6.300: Signal Processing

Complex Fourier Series

Euler's Formula: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$, where $j \triangleq \sqrt{-1}$

The **synthesis equation** tells us how to represent a periodic signal as a Fourier series.

$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$

The **analysis equation** tell us how to calculate these Fourier series coefficients.

$a_k = \frac{1}{T}\int_T f(t) e^{-jk\frac{2\pi}{T}t} dt$

February 10, 2026

Agenda for Recitation

- **Complex numbers** and **Euler's formula**
- Complex, yet much simpler: **Complex Fourier series**
- Complex Fourier series and the **delay property**
- Complex Fourier series and **symmetry**

Complex Numbers

It's useful to think about complex numbers **graphically**.

Rectangular (Real Part and Imaginary Part)

$$z = a + jb$$

where $a = \text{Re}(z)$

and $b = \text{Im}(z)$

Polar (Magnitude and Phase)

$$z = re^{j\theta}$$

where $r = \sqrt{zz^*}$

and $\tan\theta = \frac{\text{Im}(z)}{\text{Re}(z)}$

Complex Numbers

Let's get some practice working with complex numbers.

Suppose $z = a + jb$ and $a, b > 0$.

Sketch z in the complex plane. Label a, b, r, and θ.

Now, sketch z^* — the complex conjugate of z.

Finally, sketch jz. What is $\angle(jz) - \angle(z)$?

Euler's Formula

Let's get some practice working with Euler's formula.

How many of the following statements are **true**?

- $\dfrac{1}{\cos\theta + j\sin\theta} = \cos\theta - j\sin\theta$
- $(\cos\theta + j\sin\theta)^n = \cos(n\theta) + j\sin(n\theta)$
- $|2 + 2j + e^{j\pi/4}| = |2 + 2j| + |e^{j\pi/4}|$
- $\mathrm{Im}(j^j) > \mathrm{Re}(j^j)$

Complex Fourier Series: Cosine

A **Fourier series** expansion takes the form

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

where a_k denotes the k^{th} Fourier series coefficient.

Determine the Fourier series coefficients for $f(t)$.

$$f(t) = f(t+2\pi) = \cos(t)$$

Hint: Use Euler's formula. No calculus, please!

Complex Fourier Series: Cosine^2

A **Fourier series** expansion takes the form

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

where a_k denotes the k^{th} Fourier series coefficient.

Determine the Fourier series coefficients for $f(t)$.

$$f(t) = f(t+2\pi) = \cos^2(t)$$

Hint: Use Euler's formula. No calculus, please!

Complex Fourier Series: Cosine^{100}

A **Fourier series** expansion takes the form

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

where a_k denotes the k^{th} Fourier series coefficient.

Determine the Fourier series coefficients for $f(t)$.

$$f(t) = f(t+2\pi) = \cos^{100}(t)$$

Hint: Use Euler's formula. No calculus, please!

Complex Fourier Series: Gnarly Algebra

A **Fourier series** expansion takes the form

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

where a_k denotes the k^{th} Fourier series coefficient.

Determine the coefficients for a Fourier series expansion of $f(t)$. Suppose that $T = 4$.

$$f(t) = f(t+4) = \frac{-1}{\left(\sin\left(\frac{3\pi}{2}t\right) + j\cos\left(\frac{3\pi}{2}t\right)\right)^2}$$

Delay Property

Suppose that we know the Fourier series coefficients (a_k) for $f(t)$. What can we say about the Fourier series coefficients of $f(t-\tau)$, where τ is an arbitrary constant?

Delay Property: If

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

then

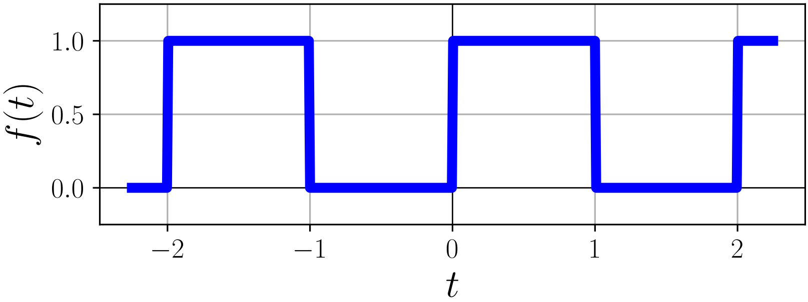
$$f(t-\tau) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\frac{2\pi}{T}\tau} e^{jk\frac{2\pi}{T}t}$$

for an arbitrary constant τ.

Meaning: Shifting $f(t)$ by τ **multiplies** each Fourier series coefficient (a_k) by a **complex-valued scalar**!

Delay Property

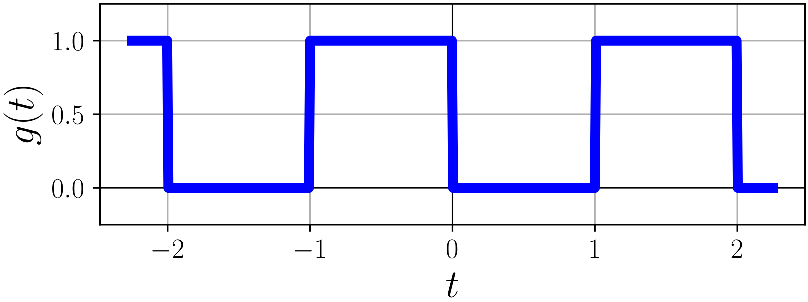
Example: The following signal is periodic in $T = 2$.



Determine the Fourier series coefficients for $f(t)$.

Delay Property

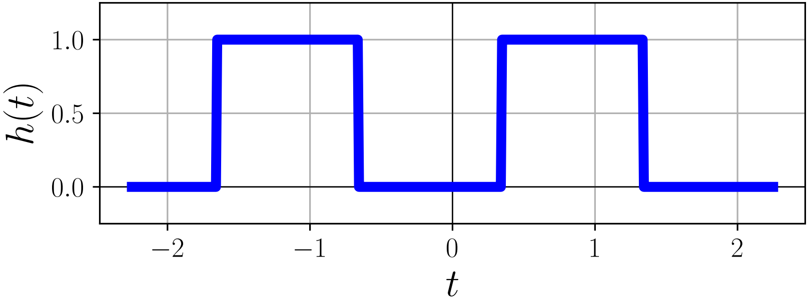
Shifting by $\tau = 1$ is easy: Use the **delay property**!



Determine the Fourier series coefficients for $g(t)$.

Delay Property

Shifting by $\tau = \frac{1}{\sqrt{\pi e}} \approx 0.34$ is easy: Use the **delay property**!



Determine the Fourier series coefficients for $h(t)$.

Complex Fourier Series and Symmetry

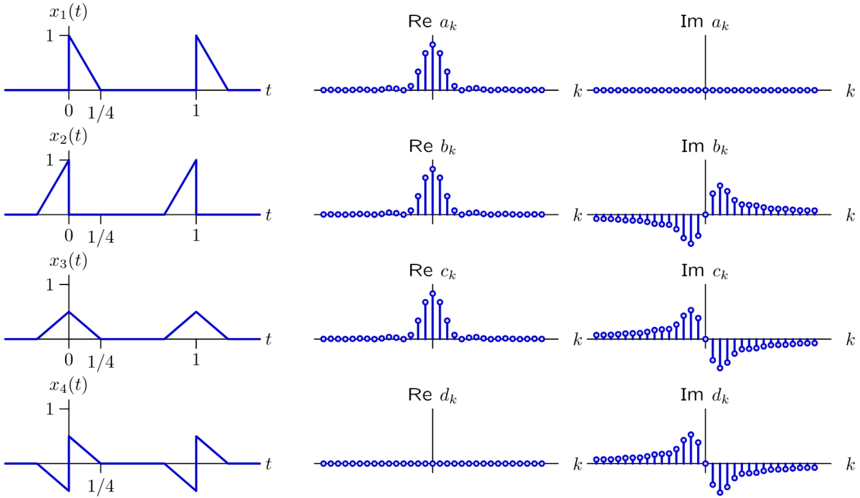
Last time, we used **symmetry** to simplify calculations.

What can we say about the Fourier series coefficients (a_k) for a **real, symmetric** signal?

What can we say about the Fourier series coefficients (a_k) for a **real, anti-symmetric** signal?

Complex Fourier Series and Symmetry

Match each signal (left) to its Fourier series coefficients.



Lessons Learned

Euler's Formula: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$, where $j \triangleq \sqrt{-1}$

Synthesis Equation: $f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$

Analysis Equation: $a_k = \frac{1}{T}\int_T f(t) e^{-jk\frac{2\pi}{T}t} dt$

Delay Property: $f(t-\tau) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\frac{2\pi}{T}\tau} e^{jk\frac{2\pi}{T}t}$

Real-valued signals: a_k are conjugate symmetric.
symmetric $f(t) \implies$ real, symmetric a_k
anti-symmetric $f(t) \implies$ imaginary, anti-symmetric a_k

Challenge: Parseval's Theorem

Our math-inclined friends might like this one.

Determine an expression for

$$\int_T f^2(t)dt$$

in terms of the Fourier series coefficients (a_k) of $f(t)$.

$$f(t) = \sum_k a_k e^{jk\frac{2\pi}{T}t}$$

If you want a hint, there's one on the next slide.

Challenge: Parseval's Theorem

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Determine an expression for

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in terms of the Fourier series coefficients (a_k) of $f(t)$.

$$f(t) = \sum_k a_k e^{jk\frac{2\pi}{T}t}$$

Hint: Substitute the Fourier series expansion of $f(t)$ into the integral. Swap the order of summation and integration. If $f(t)$ is assumed to be purely real, then its Fourier series coefficients are conjugate symmetric, i.e., $a_{-k} = a_k^*$. Consequently, $a_k a_{-k} = |a_k|^2$ for all k.