6.300: Signal Processing

Fourier Series

The **synthesis equation** tells us how to represent a periodic signal as a Fourier series.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k\frac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\frac{2\pi}{T}t\right)$$

The **analysis equations** tell us how to calculate these Fourier series coefficients.

$c_0 = \frac{1}{T}\int_T f(t)dt$ average value over a period

$c_k = \frac{2}{T}\int_T f(t)\cos\left(k\frac{2\pi}{T}t\right)dt$ for $k \geq 1$

$d_k = \frac{2}{T}\int_T f(t)\sin\left(k\frac{2\pi}{T}t\right)dt$ for $k \geq 1$

February 03, 2026

Agenda for Recitation

- Signals (and **transformations** thereof)
- **Fourier series** expansion for periodic signals

Agenda for Recitation

- Signals (and **transformations** thereof)
- **Fourier series** expansion for periodic signals

What are you hoping to get out of 6.300?

Agenda for Recitation

- Signals (and **transformations** thereof)
- **Fourier series** expansion for periodic signals

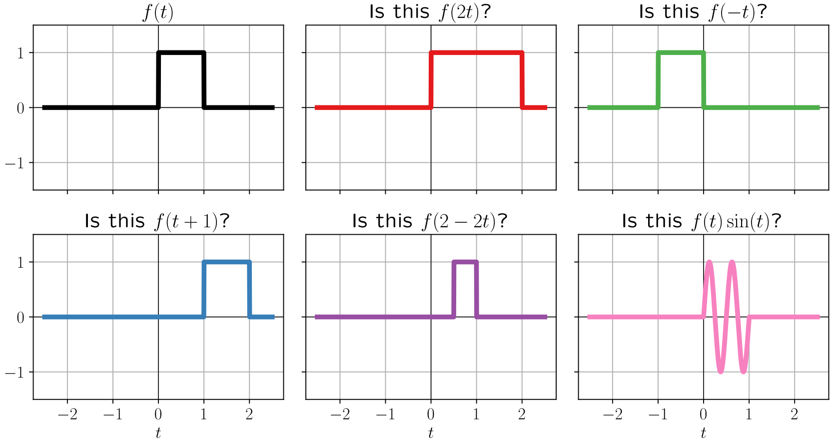
> What are you hoping to get out of 6.300?

Fourier methods are fundamental to signal processing — and to many other fields of science and engineering!

- acoustics, electromagnetics, and optics
- audio, speech, and music processing
- biomedicine (ECG, EEG, MRI)
- communications
- control
- oceanography, radio astronomy, and seismology
- remote sensing (RADAR, SONAR, LIDAR)

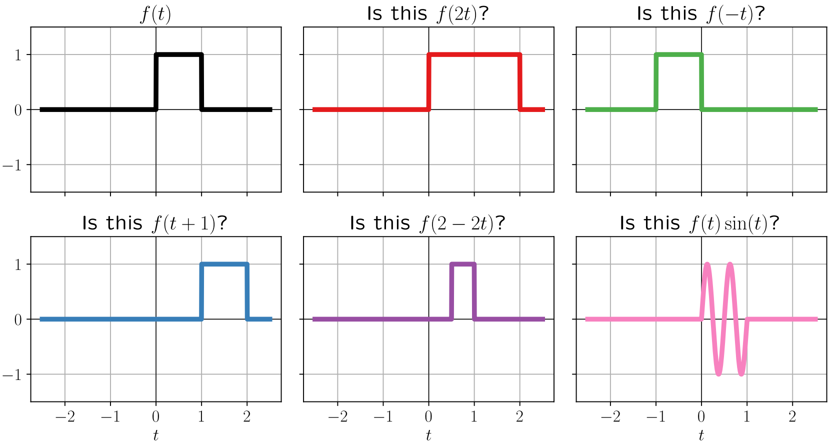
Transformations

How many expressions match the plot beneath them?



Transformations

How many expressions match the plot beneath them?
Answer: 2 $\rightarrow f(-t)$ **and** $f(2-2t)$



Series Expansions

In lecture, Professor Freeman analyzed a periodic signal $f(t) = f(t+T)$ using a **Fourier series** expansion.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\!\left(k\tfrac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\!\left(k\tfrac{2\pi}{T}t\right)$$

Series Expansions

In lecture, Professor Freeman analyzed a periodic signal $f(t) = f(t+T)$ using a **Fourier series** expansion.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\!\left(k\tfrac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\!\left(k\tfrac{2\pi}{T}t\right)$$

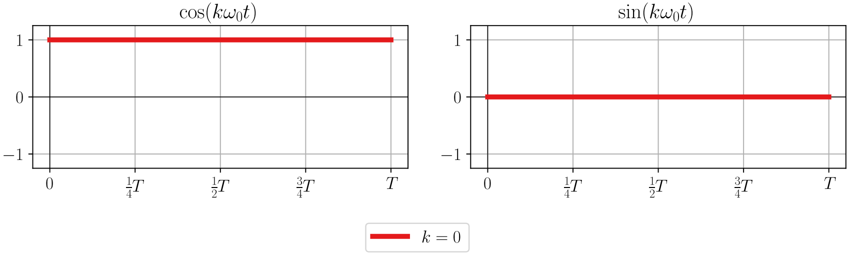
A Fourier series is a **sum of harmonically-related sinusoids**. That is, each sinusoid oscillates at an integer multiple k of the fundamental frequency $\omega_0 \triangleq 2\pi/T$.

Series Expansions

In lecture, Professor Freeman analyzed a periodic signal $f(t) = f(t+T)$ using a **Fourier series** expansion.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k\tfrac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\tfrac{2\pi}{T}t\right)$$

A Fourier series is a **sum of harmonically-related sinusoids**. That is, each sinusoid oscillates at an integer multiple k of the fundamental frequency $\omega_0 \triangleq 2\pi/T$.

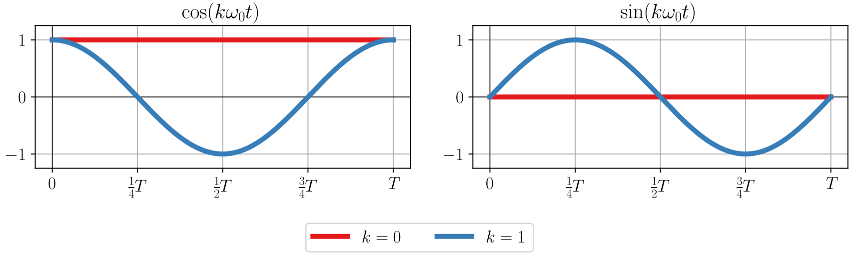


Series Expansions

In lecture, Professor Freeman analyzed a periodic signal $f(t) = f(t+T)$ using a **Fourier series** expansion.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k\tfrac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\tfrac{2\pi}{T}t\right)$$

A Fourier series is a **sum of harmonically-related sinusoids**. That is, each sinusoid oscillates at an integer multiple k of the fundamental frequency $\omega_0 \triangleq 2\pi/T$.

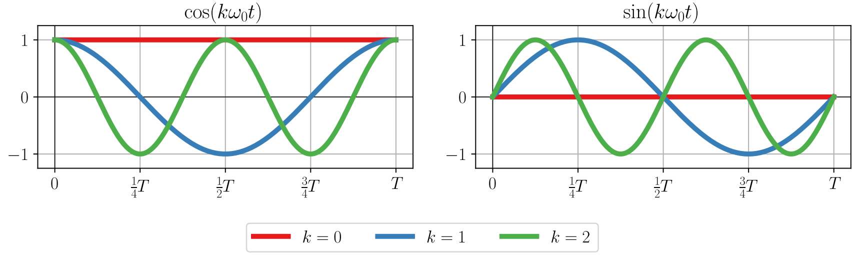


Series Expansions

In lecture, Professor Freeman analyzed a periodic signal $f(t) = f(t+T)$ using a **Fourier series** expansion.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k\tfrac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\tfrac{2\pi}{T}t\right)$$

A Fourier series is a **sum of harmonically-related sinusoids**. That is, each sinusoid oscillates at an integer multiple k of the fundamental frequency $\omega_0 \triangleq 2\pi/T$.

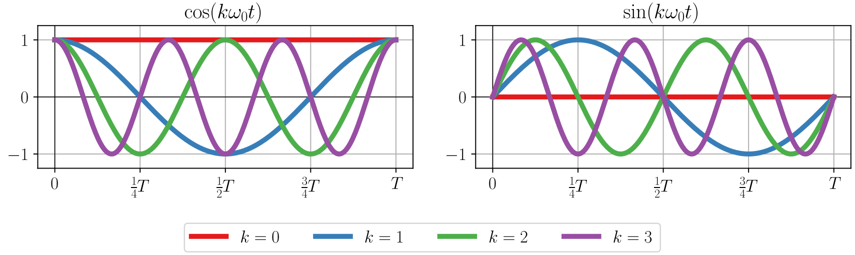


Series Expansions

In lecture, Professor Freeman analyzed a periodic signal $f(t) = f(t+T)$ using a **Fourier series** expansion.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k \tfrac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k \tfrac{2\pi}{T} t\right)$$

A Fourier series is a **sum of harmonically-related sinusoids**. That is, each sinusoid oscillates at an integer multiple k of the fundamental frequency $\omega_0 \triangleq 2\pi/T$.

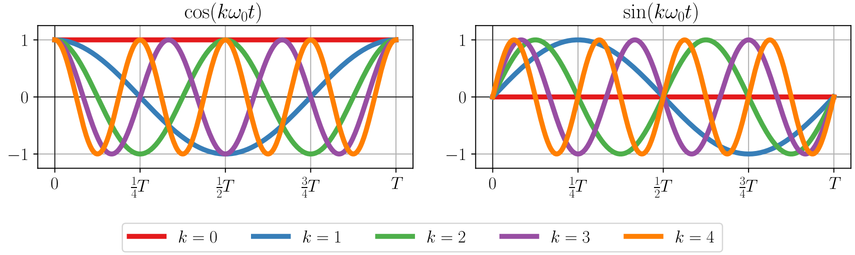


Series Expansions

In lecture, Professor Freeman analyzed a periodic signal $f(t) = f(t+T)$ using a **Fourier series** expansion.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k\tfrac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\tfrac{2\pi}{T}t\right)$$

A Fourier series is a **sum of harmonically-related sinusoids**. That is, each sinusoid oscillates at an integer multiple k of the fundamental frequency $\omega_0 \triangleq 2\pi/T$.

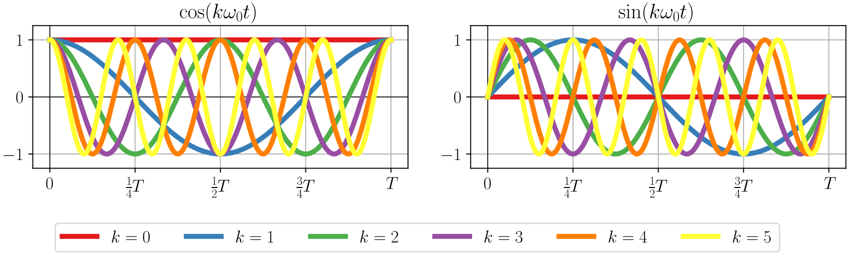


Series Expansions

In lecture, Professor Freeman analyzed a periodic signal $f(t) = f(t+T)$ using a **Fourier series** expansion.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k\frac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\frac{2\pi}{T}t\right)$$

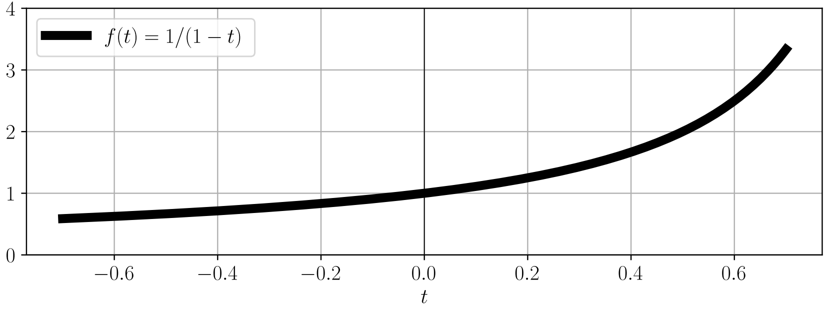
A Fourier series is a **sum of harmonically-related sinusoids**. That is, each sinusoid oscillates at an integer multiple k of the fundamental frequency $\omega_0 \triangleq 2\pi/T$.



Series Expansions

While we'll focus on **Fourier methods** in this subject, you might be familiar with another kind of expansion. A **Taylor series** yields a polynomial approximation.

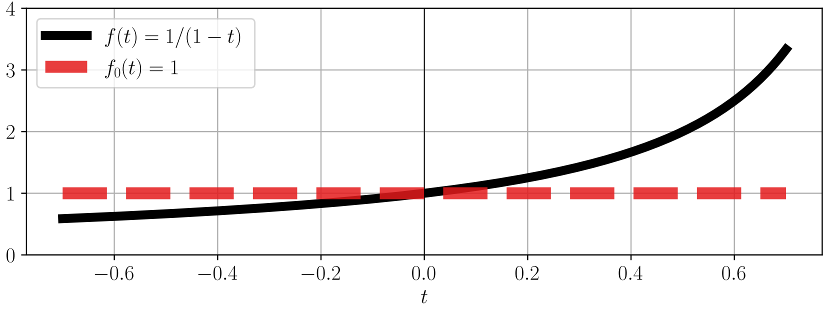
$$f(t) = \frac{1}{1-t}$$



Series Expansions

While we'll focus on **Fourier methods** in this subject, you might be familiar with another kind of expansion. A **Taylor series** yields a polynomial approximation.

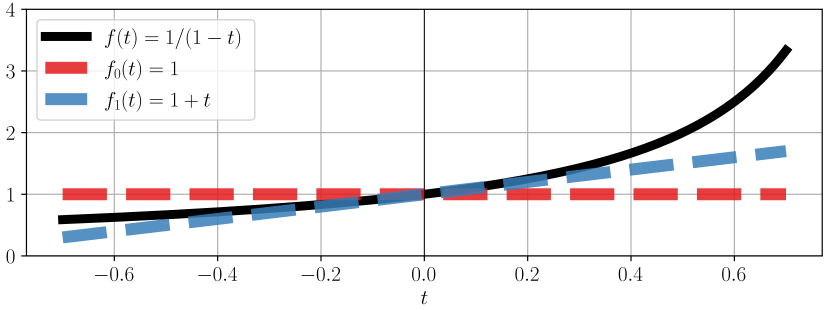
$$f(t) = \frac{1}{1-t} \approx 1 \text{ for } t \approx 0$$



Series Expansions

While we'll focus on **Fourier methods** in this subject, you might be familiar with another kind of expansion. A **Taylor series** yields a polynomial approximation.

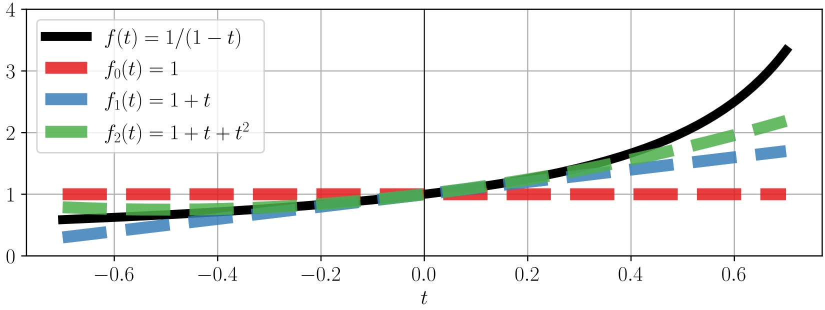
$$f(t) = \frac{1}{1-t} \approx 1 + t \text{ for } t \approx 0$$



Series Expansions

While we'll focus on **Fourier methods** in this subject, you might be familiar with another kind of expansion. A **Taylor series** yields a polynomial approximation.

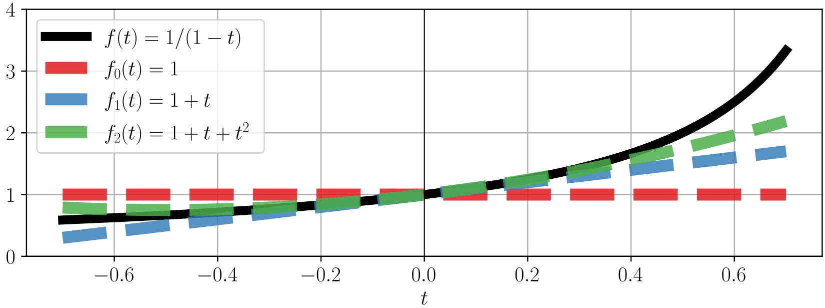
$$f(t) = \frac{1}{1-t} \approx 1 + t + t^2 \text{ for } t \approx 0$$



Series Expansions

While we'll focus on **Fourier methods** in this subject, you might be familiar with another kind of expansion. A **Taylor series** yields a polynomial approximation.

$$f(t) = \frac{1}{1-t} \approx 1 + t + t^2 \text{ for } t \approx 0$$



(No, you don't need to know this for the quiz.)

Series Expansions

Polynomials are simple to work with — hence the utility of Taylor series. So, what's the use of Fourier methods?

Why Fourier? Why not Taylor?

Fourier Series

A **Fourier series** expansion takes the form

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\!\left(k \tfrac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\!\left(k \tfrac{2\pi}{T} t\right).$$

How do we determine the **coefficients**, though?

Fourier Series

A **Fourier series** expansion takes the form

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\bigl(k \tfrac{2\pi}{T} t\bigr) + \sum_{k=1}^{\infty} d_k \sin\bigl(k \tfrac{2\pi}{T} t\bigr).$$

How do we determine the **coefficients**, though?

In lecture, Professor Freeman derived formulæ by exploiting the **orthogonality** (perpendicularity) of harmonically-related sinusoids over a length-T interval.

$$c_0 = \frac{1}{T} \int_T f(t)\, dt$$

$$c_k = \frac{2}{T} \int_T f(t) \cos\bigl(k \tfrac{2\pi}{T} t\bigr)\, dt \text{ (for } k \geq 1)$$

$$d_k = \frac{2}{T} \int_T f(t) \sin\bigl(k \tfrac{2\pi}{T} t\bigr)\, dt \text{ (for } k \geq 1)$$

Fourier Series

In lecture, Professor Freeman derived formulæ . . .

$$c_0 = \frac{1}{T}\int_T f(t)\,dt$$

$$c_k = \frac{2}{T}\int_T f(t)\cos\!\left(k\tfrac{2\pi}{T}t\right)dt \text{ (for } k \ge 1\text{)}$$

$$d_k = \frac{2}{T}\int_T f(t)\sin\!\left(k\tfrac{2\pi}{T}t\right)dt \text{ (for } k \ge 1\text{)}$$

Let $f(t) = f(t+2\pi) = \cos(t)$. Determine the coefficients for a Fourier series expansion of the form

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\!\left(k\tfrac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\!\left(k\tfrac{2\pi}{T}t\right).$$

Fourier Series

In lecture, Professor Freeman derived formulæ . . .

$$c_0 = \frac{1}{T}\int_T f(t)\,dt$$

$$c_k = \frac{2}{T}\int_T f(t)\cos\!\left(k\tfrac{2\pi}{T}t\right)dt \quad \text{(for } k \ge 1\text{)}$$

$$d_k = \frac{2}{T}\int_T f(t)\sin\!\left(k\tfrac{2\pi}{T}t\right)dt \quad \text{(for } k \ge 1\text{)}$$

Let $f(t) = f(t+2\pi) = \cos(t)$. Determine the coefficients for a Fourier series expansion of the form

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\!\left(k\tfrac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\!\left(k\tfrac{2\pi}{T}t\right).$$

$c_1 = 1$. All other coefficients are zero.

Sums of Sinusoids

A **Fourier series** expansion takes the form

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k\tfrac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\tfrac{2\pi}{T}t\right).$$

How many of the functions below have **only one** non-zero Fourier series coefficient?

- $f_1(t) = \cos\left(t - \frac{\pi}{2}\right)$
- $f_2(t) = \cos^2(t)$
- $f_3(t) = \sin(t)\cos(t)$
- $f_4(t) = 4\cos^3(t) - 3\cos(t)$
- $f_5(t) = \cos(12t)\cos(4t)\cos(2t)$

Hint: Do you need to integrate?

Trigonometric identities are given on the next slide.

Sums of Sinusoids

Trigonometric Identities

(not comprehensive)

$$\begin{aligned}
\cos^2(\theta) &= \tfrac{1}{2} + \tfrac{1}{2}\cos(2\theta) \\
\sin^2(\theta) &= \tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta) \\
\sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \\
\cos(\alpha \pm \beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \\
\cos(\alpha)\cos(\beta) &= \tfrac{1}{2}\cos(\alpha+\beta) + \tfrac{1}{2}\cos(\alpha-\beta) \\
\sin(\alpha)\sin(\beta) &= \tfrac{1}{2}\cos(\alpha-\beta) - \tfrac{1}{2}\cos(\alpha+\beta) \\
\sin(\alpha)\cos(\beta) &= \tfrac{1}{2}\sin(\alpha+\beta) + \tfrac{1}{2}\sin(\alpha-\beta) \\
\cos(\alpha)\sin(\beta) &= \tfrac{1}{2}\sin(\alpha+\beta) - \tfrac{1}{2}\sin(\alpha-\beta)
\end{aligned}$$

Don't worry if your trigonometry is rusty. We'll soon find a way to replace all this trigonometry with simple algebra.

Sums of Sinusoids

How many of the functions below have **only one** non-zero Fourier series coefficient? **3** $\rightarrow f_1, f_3, f_4$

- $f_1(t) = \cos\left(t - \frac{\pi}{2}\right) = \sin(t)$
- $f_2(t) = \cos^2(t) = \frac{1}{2} + \frac{1}{2}\cos(2t)$
- $f_3(t) = \sin(t)\cos(t) = \frac{1}{2}\sin(2t)$
- $f_4(t) = 4\cos^3(t) - 3\cos(t) = \cos(3t)$
- $f_5(t) = \cdots = \frac{1}{4}\cos(6t) + \frac{1}{4}\cos(10t) + \frac{1}{4}\cos(14t) + \frac{1}{4}\cos(18t)$

Periodicity

Let $f(t) = \cos(t) + \cos(2\pi t)$. Determine the coefficients for a Fourier series expansion of the form

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\!\left(k\tfrac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\!\left(k\tfrac{2\pi}{T}t\right).$$

If such an expansion is not possible, explain why.

Periodicity

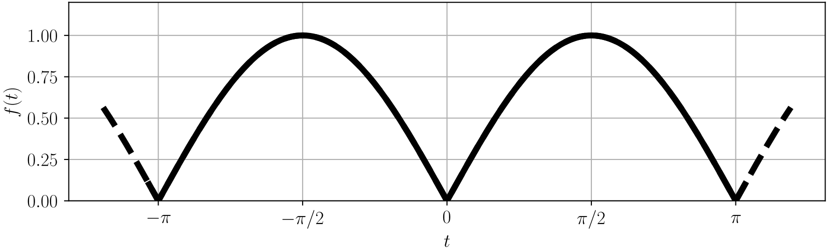
Let $f(t) = \cos(t) + \cos(2\pi t)$. Determine the coefficients for a Fourier series expansion of the form

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k \tfrac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k \tfrac{2\pi}{T} t\right).$$

If such an expansion is not possible, explain why.

$\cos(t)$ is periodic in $T_1 = 2\pi$, while $\cos(2\pi t)$ is periodic in $T_2 = 1$. There is no $T > 0$ for which $\cos(t) = \cos(t+T)$ and $\cos(2\pi t) = \cos(2\pi(t+T))$ for all t. That is, $f(t)$ is **aperiodic** — and hence $f(t)$ has no Fourier series representation.

Rectified Sine



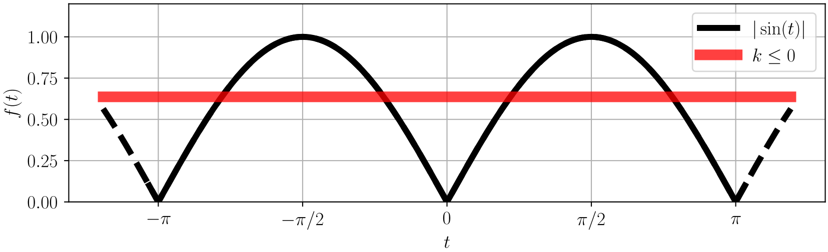
Let $f(t) = |\sin(t)|$, as shown above.

First, estimate the value of c_0 graphically.

Then determine the coefficients for a Fourier series expansion of the form

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k\frac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\frac{2\pi}{T}t\right).$$

Rectified Sine



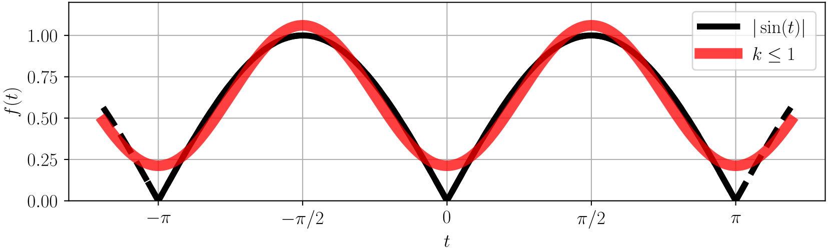
c_0 is the average value over one period. It looks like the value of c_0 should be somewhere between 0.5 and 1.

$$c_0 = \frac{1}{T}\int_T f(t)\,dt = \frac{1}{\pi}\int_0^\pi \sin(t)\,dt = \frac{2}{\pi} \approx 0.64$$

The function is symmetric about $t = 0$, so $d_k = 0$ for all k.

$$c_k = \frac{2}{\pi}\int_0^\pi \sin(t)\cos(2kt)\,dt = \frac{-4/\pi}{4k^2 - 1} \text{ for } k \geq 1$$

Rectified Sine



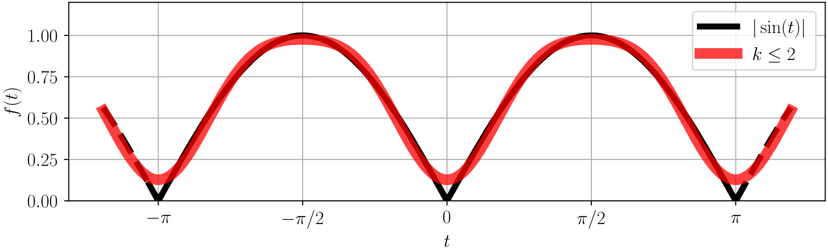
c_0 is the average value over one period. It looks like the value of c_0 should be somewhere between 0.5 and 1.

$$c_0 = \frac{1}{T}\int_T f(t)\,dt = \frac{1}{\pi}\int_0^\pi \sin(t)\,dt = \frac{2}{\pi} \approx 0.64$$

The function is symmetric about $t = 0$, so $d_k = 0$ for all k.

$$c_k = \frac{2}{\pi}\int_0^\pi \sin(t)\cos(2kt)\,dt = \frac{-4/\pi}{4k^2 - 1} \text{ for } k \geq 1$$

Rectified Sine



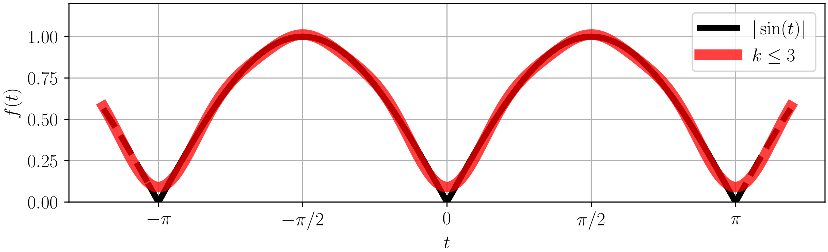
c_0 is the average value over one period. It looks like the value of c_0 should be somewhere between 0.5 and 1.

$$c_0 = \frac{1}{T}\int_T f(t)\,dt = \frac{1}{\pi}\int_0^\pi \sin(t)\,dt = \frac{2}{\pi} \approx 0.64$$

The function is symmetric about $t = 0$, so $d_k = 0$ for all k.

$$c_k = \frac{2}{\pi}\int_0^\pi \sin(t)\cos(2kt)\,dt = \frac{-4/\pi}{4k^2 - 1} \text{ for } k \geq 1$$

Rectified Sine



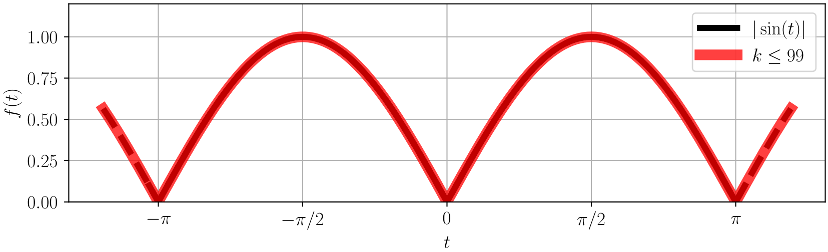
c_0 is the average value over one period. It looks like the value of c_0 should be somewhere between 0.5 and 1.

$$c_0 = \frac{1}{T}\int_T f(t)\,dt = \frac{1}{\pi}\int_0^\pi \sin(t)\,dt = \frac{2}{\pi} \approx 0.64$$

The function is symmetric about $t = 0$, so $d_k = 0$ for all k.

$$c_k = \frac{2}{\pi}\int_0^\pi \sin(t)\cos(2kt)\,dt = \frac{-4/\pi}{4k^2 - 1} \text{ for } k \geq 1$$

Rectified Sine



c_0 is the average value over one period. It looks like the value of c_0 should be somewhere between 0.5 and 1.

$$c_0 = \frac{1}{T}\int_T f(t)\,dt = \frac{1}{\pi}\int_0^{\pi} \sin(t)\,dt = \frac{2}{\pi} \approx 0.64$$

The function is symmetric about $t = 0$, so $d_k = 0$ for all k.

$$c_k = \frac{2}{\pi}\int_0^{\pi} \sin(t)\cos(2kt)\,dt = \frac{-4/\pi}{4k^2 - 1} \text{ for } k \geq 1$$

Lessons Learned

The **synthesis equation** tells us how to represent a periodic signal as a Fourier series.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k\frac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\frac{2\pi}{T}t\right)$$

The **analysis equations** tell us how to calculate these Fourier series coefficients.

$c_0 = \frac{1}{T}\int_T f(t)\,dt$ average value over a period

$c_k = \frac{2}{T}\int_T f(t)\cos\left(k\frac{2\pi}{T}t\right)dt$ for $k \geq 1$

$d_k = \frac{2}{T}\int_T f(t)\sin\left(k\frac{2\pi}{T}t\right)dt$ for $k \geq 1$

Do not mindlessly plug numbers into formulas!

Think before you calculate!

Question of the Day

Let $f(t) = \cos^2(t - \frac{\pi}{2})$.

Make a rough sketch of $f(t)$.

What is the fundamental period (T)?

What is the fundamental frequency (ω_0)?

Without resorting to calculus, determine coefficients for a Fourier series expansion of the form

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_0 t).$$