6.300: Signal Processing

Discrete-Time Fourier Series (DTFS)

Analysis: $X[k] = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\frac{2\pi}{N}n}$

Synthesis: $x[n] = \sum_{k=\langle N\rangle} X[k]e^{jk\frac{2\pi}{N}n}$

Fourier Series Properties: We can derive a number of properties by writing out the analysis or synthesis equation, making a change of variables, and matching the resulting expression to an already-known result.

February 19, 2026

Agenda for Recitation

- Review
- Discrete-time Fourier series (DTFS)
- As time allows: Deriving Fourier series properties

What questions do you have from lecture?

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What questions do you have from lecture?

Feb. 03: Fourier series — a sum of sinusoids

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(k\tfrac{2\pi}{T}t\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\tfrac{2\pi}{T}t\right)$$

Feb. 05: Fourier series and symmetry

$$f(t) = \underbrace{\left(f_{\text{symmetric}}(t)\right)}_{\text{cosines}} + \underbrace{\left(f_{\text{anti-symmetric}}(t)\right)}_{\text{sines}}$$

Feb. 10: Complex exponentials and Fourier series

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

Feb. 12: Sampling — discretizing time

$$x[n] \triangleq x(n\Delta) = x\left(\frac{n}{f_s}\right)$$

Review: Aliasing

We sample the CT waveform $x(t) = \cos(1200\pi t)$ using a sampling rate of $f_s = 1$ kHz. We listen to the resulting DT samples using the same sampling rate. What is the fundamental frequency (Hz) of the baseband CT waveform that we'd hear?

Review: Aliasing

We sample the CT waveform $x(t) = \cos(1200\pi t)$ using a sampling rate of $f_s = 1$ kHz. We listen to the resulting DT samples using the same sampling rate. What is the fundamental frequency (Hz) of the baseband CT waveform that we'd hear?

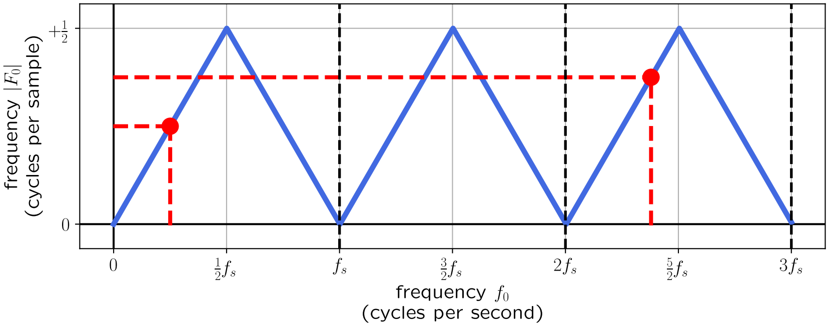
The fundamental frequency of $x(t)$ is $f_0 = 600$ Hz.

Frequencies above half the sampling rate (i.e., $f_0 > \frac{1}{2}f_s$) alias to frequencies below half the sampling rate.

Frequency folding: $f_0 = 600$ Hz $\rightarrow f_{\text{alias}} = 400$ Hz

Too many numbers? There's an easy graphical method.

Review: Aliasing



#1. Find the CT frequency f_0 along the horizontal axis.

#2. Find the corresponding DT frequency F_0 along the vertical axis. (Up, left.)

#3. Find the corresponding baseband CT frequency along the horizontal axis. (Right, down.)

P.S. You need to use a sawtooth-looking plot (not this triangle-looking plot) for complex-valued signals.

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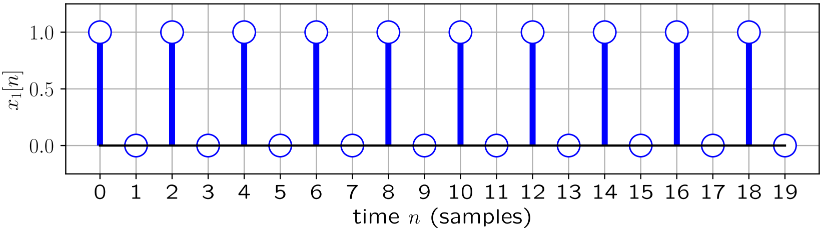
Discrete-Time Fourier Series



Determine the Fourier series coefficients for $x_1[n]$.

$$x_1[n] = \frac{1+(-1)^n}{2} = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Choose N to be the fundamental period of $x_1[n]$.

Discrete-Time Fourier Series

Determine the Fourier series coefficients for $x_1[n]$.

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Choose N to be the fundamental period of $x_1[n]$.

Choosing $N=2$ means that $e^{-jk\frac{2\pi}{N}n} = (-1)^{kn}$ for all k, n.

We can directly calculate the coefficients with a sum.

$$X_1[k] = \frac{1}{2}\sum_{n=0}^{1} x_1[n](-1)^{kn} = \frac{1+0}{2} = \frac{1}{2} \text{ for all } k$$

Note that the coefficients $X_1[k]$ are periodic in N, too.

Discrete-Time Fourier Series

Determine the Fourier series coefficients for $x_1[n]$.

$$x_1[n] = \frac{1+(-1)^n}{2} = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Choose N to be the fundamental period of $x_1[n]$.

Alternatively, we could use Euler's formula.

Choosing $N = 2$ means the fundamental is $\Omega_0 = \frac{2\pi}{2} = \pi$.

$$x_1[n] = \tfrac{1}{2} + \tfrac{1}{2}(-1)^n = \tfrac{1}{2}e^{j0\pi n} + \tfrac{1}{2}e^{j1\pi n} \implies \begin{cases} \frac{1}{2} & k \bmod 2 = 0 \\ \frac{1}{2} & k \bmod 2 = 1 \end{cases}$$

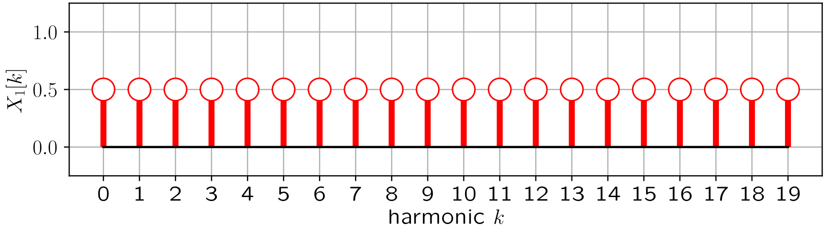
Remember: The coefficients $X_1[k]$ are periodic in N, too!

Discrete-Time Fourier Series

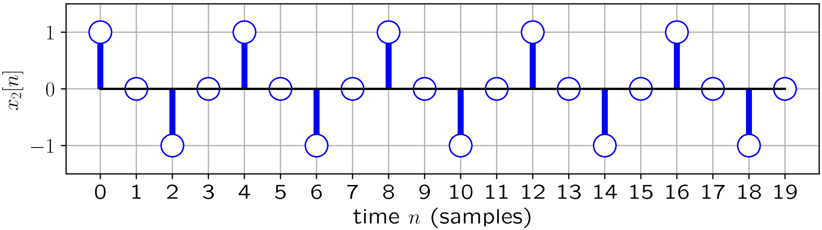
Determine the Fourier series coefficients for $x_1[n]$.

$$x_1[n] = \frac{1+(-1)^n}{2} = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Choose N to be the fundamental period of $x_1[n]$.



Discrete-Time Fourier Series



Determine the Fourier series coefficients for $x_2[n]$.

$$x_2[n] = \cos\left(\frac{\pi}{2}n\right)$$

Choose N to be the fundamental period of $x_2[n]$.

Discrete-Time Fourier Series

Determine the Fourier series coefficients for $x_2[n]$.

$$x_2[n] = \cos\left(\frac{\pi}{2}n\right)$$

Choose N to be the fundamental period of $x_2[n]$.

Choosing $N = 4$ means that $e^{-jk\frac{2\pi}{N}n} = j^{-kn}$ for all k, n.

We can directly calculate the coefficients with a sum.

$$X_2[k] = \frac{1}{4}\sum_{n=0}^{3} x_2[n]\, j^{-kn}$$

$$= \frac{1 + 0 - (-1)^{-k} + 0}{4} = \begin{cases} \frac{1}{2} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

Discrete-Time Fourier Series

Determine the Fourier series coefficients for $x_2[n]$.

$$x_2[n] = \cos\left(\frac{\pi}{2}n\right)$$

Choose N to be the fundamental period of $x_2[n]$.

Alternatively, we could use Euler's formula.

Choosing $N = 4$ means the fundamental is $\Omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$.

$$x_2[n] = \cos\left(\frac{\pi}{2}n\right) = \frac{1}{2}e^{j\frac{\pi}{2}n} + \frac{1}{2}e^{-j\frac{\pi}{2}n} = \frac{1}{2}e^{j\frac{\pi}{2}n} + \frac{1}{2}e^{j3\frac{\pi}{2}n}$$

So, $X_2[1] = X_2[3] = \frac{1}{2}$ and $X_2[0] = X_2[2] = 0$.

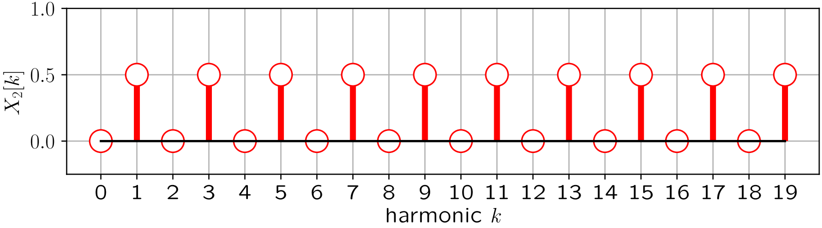
Remember: The coefficients $X_2[k]$ are periodic in N, too!

Discrete-Time Fourier Series

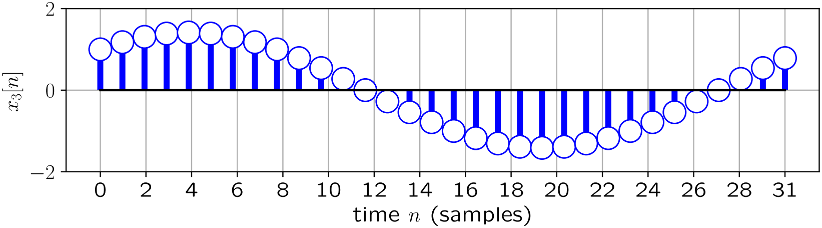
Determine the Fourier series coefficients for $x_2[n]$.

$$x_2[n] = \cos\left(\frac{\pi}{2} n\right)$$

Choose N to be the fundamental period of $x_2[n]$.



Discrete-Time Fourier Series



Determine the Fourier series coefficients for $x_3[n]$.

$$x_3[n] = \cos\left(\frac{2\pi}{32}n\right) + \sin\left(\frac{2\pi}{32}n\right)$$

Choose N to be the fundamental period of $x_3[n]$.

Discrete-Time Fourier Series

Determine the Fourier series coefficients for $x_3[n]$.

$$x_3[n] = \cos\left(\tfrac{2\pi}{32}n\right) + \sin\left(\tfrac{2\pi}{32}n\right)$$

Choose N to be the fundamental period of $x_3[n]$.

The fundamental period is $N = 32$.

I don't want to sum over 32 samples. Euler's formula!

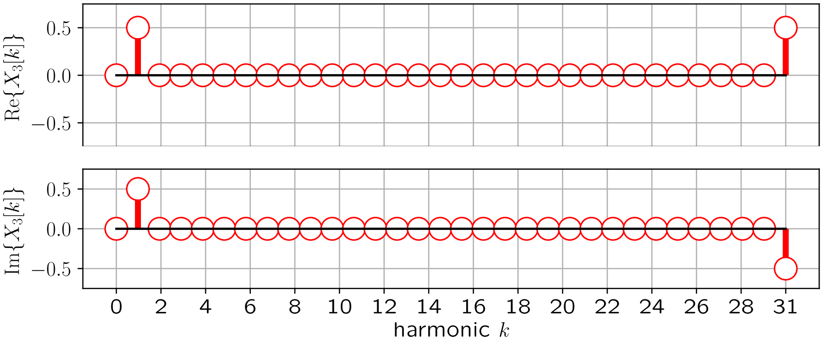
$$\begin{aligned}
x_3[n] &= \cos\left(\tfrac{2\pi}{32}n\right) + \sin\left(\tfrac{2\pi}{32}n\right) \\
&= \left(\tfrac{1}{2}e^{j\frac{2\pi}{32}n} + \tfrac{1}{2}e^{-j\frac{2\pi}{32}n}\right) + \left(\tfrac{1}{2j}e^{j\frac{2\pi}{32}n} - \tfrac{1}{2j}e^{-j\frac{2\pi}{32}n}\right) \\
&= \underbrace{\left(\tfrac{1}{2} + \tfrac{1}{2j}\right)}_{X_3[1]} e^{j\frac{2\pi}{32}n} + \underbrace{\left(\tfrac{1}{2} - \tfrac{1}{2j}\right)}_{X_3[-1]} e^{-j\frac{2\pi}{32}n}
\end{aligned}$$

Discrete-Time Fourier Series

Determine the Fourier series coefficients for $x_3[n]$.

$$x_3[n] = \cos\left(\frac{2\pi}{32}n\right) + \sin\left(\frac{2\pi}{32}n\right)$$

Choose N to be the fundamental period of $x_3[n]$.



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- Discrete-time Fourier series (DTFS)
- As time allows: Deriving Fourier series properties

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Deriving Fourier Series Properties

Let $f[n]$ denote an arbitrary N-periodic discrete-time signal. Determine Fourier series coefficients for $g[n]$.

$$g[n] = 9 - 3f[n-1]$$

Deriving Fourier Series Properties

Let $f[n]$ denote an arbitrary N-periodic discrete-time signal. Determine Fourier series coefficients for $g[n]$.

$$g[n] = 9 - 3f[n-1]$$

Start from the analysis formula.

$$G[k] = \frac{1}{N}\sum_{n=0}^{N-1}\big(9 - 3f[n-1]\big)e^{-jk\frac{2\pi}{N}n}$$

$$= 9\underbrace{\left(\frac{1}{N}\sum_{n=0}^{N-1}e^{-jk\frac{2\pi}{N}n}\right)}_{\text{We've seen this!}} - 3\underbrace{\left(\frac{1}{N}\sum_{n=0}^{N-1}f[n-1]e^{-jk\frac{2\pi}{N}n}\right)}_{\text{Use the delay property.}}$$

Combining the results, we get $G[k] = 9\delta[k] - 3e^{-jk\frac{2\pi}{N}}F[k]$.

Deriving Fourier Series Properties

Let $f[n]$ denote an arbitrary discrete-time signal with fundamental period N, and let $F[k]$ denote the Fourier series coefficients of $f[n]$ computed with period N.

$$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-jk\frac{2\pi}{N}n}$$

Let $G[k]$ denote the Fourier series coefficients of $f[n]$ computed with period $2N$. Determine a closed-form expression for $G[k]$ in terms of $F[k]$.

$$G[k] = \frac{1}{2N} \sum_{n=0}^{2N-1} f[n] e^{-jk\frac{2\pi}{2N}n}$$

Deriving Fourier Series Properties

Split the sum into the first half and the second half.

$$G[k] = \frac{1}{2}\left(\frac{1}{N}\sum_{n=0}^{N-1} f[n]e^{-jk\frac{2\pi}{2N}n}\right) + \frac{1}{2}\left(\frac{1}{N}\sum_{n=N}^{2N-1} f[n]e^{-jk\frac{2\pi}{2N}n}\right)$$

$$= \frac{1}{2}\left(\frac{1}{N}\sum_{n=0}^{N-1} f[n]e^{-jk\frac{2\pi}{2N}n}\right) + \frac{1}{2}\underbrace{\left(\frac{1}{N}\sum_{m=0}^{N-1} f[m+N]e^{-jk\frac{2\pi}{2N}(m+N)}\right)}_{\text{Change of variables: Let } n=m+N.}$$

Because f is N-periodic, $f[m+N] = f[m]$ for all m.

$$G[k] = \frac{1}{2}\left(\frac{1}{N}\sum_{n=0}^{N-1} f[n]e^{-j\left(\frac{k}{2}\right)\frac{2\pi}{N}n}\right) + \frac{(-1)^k}{2}\left(\frac{1}{N}\sum_{m=0}^{N-1} f[m]e^{-j\left(\frac{k}{2}\right)\frac{2\pi}{N}m}\right)$$

$$= \begin{cases} F\left[\frac{k}{2}\right] & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

Deriving Fourier Series Properties

Let $f[n]$ denote an arbitrary N-periodic discrete-time signal. Determine Fourier series coefficients for $g[n]$.

$$g[n] = \begin{cases} f\left[\frac{n}{2}\right] & n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Deriving Fourier Series Properties

Let $f[n]$ denote an arbitrary N-periodic discrete-time signal. Determine Fourier series coefficients for $g[n]$.

$$g[n] = \begin{cases} f\left[\frac{n}{2}\right] & n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

If $f[n]$ is N-periodic, then $g[n]$ is $2N$-periodic.

$$G[k] = \frac{1}{2N}\sum_{n=0}^{2N-1} g[n]e^{-jk\frac{2\pi}{2N}n} = \frac{1}{2N}\sum_{n\text{ even}}^{2N-1} f\left[\tfrac{n}{2}\right]e^{-jk\frac{2\pi}{2N}n}$$

Make a change of variables: Let $n = 2m$ and sum over m.

$$G[k] = \frac{1}{2}\left(\frac{1}{N}\sum_{m=0}^{N-1} f[m]e^{-jk\frac{2\pi}{N}m}\right) = \tfrac{1}{2}F[k]$$

Lessons Learned

A **discrete-time Fourier series (DTFS)** is a Fourier representation for periodic discrete-time signals.

Analysis: $X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\frac{2\pi}{N}n}$

Synthesis: $x[n] = \sum_{k=\langle N \rangle} X[k] e^{jk\frac{2\pi}{N}n}$

Fourier Series Properties: We can derive a number of properties by writing out the analysis or synthesis equation, making a change of variables, and matching the resulting expression to an already-known result.

Question of the Day

Determine the Fourier series coefficients for $x[n]$.

$$x[n] = 1 + j^n + (-1)^n + (-j)^n$$

Hint: Determine N, the period of $x[n]$. Do the DTFS basis functions for this value of N have a simple form?