# 6.300: Signal Processing

## Discrete Cosine Transform (DCT)

**Final Exam**

- on Friday, May 15
- from 9:00 a.m. to 12:00 p.m.
- in Walker Memorial (50-340)
- https://sigproc.mit.edu/spring26/q3_info

Please fill out a **subject evaluation** for this class at https://registrar.mit.edu/classes-grades-evaluations/subject-evaluation by Friday, May 15 at 9:00 a.m.

*May 5, 2026*

# Discrete Cosine Transform

The **discrete cosine transform (DCT)** is defined by analysis and synthesis equations analogous to those of the DFT.

$$X_C[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\cos\left(\frac{\pi k}{N}\left(n+\tfrac{1}{2}\right)\right) \qquad \text{(analysis)}$$

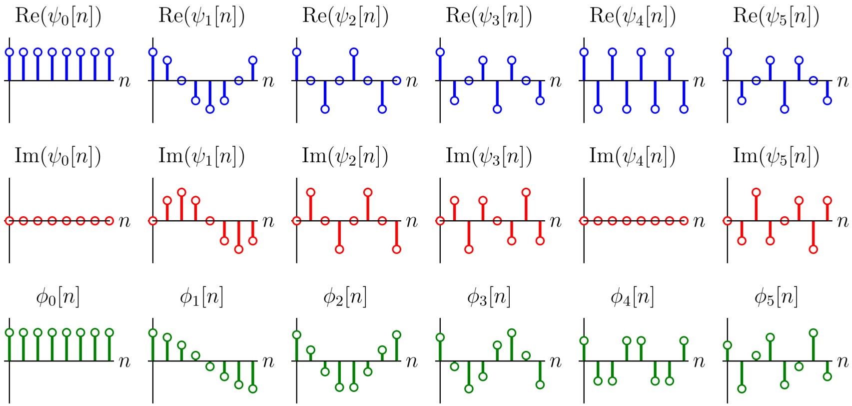
$$x[n] = X_C[0] + 2\sum_{k=1}^{N-1} X_C[k]\cos\left(\frac{\pi k}{N}\left(n+\tfrac{1}{2}\right)\right) \qquad \text{(synthesis)}$$

As described in lecture, the DCT of $x[n]$ is equal to the DFT of $x'[n]$, where $x'[n]$ is derived from $x[n]$ by

- **appending** one period of $x[n]$ in reverse order,
- **stretching** the result in time by a factor of two,
- **doubling** each value, and
- **shifting** the result right by a single sample.

# Basis Functions: DFT vs. DCT

The DFT basis functions $\psi_k[n]$ are complex exponentials.
The DCT basis functions $\phi_k[n]$ are real-valued cosines.

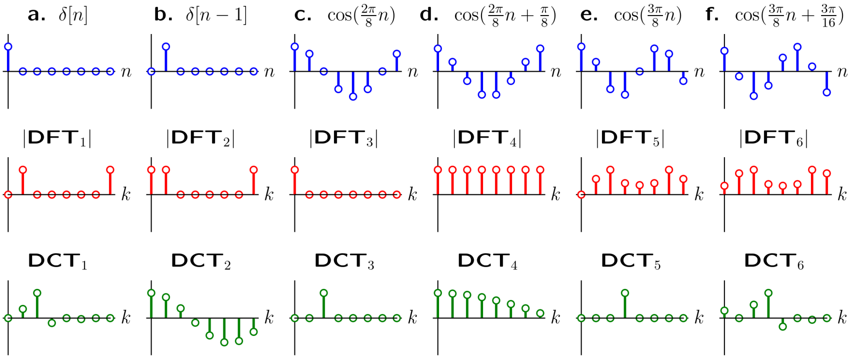


# Basis Functions: DFT vs. DCT

For each time-domain signal (**a, b, c, d, e, f**) shown below, determine which of the following plots show

- the 8-point **DFT** coefficient magnitudes and
- the 8-point **DCT** coefficients.

Note that maxima and minima may differ between plots.



# Basis Functions: DFT vs. DCT

**a.** Plug $\delta[n]$ into the DFT and DCT analysis equations.

$$F[k] = \frac{1}{8}\sum_{n=0}^{7}\delta[n]e^{-jk\frac{2\pi}{8}n} = \frac{1}{8}$$

$$F_C[k] = \frac{1}{8}\sum_{n=0}^{7}\delta[n]\cos\left(\frac{\pi k}{8}\left(n+\tfrac{1}{2}\right)\right) = \frac{1}{8}\cos\left(\frac{\pi k}{16}\right)$$

Answers: $|\mathrm{DFT}_4|$ and $\mathrm{DCT}_4$

**b.** Plug $\delta[n-1]$ into the DFT and DCT analysis equations.

$$F[k] = \frac{1}{8}\sum_{n=0}^{7}\delta[n-1]e^{-jk\frac{2\pi}{8}n} = \frac{1}{8}e^{-jk\frac{2\pi}{8}}$$

$$F_C[k] = \frac{1}{8}\sum_{n=0}^{7}\delta[n-1]\cos\left(\frac{\pi k}{8}\left(n+\tfrac{1}{2}\right)\right) = \frac{1}{8}\cos\left(\frac{3\pi k}{16}\right)$$

Answers: $|\mathrm{DFT}_4|$ and $\mathrm{DCT}_2$

# Basis Functions: DFT vs. DCT

**c.** Notice that $\cos(2\pi n/8)$ is the sum of the $k = 1$ and $k = -1$ DFT basis functions. While $\cos(2\pi n/8)$ doesn't match the form of a single DCT basis function exactly, it most closely resembles the $k = 2$ DCT basis function.

Answers: $|\text{DFT}_1|$ and $\text{DCT}_1$

**d.** A time-shift doesn't change the magnitude of the DFT magnitudes. However, a time-shift does change the DCT coefficients. In fact, $\cos(2\pi n/8 - \pi/8)$ is precisely the $k = 2$ DCT basis function.

Answers: $|\text{DFT}_1|$ and $\text{DCT}_3$

**e.** Periodic extension of this signal results in step discontinuities. In general, all DFT coefficients are non-zero, though we expect the peak around $k = \pm 1.5$. We also expect a non-zero average (DC) value. This signal most closely resembles the $k = 3$ DCT basis function.

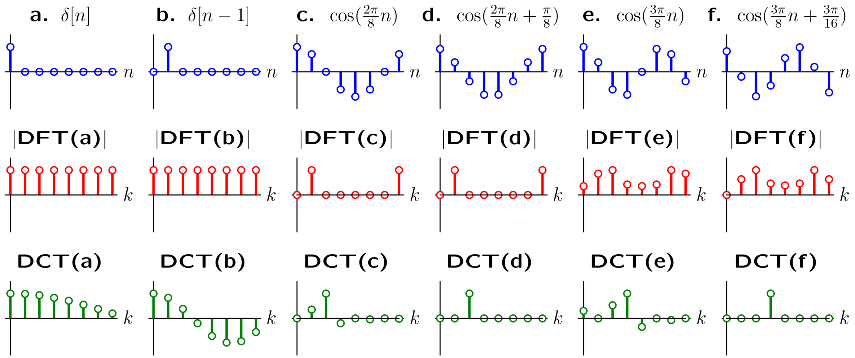
Answers: $|\text{DFT}_6|$ and $\text{DCT}_6$

# Basis Functions: DFT vs. DCT

**f.** Periodic extension of this signal results in step discontinuities. In general, all DFT coefficients are non-zero, though we expect the peak around $k = \pm 1.5$. We also expect the average (DC) value to be zero. This signal matches the $k = 3$ DCT basis function exactly.

Answers: $|\text{DFT}_5|$ and $\text{DCT}_5$

The |DFT| and DCT coefficients are shown below each signal.



# Basis Functions: DFT vs. DCT

Much of the utility of Fourier transforms — and, in particular, the DFT — stems from properties of the Fourier basis functions.

**DTFS:** $e^{jk\frac{2\pi}{N}n}$ **DTFT:** $e^{j\Omega n}$ **DFT:** $e^{jk\frac{2\pi}{N}n}$

To better understand the DCT, we must understand the DCT basis functions.

**DCT:** $\cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right)$

# DCT Basis Functions: Symmetry

The $k^{\text{th}}$ DCT basis function of order $N$ is given by

$$\phi_k[n] = \cos\left(\frac{\pi k}{N}\left(n + \tfrac{1}{2}\right)\right)$$

where $k \in \{0, 1, 2, \ldots, N-1\}$.

How many of the symmetries below are true?

- $\phi_k[n + 2N] = \phi_k[n]$
- $\phi_k[n + N] = (-1)^k \phi_k[n]$
- $\phi_k[n - N] = (-1)^k \phi_k[n]$
- $\phi_k[(N-1) - n] = (-1)^k \phi_k[n]$

# DCT Basis Functions: Symmetry

The $k^{\text{th}}$ DCT basis function of order $N$ is given by

$$\phi_k[n] = \cos\left(\frac{\pi k}{N}\left(n + \tfrac{1}{2}\right)\right)$$

where $k \in \{0, 1, 2, \ldots, N-1\}$.

How many of the symmetries below are true? **4**

- $\phi_k[n+2N] = \phi_k[n]$ **true**
- $\phi_k[n+N] = (-1)^k \phi_k[n]$ **true**
- $\phi_k[n-N] = (-1)^k \phi_k[n]$ **true**
- $\phi_k[(N-1)-n] = (-1)^k \phi_k[n]$ **true**

These symmetries are straightforward to derive using trigonometric identities — or by looking at plots of the DCT basis functions.

# DCT Basis Functions: Area

Show that

$$\sum_{n=0}^{N-1} \phi_k[n] = N\delta[k] = \begin{cases} N & k = 0 \\ 0 & k \neq 0. \end{cases}$$

# DCT Basis Functions: Area

Show that

$$\sum_{n=0}^{N-1} \phi_k[n] = N\delta[k] = \begin{cases} N & k = 0 \\ 0 & k \neq 0. \end{cases}$$

Graphical argument: The "area" under the curve traced out by the $k^{\text{th}}$ DCT basis function of order $N$ is zero — unless $k = 0$, in which case the "area" is $N$.

# DCT Basis Functions: Orthogonality

The DCT basis functions are orthogonal. Show that

$$\sum_{n=0}^{N-1} \phi_k[n]\phi_\ell[n] = \begin{cases} N & k = \ell = 0 \\ \frac{1}{2}N & k = \ell \neq 0 \\ 0 & k \neq \ell. \end{cases}$$

# DCT Basis Functions: Orthogonality

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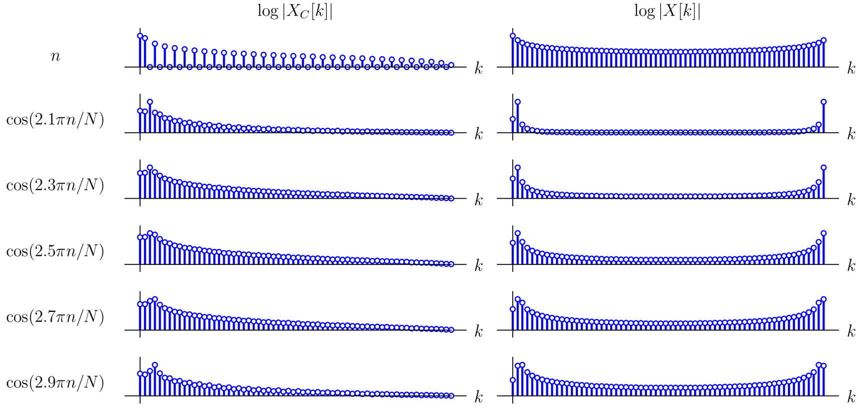
$$\begin{aligned} \sum_{n=0}^{N-1} \phi_k[n]\phi_\ell[n] &= \frac{1}{2}\sum_{n=0}^{N-1} \phi_{k-\ell}[n] + \frac{1}{2}\sum_{n=0}^{N-1} \phi_{k+\ell}[n] \\ &= \frac{N}{2}\delta[k-\ell] + \frac{N}{2}\delta[k+\ell] \\ &= \begin{cases} N & k = \ell = 0 \\ \frac{1}{2}N & k = \ell \neq 0 \\ 0 & k \neq \ell \end{cases} \end{aligned}$$

# DFT vs. DCT: Compaction

If a signal has predominately low-frequency content, then the higher-order coefficients of the DCT tend to decrease faster than the corresponding coefficients of the DFT. Similar compaction results for sinusoidal signals.



# DFT vs. DCT: Eigenfunctions

The Fourier basis functions $e^{j\Omega n}$ are **eigenfunctions** of linear, time-invariant systems. The **eigenvalues** are $H(\Omega)$.

$$e^{j\Omega n} \to \boxed{\text{LTI}} \to H(\Omega) e^{j\Omega n}$$

The DCT basis functions are not eigenfunctions of linear, time-invariant systems, however. So, the DCT cannot be used for filtering.

# Question of the Day

The DCT provides significant data compaction for images. So, why didn't we spend half the semester talking about the DCT, rather than the DFT? Why is the DFT still important? Explain in a few sentences. (**Hint:** Basis functions . . . Eigenfunctions . . . )