# 6.300: Signal Processing

## Frequency Response and Filtering

An LTI system's **frequency response** is the Fourier transform of the system's unit-sample response.

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} \iff h[n] = \frac{1}{2\pi}\int_{2\pi} H(\Omega)e^{j\Omega n}d\Omega$$

**Filtering:** Convolution ($*$) in the time domain corresponds to multiplication ($\times$) in the frequency domain.

$$x[n] \rightarrow \boxed{\text{LTI}} \rightarrow y[n] = (h * x)[n]$$

$$X(\Omega) \rightarrow \boxed{\text{LTI}} \rightarrow Y(\Omega) = H(\Omega)X(\Omega)$$

*March 17, 2026*

# Agenda for Recitation

- Filters: Low-pass, high-pass, band-pass, and band-stop
- Connecting systems in series: Cascaded system
- Analog and digital systems; filter design (optional)
- Fourier transforms in probability theory (optional)

What questions do you have from lecture?

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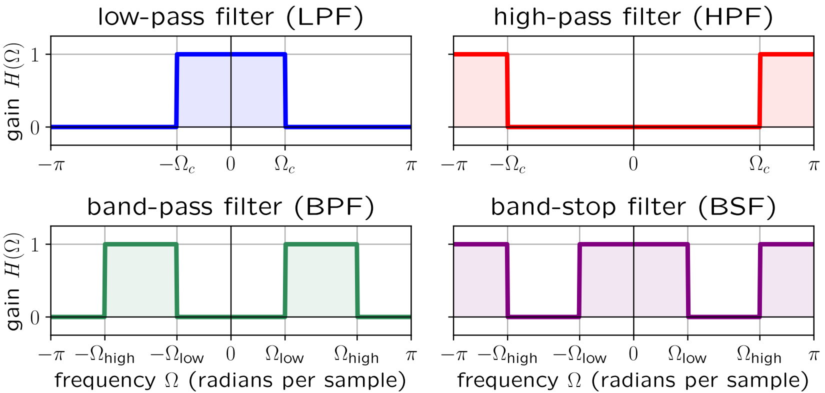
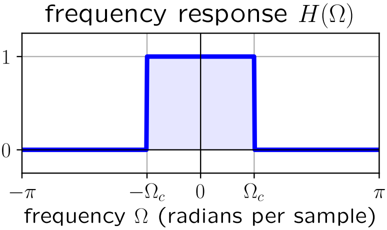
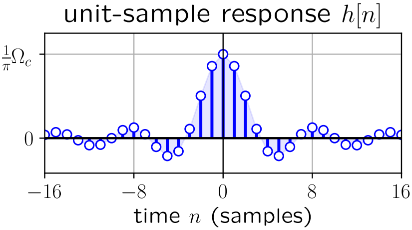
# Filters




- **LPF:** pass low frequencies, reject high frequencies
- **HPF:** pass high frequencies, reject low frequencies
- **BPF:** pass specified band of frequencies, reject others
- **BSF:** reject specified band of frequencies, pass others

# Filters




Suppose that we low-pass filter a signal.

$$X(\Omega) \rightarrow \boxed{\text{LPF}} \rightarrow Y(\Omega) = H(\Omega)X(\Omega)$$

**or** $$x[n] \rightarrow \boxed{\text{LPF}} \rightarrow y[n] = (h * x)[n]$$

Multiplication ($\times$) by $H(\Omega)$ in the frequency domain is equivalent to convolution ($*$) with $h[n]$ in the time domain. Determine a closed-form expression for $h[n]$.

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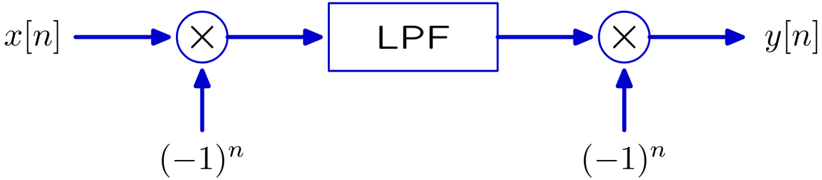
# Cascaded System

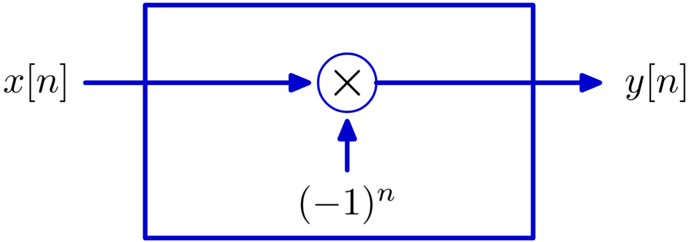



Consider the cascaded system shown above.

How many statements below are true?

- The cascaded system is **linear**.
- The cascaded system is **time-invariant**.

# Cascaded System: Multiplier




Let's break the problem down into sub-problems.
Consider the multiplier sub-system shown above.

How many statements below are true?

- The multiplier is **linear**.
- The multiplier is **time-invariant**.

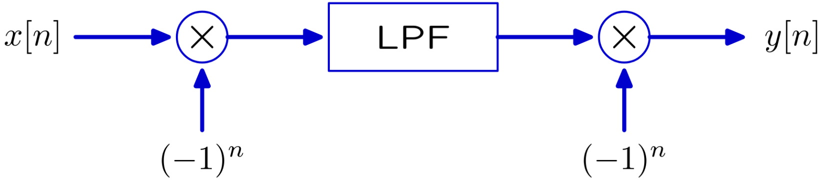
# Cascaded System: Low-Pass Filter


Let's break the problem down into sub-problems.
Consider the low-pass filter sub-system shown above.

How many statements below are true?

- The low-pass filter is **linear**.
- The low-pass filter is **time-invariant**.

# Cascaded System




Consider the cascaded system shown above.

How many statements below are true?

- The cascaded system is **linear**.
- The cascaded system is **time-invariant**.

# Cascaded System: High-Pass Filter?

The time-domain relation between input and output is

$$\begin{aligned} y[n] &= (-1)^n \sum_m (-1)^m x[m] h[n-m] \\ &= \sum_m (-1)^{n+m} x[m] h[n-m] \\ &= \sum_m (-1)^{n-m} x[m] h[n-m] \\ &= \sum_m x[m] (-1)^{n-m} h[n-m] \\ &= \sum_m x[m] h'[n-m] \end{aligned}$$

So, is $h'[n] \triangleq (-1)^n h[n]$ a high-pass filter?

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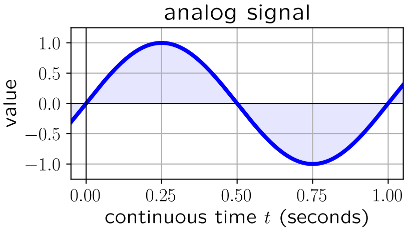
# Analog and Digital Systems

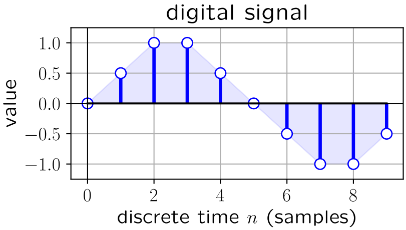
**Analog** means "continuous in both the independent and dependent variables." **Digital** means "discrete in both the independent and dependent variables."

We often think of time ($t$, $n$) or frequency ($\omega$, $\Omega$) as the independent variables.

**Digitization** (analog $\rightarrow$ digital) entails two steps.

- **sampling** (e.g., continuous time $t \rightarrow$ discrete time $n$)
- **quantization** (storing values with finite register length)



# Analog and Digital Systems

## Analog Filters

- electrical (resistor, capacitor, inductor)
- mechanical (mass, spring, damper)
- acoustical (mass, compliance, resistance)
- biological (e.g., phosphorylation cycle[1])

## Digital Filters

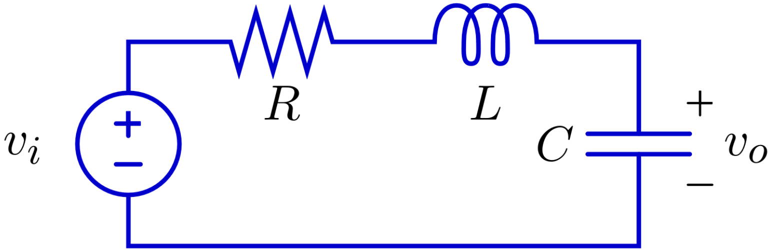
- array(s) of numbers
  - coefficients $(b_0, b_1, \ldots, a_0, a_1, \ldots)$
  - zeros, poles, and gain $(\xi_1, \xi_2, \ldots, p_1, p_2, \ldots, K)$
  - state-space representation $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$

[1] *Biomolecular Feedback Systems* by Del Vecchio and Murray

# Analog and Digital Systems

## Electrical System: Series RLC Circuit

- input voltage $v_i = v_i(t)$ volts (V)
- output voltage $v_o = v_o(t)$ volts (V)
- resistor with resistance $R$ ohms ($\Omega$)
- capacitor with capacitance $C$ farads (F)
- inductor with inductance $L$ henries (H)



## Electrical System: Series RLC Circuit

The relation between $v_i$ and $v_o$ takes the form of a linear, constant-coefficient differential equation.

$$LC\frac{d^2v_o(t)}{dt^2} + RC\frac{dv_o(t)}{dt} + v_o(t) = v_i(t)$$

The frequency response may be written as

$$H(\omega) \triangleq \frac{V_o(\omega)}{V_i(\omega)} = \frac{\omega_0^2}{\omega_0^2 + \frac{1}{\tau}j\omega - \omega^2}$$

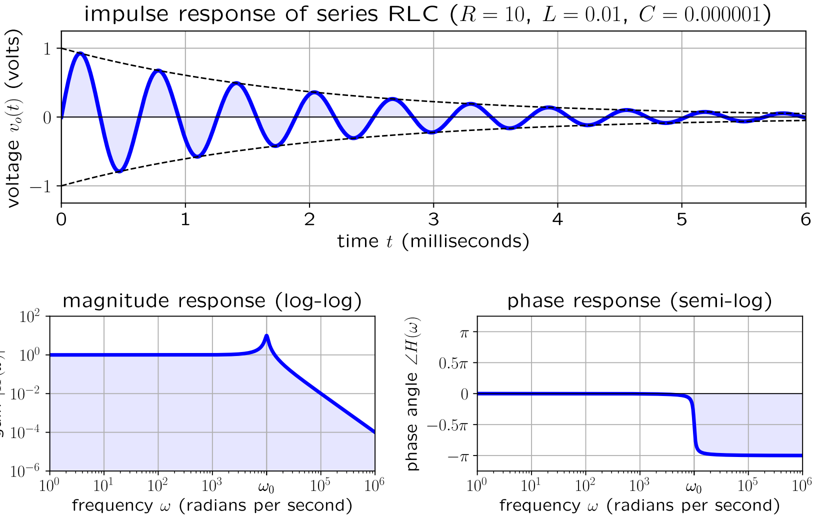
where

$$\omega_0 \triangleq \sqrt{\frac{1}{LC}} \text{ and } \tau \triangleq \frac{L}{R}$$

denote the **natural frequency** and **time constant**, respectively.

# Analog and Digital Systems

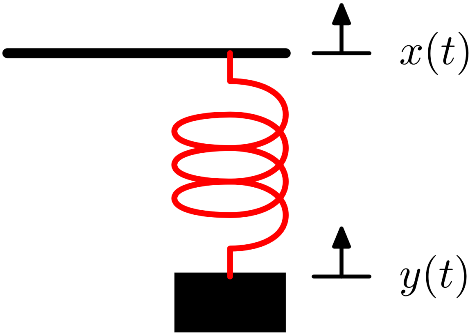
## Electrical System: Series RLC Circuit



# Analog and Digital Systems

## Mechanical System: Mass on Spring

- input position $x(t)$ — meters (m)
- output position $y(t)$ — meters (m)
- mass $M$ — kilograms (kg)
- spring constant $K$ — newtons per meter (N/m)



# Analog and Digital Systems

## Mechanical System: Mass on Spring

The relation between $x$ and $y$ takes the form of a linear, constant-coefficient differential equation.

$$M \frac{d^2 y(t)}{dt^2} = K\big(x(t) - y(t)\big)$$

The frequency response may be written as

$$H(\omega) \triangleq \frac{Y(\omega)}{X(\omega)} = \frac{\omega_0^2}{\omega_0^2 - \omega^2}$$

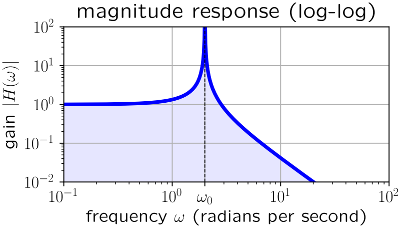
where

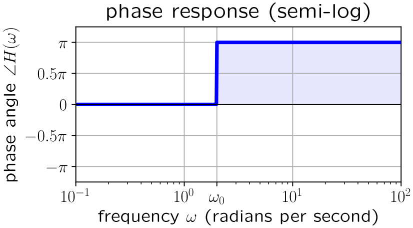
$$\omega_0 \triangleq \sqrt{\frac{K}{M}}$$

denotes the **natural frequency**.

# Analog and Digital Systems

## Mechanical System: Mass on Spring




**Resonance** occurs at the natural frequency $\omega_0$. Pure resonance may only be attained without any damping.

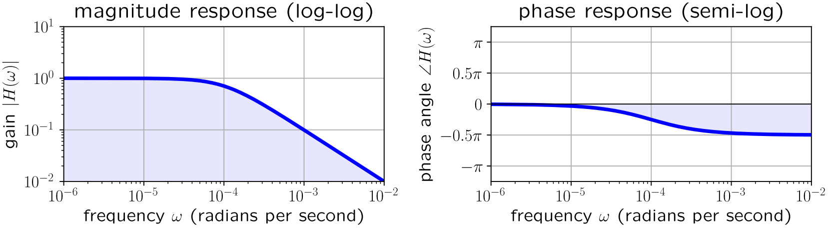
**Electrical-mechanical analogies:** For linear **electrical systems**, the "basic building blocks" (i.e., lumped elements) are resistors, inductors, and capacitors. For linear **mechanical systems**, the "basic building blocks" (i.e., lumped elements) are springs, masses, and dampers.

# Analog and Digital Systems

## Biological System: Phosphorylation Cycle[2]

- kinase $x(t)$
- phosphorylated substrate $y(t)$
- production rate $\beta$
- decay rate $\gamma$

$$\frac{dy(t)}{dt} = \beta x(t) - \gamma y(t)$$




[2] *Biomolecular Feedback Systems* by Del Vecchio and Murray

# Analog and Digital Systems

## Digital Processing of Analog Signals

- anti-aliasing filter — analog low-pass filter
- analog-to-digital converter — sampling, quantization
- digital signal processor — digital filter
- digital-to-analog converter — interpolation
- anti-imaging filter — analog low-pass filter

$$x(t) \rightarrow \boxed{\text{LPF}} \rightarrow \boxed{\text{A/D}} \rightarrow \boxed{\text{DSP}} \rightarrow \boxed{\text{D/A}} \rightarrow \boxed{\text{LPF}} \rightarrow y(t)$$

Is digital signal processing (DSP) worth all this hassle?
Oftentimes, yes!

# Analog and Digital Systems

Per Professor Al Oppenheim, pre-1960s signal processing was

> *essentially **analog**. People built signal processing systems with . . . resistors, inductors, capacitors, op amps, and things like that.*[3]

J. W. Cooley and J. Tukey's re-discovery of the **FFT** circa 1965 marked the advent of **digital signal processing (DSP)** as we now know it — processing signals with digital computers.

Dr. Tom Barnwell (of Georgia Tech) once joked,

> *DSP is a discpline that allows us to replace a simple resistor and capacitor with two anti-aliasing filters, an [analog-to-digital] and [digital-to-analog] converter, and a general purpose computer . . . as long as the signal we are interested in doesn't vary too quickly.*[4]

[3] `https://ethw.org/Oral-History:Alan_Oppenheim`

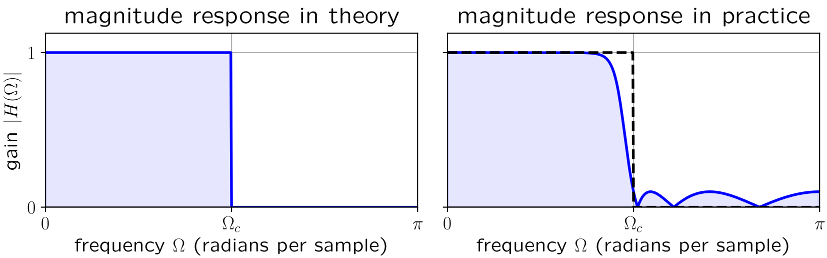
[4] `https://www.eecs.mit.edu/signal-processing-how-did-we-get-to-where-were-going/`

# Digital Filter Design

"Ideal filters" simplify analyses. They have

- perfectly flat **passbands**,[5]
- perfectly flat **stopbands**,[6] and
- arbitrarily narrow **transition bands**,[7]

but they are not realizable in practice.




[5]range of frequencies you want to keep
[6]range of frequencies you want to reject
[7]region between passband and stopband

# Digital Filter Design

**Filter design:** Design a filter to satisfy specifications. Some compromises or trade-offs may need to be made.

**#1.** Define the desired response in time or frequency.

- e.g., **passband** and **stopband** frequencies

**#2.** Define **specifications**. (You must allow some "wiggle room" for a practical implementation.)

- maximum and minimum **passband gain**
- maximum and minimum **stopband gain**
- **filter order** (i.e., maximum length of $h[n]$)

**#3.** Choose a design method.

- **ad hoc** (e.g., intuition, experimentation, . . . )
- **optimal** (Solve an optimization problem.)[8]

[8] **Optimal** means that you solved some optimization problem. It doesn't mean that the filter you designed is any good, though.

# Digital Filter Design[9]

## Infinite Impulse Response (IIR)

- Difference equation: $\sum_k a_k y[n-k] = \sum_k b_k x[n-k]$
- Can be unstable: $\sum_n |(h * x)[n]| \to \infty$ even if $\sum_n |x[n]| < \infty$
- Can be difficult to control phase response $\angle H(\Omega)$
- Generally fewer adds, multiplies required than FIR filter
- Significant history in analog filter design

## Finite Impulse Response (FIR)

- Difference equation: $y[n] = \sum_k b_k x[n-k]$
- Always stable: If $\sum_n |x[n]| < \infty$, then $\sum_n |(h * x)[n]| < \infty$
- Linear phase $\angle H(\Omega) = \alpha_0 \Omega + \beta_0$ usually achievable
- Generally more adds, multiplies required than IIR filter
- Potential for polyphase filter bank implementation
- Can always be made causal: $h[n] = 0$ for $n < 0$
- Some relevant history in antenna array design

[9] from 6.7000: Discrete-Time Signal Processing (2024)

# Digital Filter Design

## FIR Design Methods

- Window method: Truncate unit-sample response.
- Frequency sampling: Sample $H(\Omega)$; compute $\tilde{h}[n]$.
- Least squares: Minimize squared error.
- Constrained least squares (e.g., peak gain constraints)
- Minimax: Minimize the maximum deviation.
  - e.g., via Parks-McClellan/Remez exchange algorithm

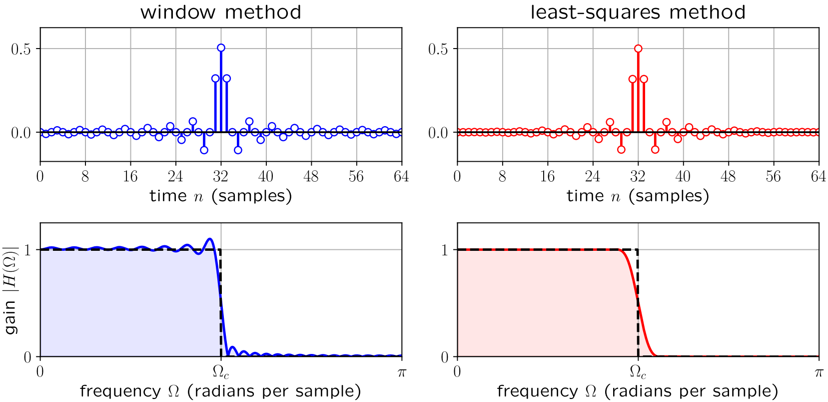
## IIR Design Methods

- Impulse invariance: Sample $h(t)$ to get $h[n] \triangleq h(n\Delta)$.
- Bilinear transform: Map $\omega \in (-\infty, \infty)$ to $\Omega \in [-\pi, \pi]$.
  - Butterworth
  - Chebyshev (Type I and Type II)
  - Elliptic (Cauer)

. . . and more!

# Digital Filter Design

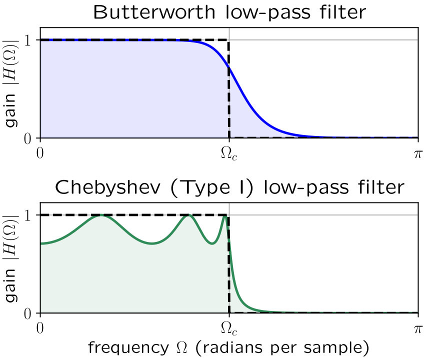
## Examples of FIR Digital Filters




- cut-off frequency: $0.5\pi$ radians per sample
- transition width: $0.1\pi$ radians per sample
- filter order (number of non-zero taps): 65

# Digital Filter Design

## Examples of IIR Digital Filters




- cut-off frequency: $0.5\pi$ radians per sample
- maximum passband ripple: 3 dB
- minimum stopband attenuation: 20 dB
- filter order: 6

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# Fourier Transforms in Probability

A **probability density function (PDF)** $p_X(t)$ defines the probability distribution of a **random variable** $X$.

Take on faith the fact that the random variable $Z = X + Y$ has probability density function $p_Z(t) = (p_X * p_Y)(t)$.

The Fourier transform of the probability density function is called the **characteristic function**.

$$\text{PDF } p_X(t) \iff \text{characteristic function } \varphi_X(\omega)$$

While the moment generating function (MGF) may not always exist, the characteristic function always exists.

Convolution of PDFs is equivalent to multiplication of the corresponding characteristic functions.

$$p_Z(t) = (p_X * p_Y)(t) \iff \varphi_Z(\omega) = \varphi_X(\omega)\varphi_Y(\omega)$$

# And We're Back

We now return to our regularly-scheduled programming.

# Lessons Learned

We may analyze and design LTI systems in both the time ($n$) and frequency ($\Omega$) domains. It's natural to describe many systems in terms of a frequency response $H(\Omega)$.

An LTI system's **frequency response** is the Fourier transform of the system's unit-sample response.

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} \iff h[n] = \frac{1}{2\pi}\int_{2\pi} H(\Omega)e^{j\Omega n}d\Omega$$

**Filtering:** Convolution ($*$) in the time domain corresponds to multiplication ($\times$) in the frequency domain.

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$$X(\Omega) \rightarrow \boxed{\text{LTI}} \rightarrow Y(\Omega) = H(\Omega)X(\Omega)$$