6.300: Signal Processing

Discrete-Time Fourier Transform (DTFT)

Analysis: $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

Synthesis: $x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$

Time Delay: $x[n - n_0] \iff e^{-j\Omega n_0} X(\Omega)$

Frequency Derivative: $nx[n] \iff j\frac{d}{d\Omega} X(\Omega)$

Periodic Signals: $X(\Omega) = \sum_k 2\pi X[k] \delta(\Omega - k\Omega_0)$

March 5, 2026

Agenda for Recitation

- Discrete-time Fourier transform (DTFT)

What questions do you have from lecture?

Overall, we were very pleased with the quiz scores! This was not an easy quiz, and you rose to the occasion.

If you didn't do as well as you had hoped, remember that Quiz #1 is worth only (?) 10% of your course grade.
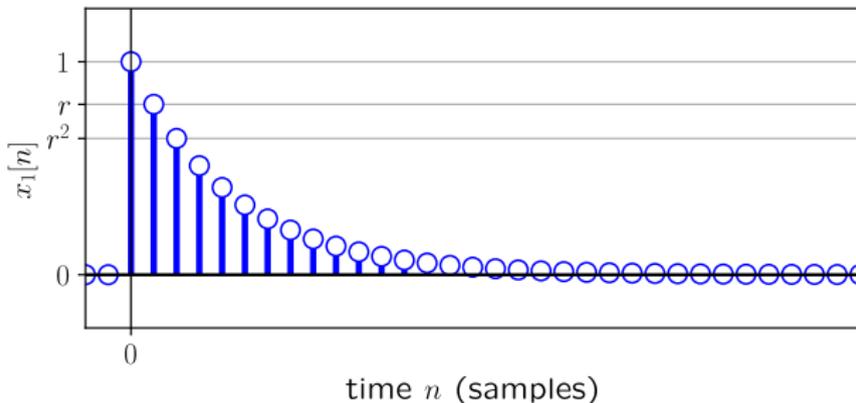
If you're concerned about your performance in the course for any reason or simply want to talk with someone, we're here for you. Talk to us in person or send an e-mail to 6.300-instructors@mit.edu. We can make a plan for the future.

Fourier Transforms

Determine $X_1(\Omega)$, the Fourier transform of $x_1[n]$.

$$x_1[n] = \begin{cases} r^n & n \geq 0 \\ 0 & n < 0 \end{cases} \text{ where } |r| < 1$$

Sketch $|X_1(\Omega)|$ and $\angle X_1(\Omega)$.

Fourier Transforms

Determine $X_1(\Omega)$, the Fourier transform of $x_1[n]$.

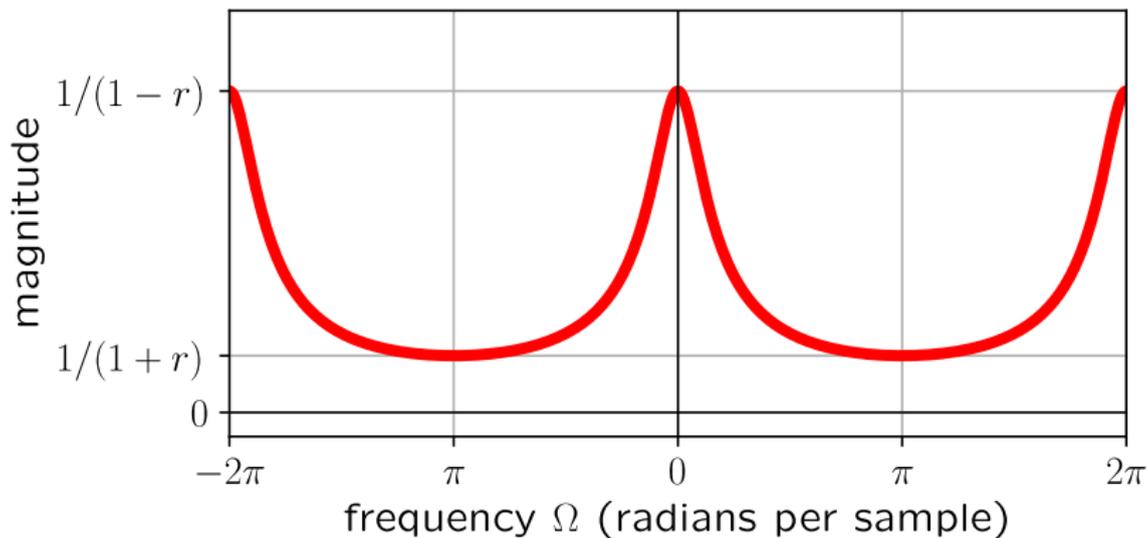$$x_1[n] = \begin{cases} r^n & n \geq 0 \\ 0 & n < 0 \end{cases} \text{ where } |r| < 1$$

Sketch $|X_1(\Omega)|$ and $\angle X_1(\Omega)$.

Compute the Fourier transform.

$$X_1(\Omega) = \sum_{n=0}^{\infty} r^n e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(re^{-j\Omega}\right)^n = \frac{1}{1 - re^{-j\Omega}}$$

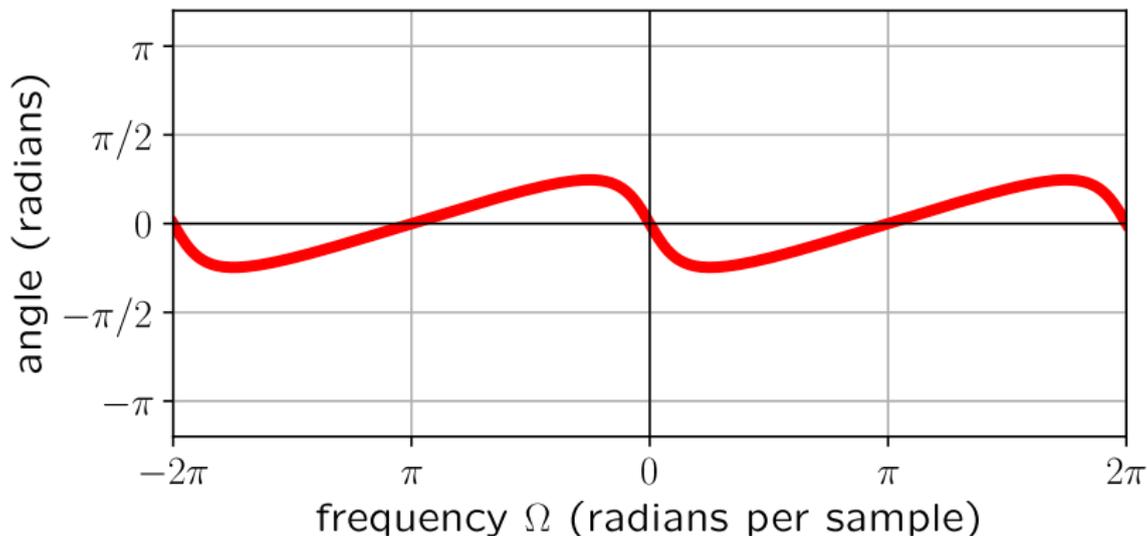Sketch the magnitude and phase using a graphical method: Draw a complex-plane vector diagram.

Fourier Transforms

The magnitude $|X_1(\Omega)|$ is shown above.

Because $e^{j\Omega}$ is periodic in 2π, so too is $|X_1(\Omega)|$.

Fourier Transforms

The angle $\angle X_1(\Omega)$ is shown above.

Because $e^{j\Omega}$ is periodic in 2π, so too is $\angle X_1(\Omega)$.

Fourier Transforms

Determine $X_1(\Omega)$, the Fourier transform of $x_1[n]$.

$$x_1[n] = \begin{cases} r^n & n \geq 0 \\ 0 & n < 0 \end{cases} \text{ where } |r| < 1$$

Sketch $|X_1(\Omega)|$ and $\angle X_1(\Omega)$.

As an aside, it's interesting to note that **geometric sequences** are the discrete-time (DT) analogue of continuous-time (CT) exponential functions.

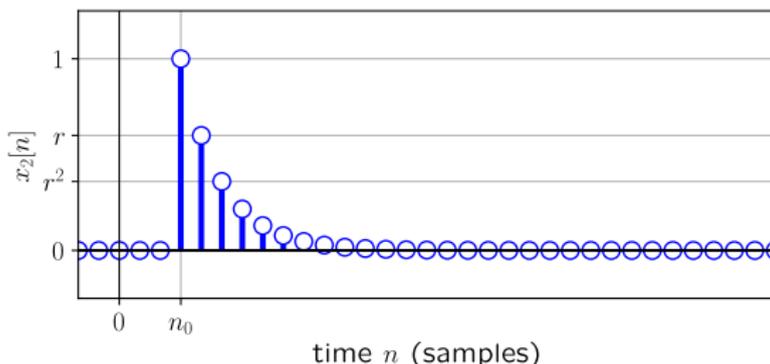$$e^{at}\big|_{t=n\Delta} = e^{an\Delta} = \underbrace{(e^{a\Delta})^n}_{r^n} = r^n$$

Exponential functions e^{at} solve differential equations, while geometric sequences r^n solve difference equations.

Fourier Transforms

Determine $X_2(\Omega)$, the Fourier transform of $x_2[n]$.

$$x_2[n] = x_1[n - n_0] = \begin{cases} r^{n-n_0} & n \geq n_0 \\ 0 & n < n_0 \end{cases}$$

Sketch $|X_2(\Omega)|$ and $\angle X_2(\Omega)$. How are $|X_2(\Omega)|$ and $|X_1(\Omega)|$ related? How are $\angle X_2(\Omega)$ and $\angle X_1(\Omega)$ related?

time n (samples)

Fourier Transforms

Determine $X_2(\Omega)$, the Fourier transform of $x_2[n]$.

$$x_2[n] = x_1[n - n_0] = \begin{cases} r^{n-n_0} & n \geq n_0 \\ 0 & n < n_0 \end{cases}$$
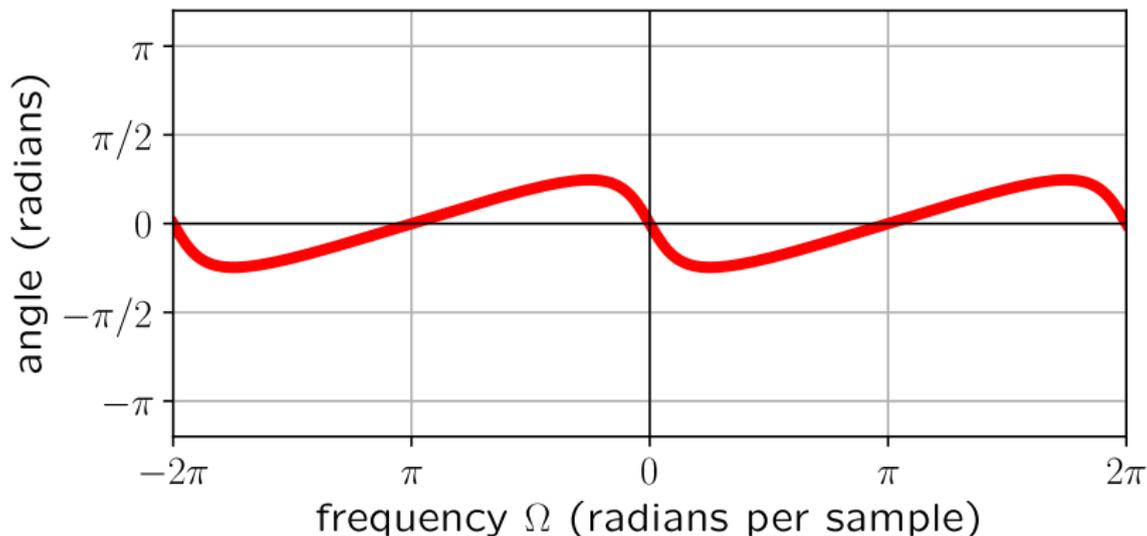
Sketch $|X_2(\Omega)|$ and $\angle X_2(\Omega)$. How are $|X_2(\Omega)|$ and $|X_1(\Omega)|$ related? How are $\angle X_2(\Omega)$ and $\angle X_1(\Omega)$ related?

Delay property: $x_2[n] = x_1[n - n_0] \iff X_2(\Omega) = e^{-j\Omega n_0} X_1(\Omega)$.

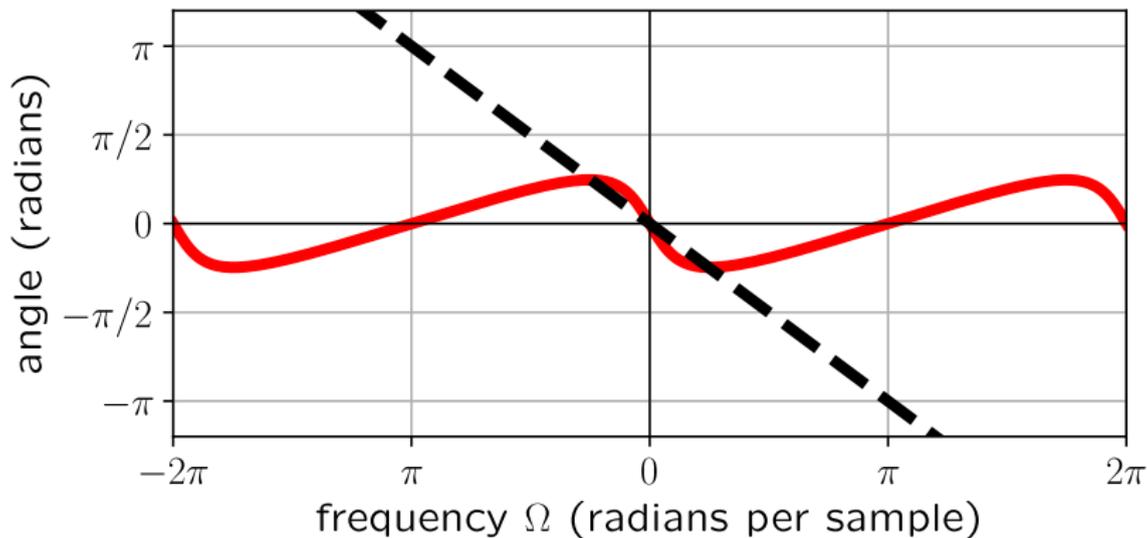Magnitudes multiply: $|X_2(\Omega)| = |e^{-j\Omega n_0}||X_1(\Omega)| = |X_1(\Omega)|$.

Angles add: $\angle X_2(\Omega) = \angle e^{-j\Omega n_0} + \angle X_1(\Omega) = -\Omega n_0 + \angle X_1(\Omega)$.
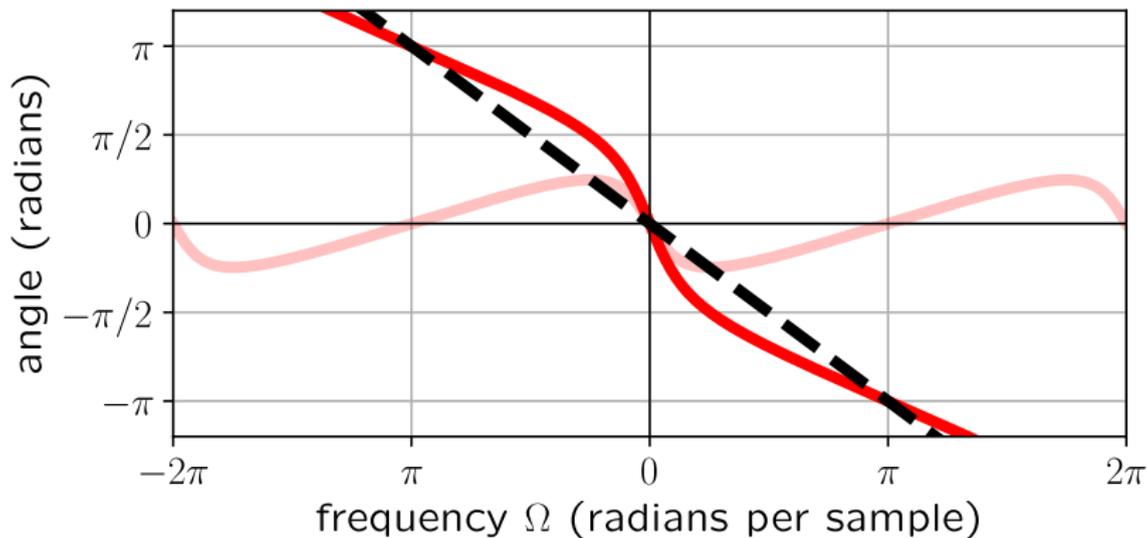
Fourier Transforms

To sketch $\angle X_2(\Omega)$, start by sketching $\angle X_1(\Omega)$.
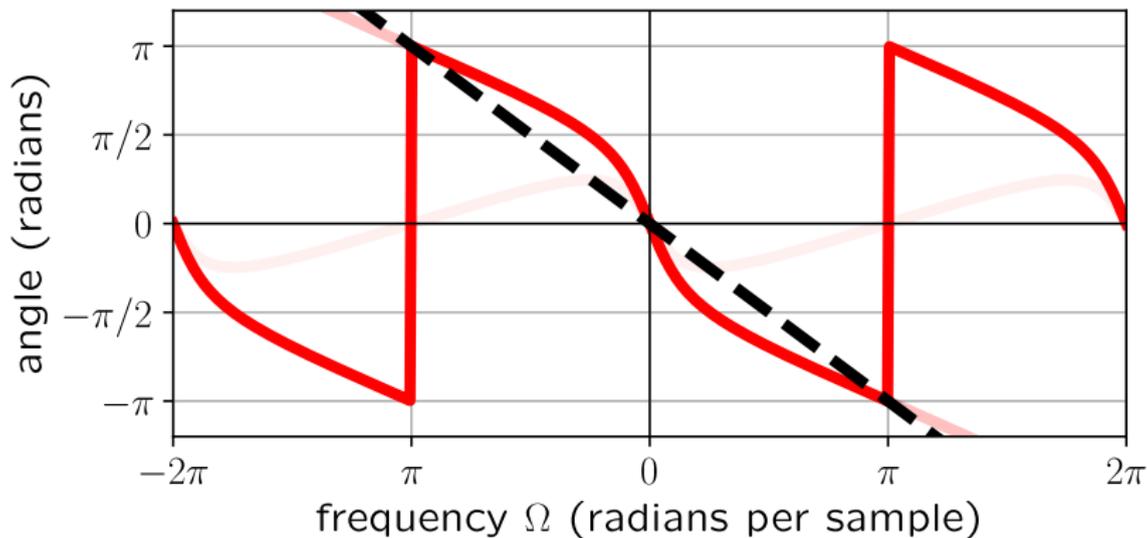
Fourier Transforms

Sketch the $-\Omega n_0$ term.
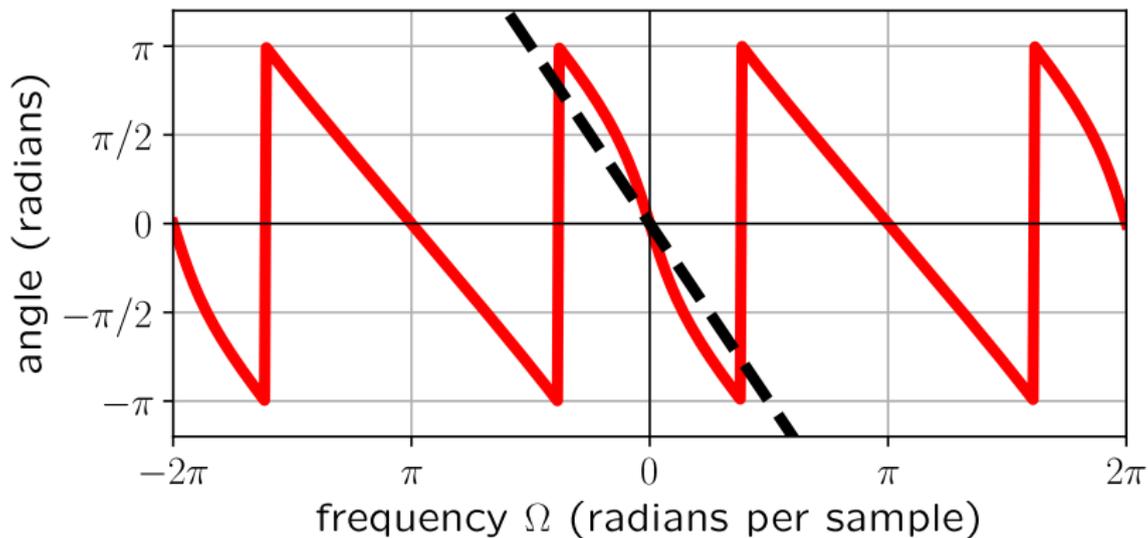
Fourier Transforms

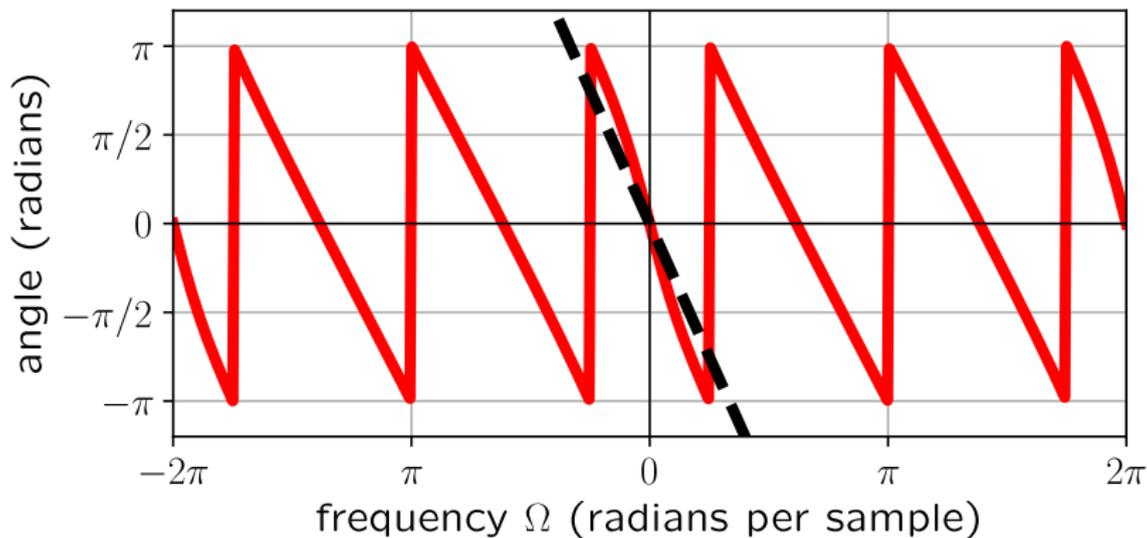Angles add: $\angle X_2(\Omega) = -\Omega n_0 + \angle X_1(\Omega)$.

Fourier Transforms

Finally, wrap $\angle X_2(\Omega)$ into the $[-\pi, \pi]$ range.

Fourier Transforms

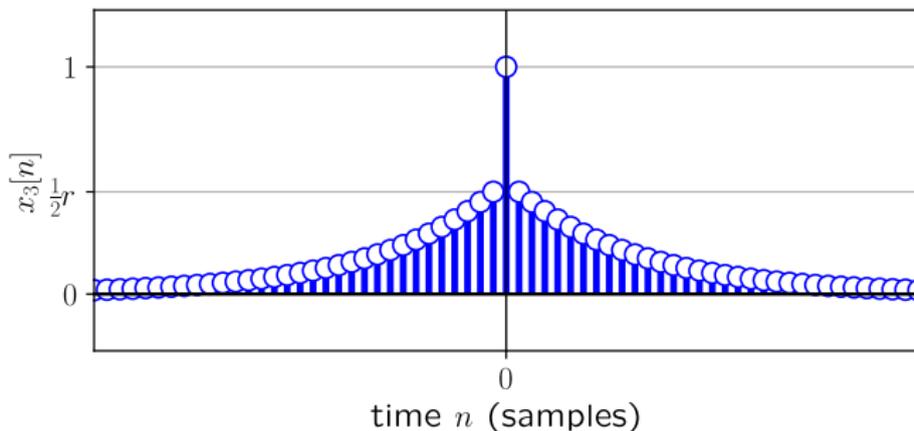As $n_0 \to \infty$, the $-\Omega n_0$ line slopes downward more steeply.

Fourier Transforms

As $n_0 \to \infty$, the $-\Omega n_0$ line slopes downward more steeply.

Fourier Transforms

Determine $X_3(\Omega)$, the Fourier transform of $x_3[n]$.

$$x_3[n] = \mathsf{Symmetric}\{x_1[n]\}$$

If a time-domain signal is **real** and **symmetric**, what can you say about the Fourier transform?

Fourier Transforms

Determine $X_3(\Omega)$, the Fourier transform of $x_3[n]$.

$$x_3[n] = \text{Symmetric}\{x_1[n]\}$$

If a time-domain signal is **real** and **symmetric**, what can you say about the Fourier transform?

The Fourier transform is linear.

$$x_3[n] = \tfrac{1}{2}x_1[n] + \tfrac{1}{2}x_1[-n]$$

$$X_3(\Omega) = \tfrac{1}{2}X_1(\Omega) + \tfrac{1}{2}X_1(-\Omega) = \frac{1 - r\cos(\Omega)}{1 - 2r\cos(\Omega) + r^2}$$
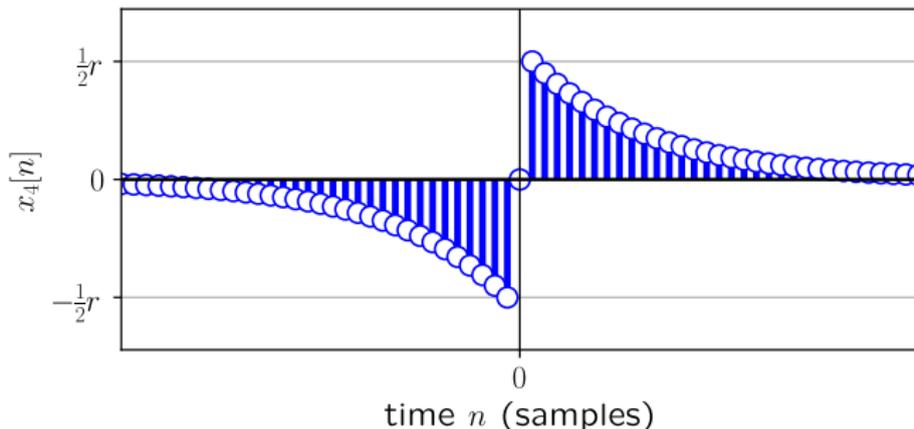
Notice that $X_3(\Omega)$ is real and symmetric.

Fourier Transforms

Determine $X_4(\Omega)$, the Fourier transform of $x_4[n]$.

$$x_4[n] = \text{Anti-symmetric}\{x_1[n]\}$$

If a time-domain signal is **real** and **anti-symmetric**, what can you say about the Fourier transform?

Fourier Transforms

Determine $X_4(\Omega)$, the Fourier transform of $x_4[n]$.

$$x_4[n] = \mathsf{Anti\text{-}symmetric}\{x_1[n]\}$$

If a time-domain signal is **real** and **anti-symmetric**, what can you say about the Fourier transform?

The Fourier transform is linear.

$$x_4[n] = \tfrac{1}{2}x_1[n] - \tfrac{1}{2}x_1[-n]$$

$$X_4(\Omega) = \tfrac{1}{2}X_1(\Omega) - \tfrac{1}{2}X_1(-\Omega) = \frac{-jr\sin(\Omega)}{1 - 2r\cos(\Omega) + r^2}$$

Notice that $X_4(\Omega)$ is imaginary and anti-symmetric.

Fourier Transforms

Determine $X_5(\Omega)$, the Fourier transform of $x_5[n]$.

$$x_5[n] = nx_1[n]$$

Fourier Transforms

Determine $X_5(\Omega)$, the Fourier transform of $x_5[n]$.

$$x_5[n] = n x_1[n]$$

Differentiate in the frequency domain.

$$X_1(\Omega) = \sum_{n=-\infty}^{\infty} x_1[n] e^{-j\Omega n}$$

$$\frac{d}{d\Omega} X_1(\Omega) = \sum_{n=-\infty}^{\infty} (-jn) x_1[n] e^{-j\Omega n}$$
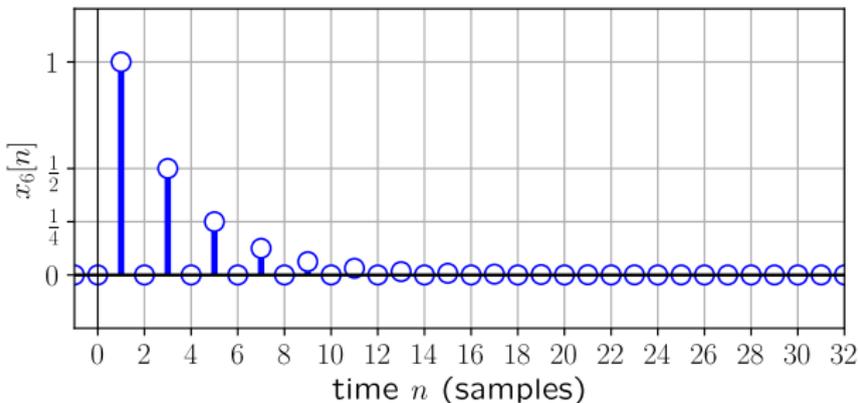
$$\underbrace{j\frac{d}{d\Omega} X_1(\Omega)}_{X_5(\Omega)} = \sum_{n=-\infty}^{\infty} \underbrace{(n x_1[n])}_{x_5[n]} e^{-j\Omega n} = \frac{r e^{-j\Omega}}{(1 - r e^{-j\Omega})^2}$$

Fourier Transforms

Determine $X_6(\Omega)$, the Fourier transform of $x_6[n]$.

$$x_6[n] = \begin{cases} \left(\frac{1}{2}\right)^{\frac{n}{2}} & n \in \{0, 2, 4, 6, 8, 10, 12, \dots\} \\ 0 & \text{otherwise} \end{cases}$$

What does **"stretching in time"** do in frequency?

Fourier Transforms

Determine $X_6(\Omega)$, the Fourier transform of $x_6[n]$.

$$x_6[n] = \begin{cases} \left(\frac{1}{2}\right)^{\frac{n}{2}} & n \in \{0, 2, 4, 6, 8, 10, 12, \dots\} \\ 0 & \text{otherwise} \end{cases}$$

What does **"stretching in time"** do in frequency?

Make a change of variables: $n = 2m$.

$$X_6(\Omega) = \sum_{n \text{ even}}^{\infty} \left(\tfrac{1}{2}\right)^{\frac{n}{2}} e^{-j\Omega n} = \sum_{m=0}^{\infty} \left(\tfrac{1}{2}\right)^m e^{-j\Omega(2m)} = \sum_{m=0}^{\infty} \left(\tfrac{1}{2} e^{-j2\Omega}\right)^m$$

Stretching in time? Compressing in frequency!

$$X_6(\Omega) = \sum_{m=0}^{\infty} \left(\tfrac{1}{2} e^{-j2\Omega}\right)^m = \frac{1}{1 - \tfrac{1}{2} e^{-j2\Omega}}$$

Fourier Transforms

$x_7[n]$ has Fourier transform $X_7(\Omega)$. Determine $x_7[n]$.
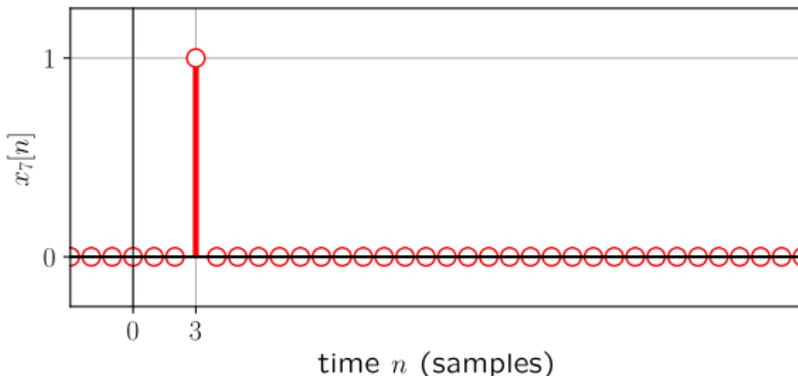
$$X_7(\Omega) = e^{-j3\Omega}$$

Fourier Transforms

$x_7[n]$ has Fourier transform $X_7(\Omega)$. Determine $x_7[n]$.

$$X_7(\Omega) = e^{-j3\Omega}$$

Plug in to the synthesis equation.

$$x_7[n] = \frac{1}{2\pi} \int_{2\pi} e^{-j3\Omega} e^{j\Omega n} d\Omega = \frac{1}{2\pi} \int_{2\pi} e^{j\Omega(n-3)} d\Omega = \begin{cases} 1 & n = 3 \\ 0 & n \neq 3 \end{cases}$$

Lessons Learned

The **discrete-time Fourier transform (DTFT)** is a Fourier representation for aperiodic and periodic discrete-time signals. It has many useful properties.

Analysis: $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

Synthesis: $x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$

Time Delay: $x[n - n_0] \iff X(\Omega) e^{-j\Omega n_0}$

Frequency Derivative: $nx[n] \iff j\frac{d}{d\Omega} X(\Omega)$

Periodic Signals: $X(\Omega) = \sum_k 2\pi X[k] \delta(\Omega - k\Omega_0)$

Question of the Day

Previously, we derived the **time-derivative property** for the continuous-time Fourier transform.

$$\frac{d}{dt}x(t) \iff j\omega X(\omega)$$

Does the **discrete-time Fourier transform** have a time-derivative property? Why or why not?