6.300: Signal Processing

Complex Fourier Series

Euler's Formula: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$, where $j \triangleq \sqrt{-1}$

The **synthesis equation** tells us how to represent a periodic signal as a Fourier series.

$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$

The **analysis equation** tell us how to calculate these Fourier series coefficients.

$a_k = \frac{1}{T}\int_T f(t) e^{-jk\frac{2\pi}{T}t} dt$

February 10, 2026

Agenda for Recitation

- **Complex numbers** and **Euler's formula**
- Complex, yet much simpler: **Complex Fourier series**
- Complex Fourier series and the **delay property**
- Complex Fourier series and **symmetry**

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What questions do you have from lecture?

Complex Numbers

It's useful to think about complex numbers **graphically**.

Rectangular (Real Part and Imaginary Part)

$$z = a + jb$$

where $a = \text{Re}(z)$

and $b = \text{Im}(z)$

Polar (Magnitude and Phase)

$$z = re^{j\theta}$$

where $r = \sqrt{zz^*}$

and $\tan\theta = \frac{\text{Im}(z)}{\text{Re}(z)}$

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where $a = \text{Re}(z) = r\cos(\theta)$

and $b = \text{Im}(z) = r\sin(\theta)$

Polar (Magnitude and Phase)

$$z = re^{j\theta}$$

where $r = \sqrt{zz^*} = \sqrt{a^2 + b^2}$

and $\tan\theta = \dfrac{\text{Im}(z)}{\text{Re}(z)} = \dfrac{b}{a}$

Complex Numbers

Let's get some practice working with complex numbers.

Suppose $z = a + jb$ and $a, b > 0$.

Sketch z in the complex plane. Label $a, b, r,$ and θ.

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Finally, sketch jz. What is $\angle(jz) - \angle(z)$?

$z^* = re^{-j\theta}$ — conjugation **negates** the angle.

Multiplication by $j = e^{j\pi/2}$ rotates z by 90° counterclockwise. So, $\angle(jz) - \angle(z) = 90^\circ$ for any $z \neq 0$.

Euler's Formula

Let's get some practice working with Euler's formula.

How many of the following statements are **true**?

- $\dfrac{1}{\cos\theta + j\sin\theta} = \cos\theta - j\sin\theta$
- $(\cos\theta + j\sin\theta)^n = \cos(n\theta) + j\sin(n\theta)$
- $|2 + 2j + e^{j\pi/4}| = |2 + 2j| + |e^{j\pi/4}|$
- $\text{Im}(j^j) > \text{Re}(j^j)$

Euler's Formula

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- $\dfrac{1}{\cos\theta + j\sin\theta} = \cos\theta - j\sin\theta$ **true**
- $(\cos\theta + j\sin\theta)^n = \cos(n\theta) + j\sin(n\theta)$ **true**
- $|2 + 2j + e^{j\pi/4}| = |2 + 2j| + |e^{j\pi/4}|$ **true**
- $\mathrm{Im}(j^j) > \mathrm{Re}(j^j)$ **false**

The first **3** statements are **true**. (Use Euler's formula.)
Note that the third statement does not hold in general.
The last statement is **false**: $j^j = e^{-\pi/2}$ is purely real.

Complex Fourier Series: Cosine

A **Fourier series** expansion takes the form

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

where a_k denotes the k^{th} Fourier series coefficient.

Determine the Fourier series coefficients for $f(t)$.

$$f(t) = f(t+2\pi) = \cos(t)$$

Hint: Use Euler's formula. No calculus, please!

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Use Euler's formula.

$$f(t) = \cos(t) = \frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt}$$

So, $a_1 = a_{-1} = \frac{1}{2}$. $a_k = 0$ for all $k \neq \pm 1$.

Complex Fourier Series: Cosine^2

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$$f(t) = f(t+2\pi) = \cos^2(t)$$

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Determine the Fourier series coefficients for $f(t)$.

$$f(t) = f(t+2\pi) = \cos^2(t)$$

Hint: Use Euler's formula. No calculus, please!

Use Euler's formula and the binomial theorem.

$$f(t) = \cos^2(t) = \left(\tfrac{1}{2}e^{jt} + \tfrac{1}{2}e^{-jt}\right)^2 = \tfrac{1}{4}e^{j2t} + \tfrac{1}{2} + \tfrac{1}{4}e^{-j2t}$$

So, $a_2 = a_{-2} = \frac{1}{4}$ and $a_0 = \frac{1}{2}$. $a_k = 0$ for all $k \notin \{0, \pm 2\}$.

Complex Fourier Series: Cosine^{100}

A **Fourier series** expansion takes the form

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

where a_k denotes the k^{th} Fourier series coefficient.

Determine the Fourier series coefficients for $f(t)$.

$$f(t) = f(t+2\pi) = \cos^{100}(t)$$

Hint: Use Euler's formula. No calculus, please!

Complex Fourier Series: Cosine^{100}

Determine the Fourier series coefficients for $f(t)$.

$$f(t) = f(t+2\pi) = \cos^{100}(t)$$

Hint: Use Euler's formula. No calculus, please!

Use Euler's formula and the binomial theorem.

$$\cos^{100}(t) = \left(\tfrac{1}{2}e^{jt} + \tfrac{1}{2}e^{-jt}\right)^{100} = 2^{-100}\left(e^{j100t} + \cdots\right)$$

$$a_k = \begin{cases} 2^{-100}\binom{100}{50-\frac{1}{2}k} & k \in \{-100, -98, -96, \ldots, 96, 98, 100\} \\ 0 & \text{otherwise} \end{cases}$$

Paradoxically, complex numbers make math simpler!

Complex Fourier Series: Gnarly Algebra

A **Fourier series** expansion takes the form

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

where a_k denotes the k^{th} Fourier series coefficient.

Determine the coefficients for a Fourier series expansion of $f(t)$. Suppose that $T = 4$.

$$f(t) = f(t+4) = \frac{-1}{\left(\sin\left(\frac{3\pi}{2}t\right) + j\cos\left(\frac{3\pi}{2}t\right)\right)^2}$$

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$$a_k = \begin{cases} 1 & k = 6 \\ 0 & k \neq 6 \end{cases}$$ (Euler's formula and algebra)

Delay Property

Suppose that we know the Fourier series coefficients (a_k) for $f(t)$. What can we say about the Fourier series coefficients of $f(t-\tau)$, where τ is an arbitrary constant?

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Suppose that we know the Fourier series coefficients (a_k) for $f(t)$. What can we say about the Fourier series coefficients of $f(t-\tau)$, where τ is an arbitrary constant?

Delay Property: If

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

then

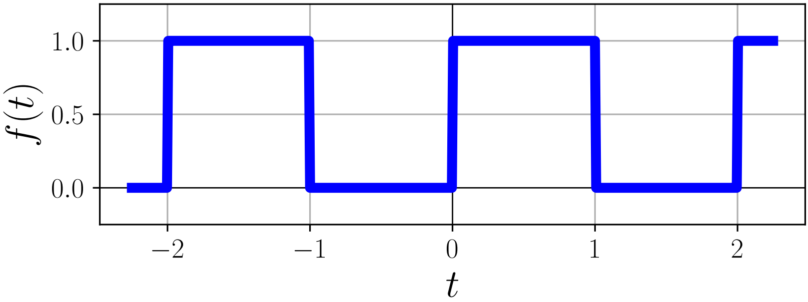
$$f(t-\tau) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\frac{2\pi}{T}\tau} e^{jk\frac{2\pi}{T}t}$$

for an arbitrary constant τ.

Meaning: Shifting $f(t)$ by τ **multiplies** each Fourier series coefficient (a_k) by a **complex-valued scalar**!

Delay Property

Example: The following signal is periodic in $T = 2$.



Determine the Fourier series coefficients for $f(t)$.

Delay Property

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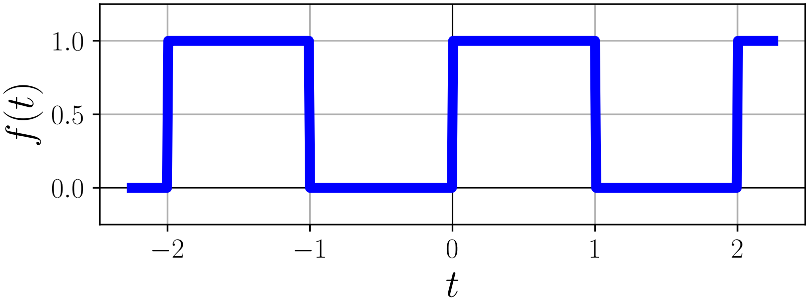


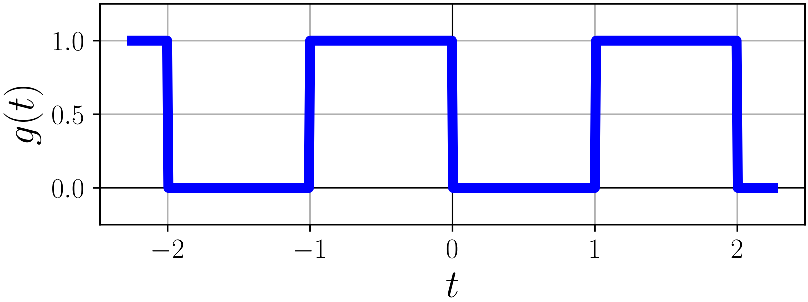
The Fourier series coefficients are computed as

$$a_k = \frac{1}{2}\int_0^1 e^{-jk\pi t}dt = \frac{1}{2}\left[\frac{e^{-jk\pi t}}{-jk\pi}\right]_0^1 = \frac{1-(-1)^k}{j2\pi k}$$

What about $k = 0$? The average value over a period is $\frac{1}{2}$.

Delay Property

Shifting by $\tau = 1$ is easy: Use the **delay property**!



Determine the Fourier series coefficients for $g(t)$.

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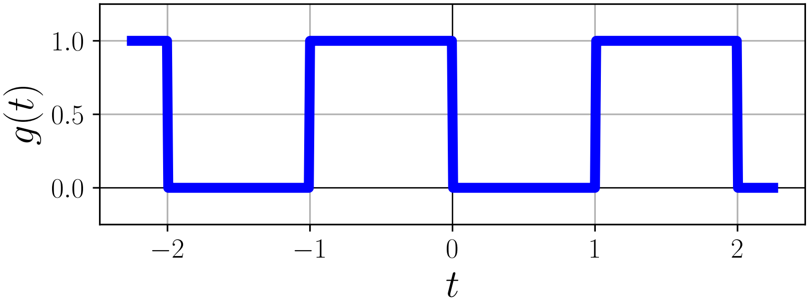
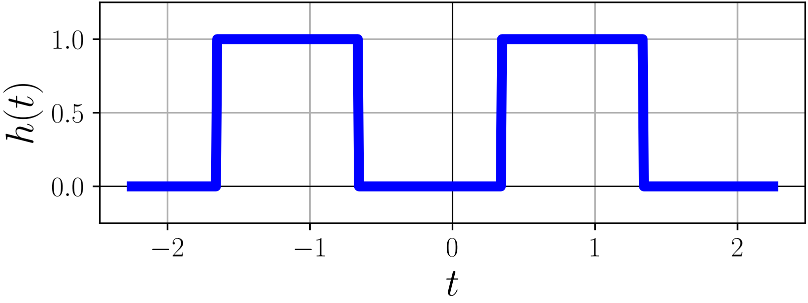


$$g(t) = f(t - \tau) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\frac{2\pi}{T}\tau} e^{jk\frac{2\pi}{T}t}$$

$$= f(t-1) = \sum_{k=-\infty}^{\infty} a_k (-1)^k e^{jk2\pi t}$$

Delay Property

Shifting by $\tau = \frac{1}{\sqrt{\pi e}} \approx 0.34$ is easy: Use the **delay property**!



Determine the Fourier series coefficients for $h(t)$.

Delay Property

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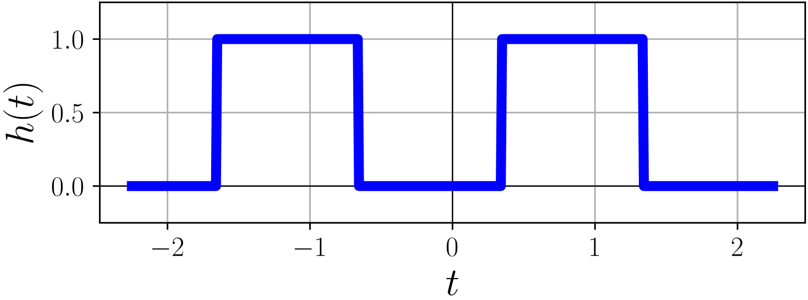

$$h(t) = f(t-\tau) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\frac{2\pi}{T}\tau} e^{jk\frac{2\pi}{T}t}$$

$$= f\left(t - \frac{1}{\sqrt{\pi e}}\right) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\frac{2\pi}{\sqrt{\pi e}}} e^{jk2\pi t}$$

Complex Fourier Series and Symmetry

Last time, we used **symmetry** to simplify calculations.

What can we say about the Fourier series coefficients (a_k) for a **real, symmetric** signal?

What can we say about the Fourier series coefficients (a_k) for a **real, anti-symmetric** signal?

Complex Fourier Series and Symmetry

Last time, we used **symmetry** to simplify calculations.

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What can we say about the Fourier series coefficients (a_k) for a **real, anti-symmetric** signal?

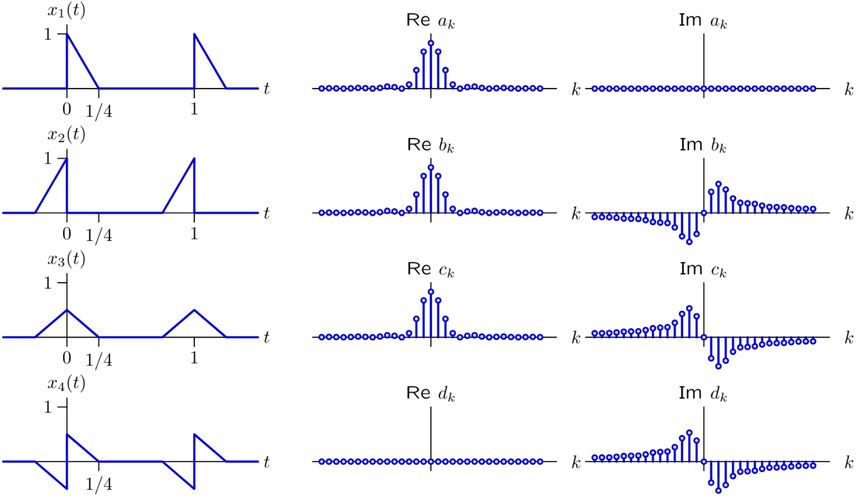
If $f(t)$ is real and symmetric, then
a_k is a **purely real, symmetric** function of k.

If $f(t)$ is real and anti-symmetric, then
a_k is a **purely imaginary, anti-symmetric** function of k.

In general, for a real signal, the Fourier series coefficients are **conjugate (Hermitian) symmetric:** $a_{-k} = a_k^*$.

Complex Fourier Series and Symmetry

Match each signal (left) to its Fourier series coefficients.



Complex Fourier Series and Symmetry

$x_3(t)$ is real and symmetric, so the Fourier series coefficients must be purely real: a_k.

$x_4(t)$ is real and anti-symmetric, so the Fourier series coefficients must be purely imaginary: d_k.

If we add signals, the Fourier series coefficients add, too.
$x_1(t) = x_3(t) + x_4(t)$: The Fourier series coefficients are c_k.

If we add signals, the Fourier series coefficients add, too.
$x_2(t) = x_3(t) - x_4(t)$: The Fourier series coefficients are b_k.

Lessons Learned

Euler's Formula: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$, where $j \triangleq \sqrt{-1}$

Synthesis Equation: $f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$

Analysis Equation: $a_k = \frac{1}{T}\int_T f(t) e^{-jk\frac{2\pi}{T}t} dt$

Delay Property: $f(t-\tau) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\frac{2\pi}{T}\tau} e^{jk\frac{2\pi}{T}t}$

Real-valued signals: a_k are conjugate symmetric.
symmetric $f(t) \implies$ real, symmetric a_k
anti-symmetric $f(t) \implies$ imaginary, anti-symmetric a_k

Question of the Day

Determine the Fourier series coefficients (a_k) for $f(t)$.

$$f(t) = f(t+\pi) = \sin^2(t)$$

Hint: $e^{j\theta} = \cos\theta + j\sin\theta$.

Challenge: Parseval's Theorem

Our math-inclined friends might like this one.

Determine an expression for

$$\int_T f^2(t)\,dt$$

in terms of the Fourier series coefficients (a_k) of $f(t)$.

$$f(t) = \sum_k a_k e^{jk\frac{2\pi}{T}t}$$

If you want a hint, there's one on the next slide.

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Hint: Substitute the Fourier series expansion of $f(t)$ into the integral. Swap the order of summation and integration. If $f(t)$ is assumed to be purely real, then its Fourier series coefficients are conjugate symmetric, i.e., $a_{-k} = a_k^*$. Consequently, $a_k a_{-k} = |a_k|^2$ for all k.