6.300: Signal Processing

Fourier Series

The **synthesis equation** tells us how to represent a periodic signal as a Fourier series.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k\frac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\frac{2\pi}{T}t\right)$$

The **analysis equations** tell us how to calculate these Fourier series coefficients.

$c_0 = \frac{1}{T}\int_T f(t)dt$ average value over a period

$c_k = \frac{2}{T}\int_T f(t)\cos\left(k\frac{2\pi}{T}t\right)dt$ for $k \geq 1$

$d_k = \frac{2}{T}\int_T f(t)\sin\left(k\frac{2\pi}{T}t\right)dt$ for $k \geq 1$

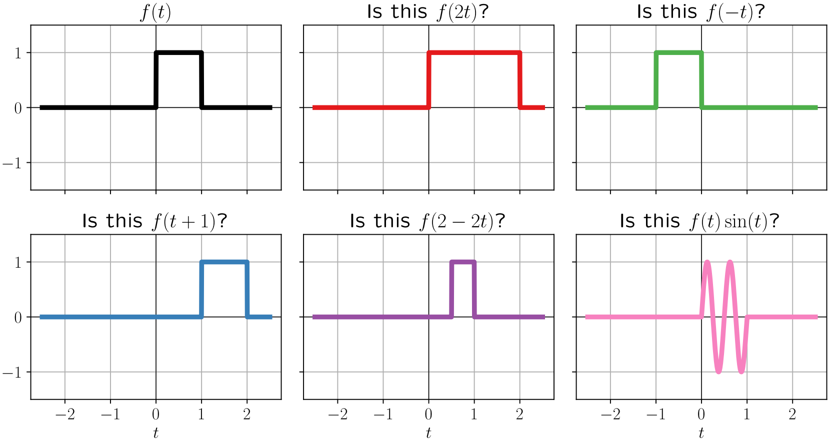
February 03, 2026

Agenda for Recitation

- Signals (and **transformations** thereof)
- **Fourier series** expansion for periodic signals

Transformations

How many expressions match the plot beneath them?

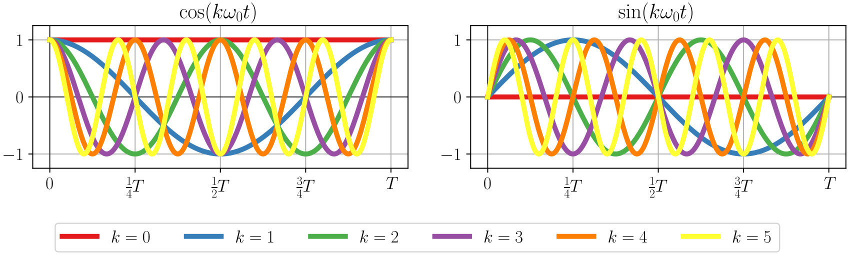


Series Expansions

In lecture, Professor Freeman analyzed a periodic signal $f(t) = f(t + T)$ using a **Fourier series** expansion.

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k \tfrac{2\pi}{T} t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k \tfrac{2\pi}{T} t\right)$$

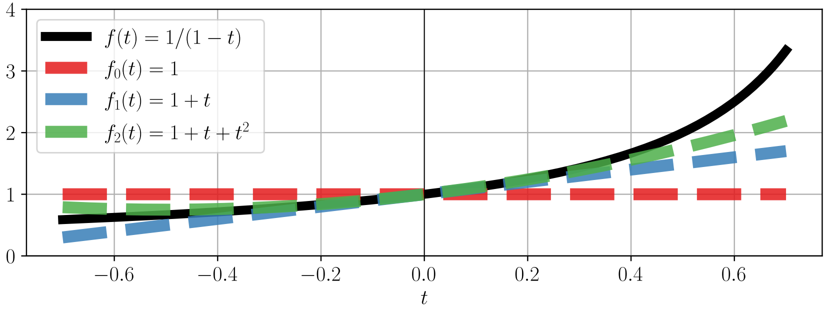
A Fourier series is a **sum of harmonically-related sinusoids**. That is, each sinusoid oscillates at an integer multiple k of the fundamental frequency $\omega_0 \triangleq 2\pi/T$.



Series Expansions

While we'll focus on **Fourier methods** in this subject, you might be familiar with another kind of expansion. A **Taylor series** yields a polynomial approximation.

$$f(t) = \frac{1}{1-t} \approx 1 + t + t^2 \text{ for } t \approx 0$$



Series Expansions

Polynomials are simple to work with — hence the utility of Taylor series. So, what's the use of Fourier methods?

Why Fourier? Why not Taylor?

Fourier Series

A **Fourier series** expansion takes the form

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\big(k\tfrac{2\pi}{T}t\big) + \sum_{k=1}^{\infty} d_k \sin\big(k\tfrac{2\pi}{T}t\big).$$

How do we determine the **coefficients**, though?

In lecture, Professor Freeman derived formulæ by exploiting the **orthogonality** (perpendicularity) of harmonically-related sinusoids over a length-T interval.

$$c_0 = \frac{1}{T}\int_T f(t)\,dt$$

$$c_k = \frac{2}{T}\int_T f(t)\cos\big(k\tfrac{2\pi}{T}t\big)\,dt \ \text{(for } k \geq 1)$$

$$d_k = \frac{2}{T}\int_T f(t)\sin\big(k\tfrac{2\pi}{T}t\big)\,dt \ \text{(for } k \geq 1)$$

Fourier Series

In lecture, Professor Freeman derived formulæ ...

$$c_0 = \frac{1}{T}\int_T f(t)\,dt$$

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$$d_k = \frac{2}{T}\int_T f(t)\sin\!\left(k\tfrac{2\pi}{T}t\right)dt \text{ (for } k \geq 1\text{)}$$

Let $f(t) = f(t+2\pi) = \cos(t)$. Determine the coefficients for a Fourier series expansion of the form

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\!\left(k\tfrac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\!\left(k\tfrac{2\pi}{T}t\right).$$

Sums of Sinusoids

A **Fourier series** expansion takes the form

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k\tfrac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\tfrac{2\pi}{T}t\right).$$

How many of the functions below have **only one** non-zero Fourier series coefficient?

- $f_1(t) = \cos\left(t - \frac{\pi}{2}\right)$
- $f_2(t) = \cos^2(t)$
- $f_3(t) = \sin(t)\cos(t)$
- $f_4(t) = 4\cos^3(t) - 3\cos(t)$
- $f_5(t) = \cos(12t)\cos(4t)\cos(2t)$

Hint: Do you need to integrate?

Trigonometric identities are given on the next slide.

Sums of Sinusoids

Trigonometric Identities

(not comprehensive)

$$\begin{aligned}
\cos^2(\theta) &= \tfrac{1}{2} + \tfrac{1}{2}\cos(2\theta) \\
\sin^2(\theta) &= \tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta) \\
\sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \\
\cos(\alpha \pm \beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \\
\cos(\alpha)\cos(\beta) &= \tfrac{1}{2}\cos(\alpha+\beta) + \tfrac{1}{2}\cos(\alpha-\beta) \\
\sin(\alpha)\sin(\beta) &= \tfrac{1}{2}\cos(\alpha-\beta) - \tfrac{1}{2}\cos(\alpha+\beta) \\
\sin(\alpha)\cos(\beta) &= \tfrac{1}{2}\sin(\alpha+\beta) + \tfrac{1}{2}\sin(\alpha-\beta) \\
\cos(\alpha)\sin(\beta) &= \tfrac{1}{2}\sin(\alpha+\beta) - \tfrac{1}{2}\sin(\alpha-\beta)
\end{aligned}$$

Don't worry if your trigonometry is rusty. We'll soon find a way to replace all this trigonometry with simple algebra.

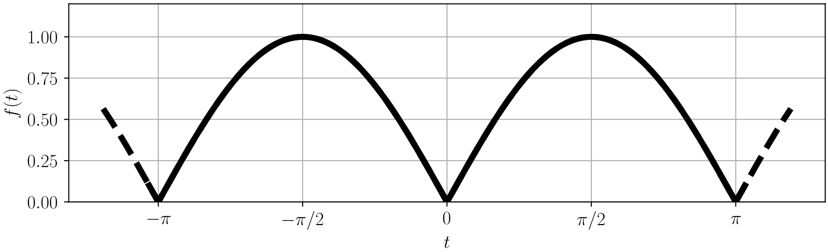
Periodicity

Let $f(t) = \cos(t) + \cos(2\pi t)$. Determine the coefficients for a Fourier series expansion of the form

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k\frac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\frac{2\pi}{T}t\right).$$

If such an expansion is not possible, explain why.

Rectified Sine



Let $f(t) = |\sin(t)|$, as shown above.

First, estimate the value of c_0 graphically.

Then determine the coefficients for a Fourier series expansion of the form

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Lessons Learned

The **synthesis equation** tells us how to represent a periodic signal as a Fourier series.

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Do not mindlessly plug numbers into formulas!

Think before you calculate!