Aliasing and the Sampling Theorem

When we sample, we throw away some information that was in the CT signal. What if we throw away too much?

Answer: **Aliasing!** It occurs if we sample too slowly.

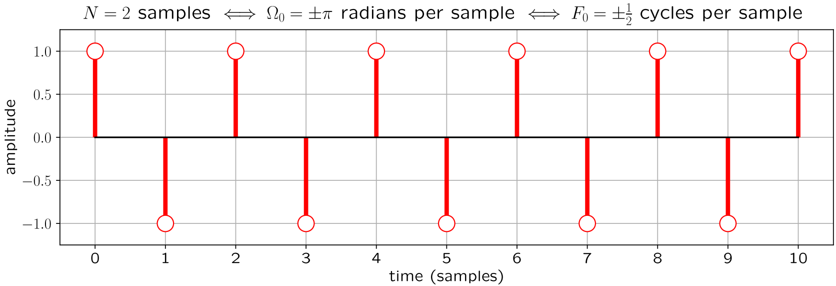
Aliasing: Sometimes, $x_1[n] = x_2[n]$ even if $x_1(t) \neq x_2(t)$. Different continuous-time signals may yield the same set of discrete-time samples! (Generally undesirable.)



Aliasing and the Sampling Theorem

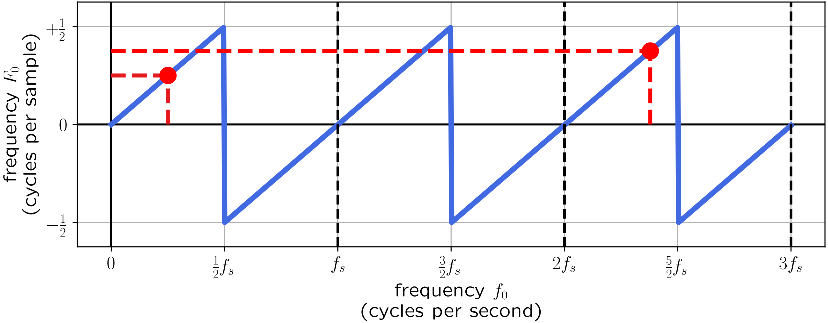
There is no smallest unit of continuous time, but there is indeed a smallest unit of discrete time — a single sample.

Similarly, in continuous time, there is no such thing as a "highest frequency" — but in discrete time, there is!



The highest discrete-time frequency is $\Omega_0 = \pm\pi$ radians per sample — equivalently, $F_0 = \pm\frac{1}{2}$ cycles per sample.

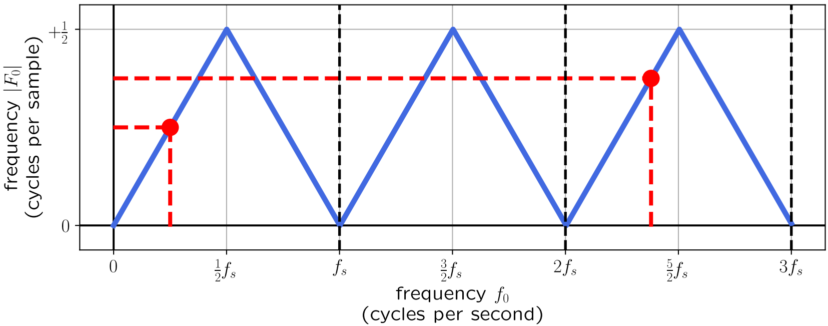
Aliasing and the Sampling Theorem



Which continuous-time frequencies (e.g., f_0 and ω_0) these discrete-time frequencies (e.g., F_0 and Ω_0) correspond to is determined by the sampling rate f_s.

$$\underbrace{-\tfrac{1}{2}f_s \le f_0 \le \tfrac{1}{2}f_s}_{\text{cycles per second}} \iff \underbrace{-\tfrac{1}{2} \le F_0 \le \tfrac{1}{2}}_{\text{cycles per sample}} \iff \underbrace{-\pi \le \Omega_0 \le \pi}_{\text{radians per sample}}$$

Aliasing and the Sampling Theorem

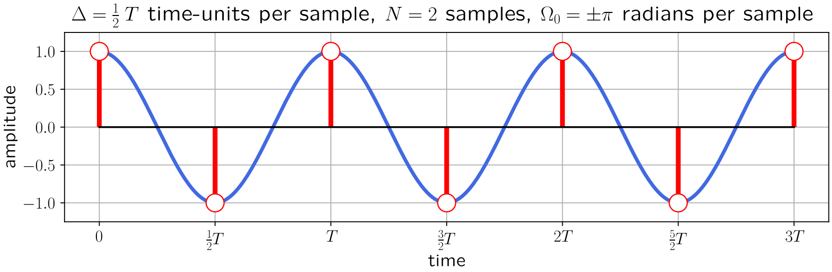


At times, we may only be sensitive to the magnitude (and not the sign) of the frequency — like when listening to audio. So, sometimes we can simplify the previous plot.

Demonstration: What does aliasing sound like?

Aliasing and the Sampling Theorem

Hand-wavy "proof by picture" of the sampling theorem: It takes **at least 2 samples per cycle** to preserve the information in a sinusoid. $\Delta \leq \frac{1}{2}T \iff f_s \geq 2f_0$. QED.



Sampling theorem: Suppose that f_{max} is the highest non-zero frequency in a CT signal. A sampling rate $f_s \geq 2f_{\text{max}}$ will prevent aliasing in frequencies $|f| \leq f_{\text{max}}$.

Lessons Learned

Discrete-time (DT) signals result from **sampling** continuous-time (CT) signals.

Sampling: $x[n] \triangleq x(n\Delta)$, where $\Delta = \frac{1}{f_s}$ is the sampling interval. Brackets matter! Unless $\Delta = 1$, $x[n] \neq x(n)$.

Aliasing: Sometimes, $x_1[n] = x_2[n]$ even if $x_1(t) \neq x_2(t)$. Different continuous-time signals may yield the same set of discrete-time samples! (Generally undesirable.)

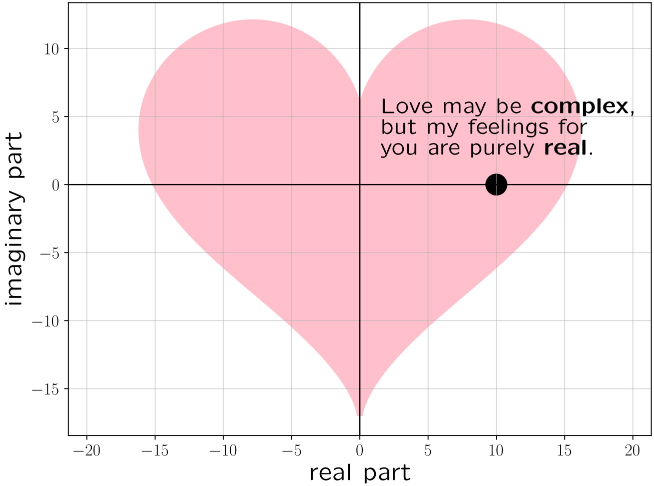
Sampling theorem: Suppose that f_{max} is the highest non-zero frequency in a CT signal. A sampling rate $f_s \geq 2f_{\text{max}}$ will prevent aliasing in frequencies $|f| \leq f_{\text{max}}$.

Question of the Day

True or false? The upper threshold of human hearing is approximately 20 kHz, so a sampling rate of 40 kHz ought to suffice for all digital audio applications.

If **false**, briefly explain your reasoning.

Valentine's Day at MIT



Getting mixed **signals**? Well, now you can do a **Fourier analysis** — and your homework on **CT and DT signals**.