# 6.300: Signal Processing

## Communication Systems

**Frequency Shift Property:** $x(t)e^{j\omega_0 t} \iff X(\omega-\omega_0)$

**Convolution Theorem:** Convolution in one domain corresponds to multiplication in the other domain.

$$(f*g)(t) \iff F(\omega)G(\omega) \qquad f(t)g(t) \iff \frac{1}{2\pi}(F*G)(\omega)$$

**Sinusoidal Carrier Modulation:** $s(t) = A\cos(\omega t - \phi)$

- amplitude modulation: time-varying $A = A(t)$
- frequency modulation: time-varying $\omega = \omega(t)$
- phase modulation: time-varying $\phi = \phi(t)$

*March 19, 2026*

# Agenda for Recitation

- Time-frequency duality
- Modulation schemes: DSB-AM and SSB-SC
- Implementing a band-pass filter (BPF)

What questions do you have from lecture?

# Agenda for Recitation

- Time-frequency duality
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# Time-Frequency Duality

Notice that the analysis and synthesis formulæ for Fourier transforms look very similar.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Consequently, most of our results work "both ways" — with relatively minor adjustments. This is called **duality**.

Example: Given one transform pair, get another for free!

$$x(t) \iff X(\omega)$$
$$X(t) \iff 2\pi x(-\omega)$$

e.g.,

$$x(t) = e^{-|t|} \iff X(\omega) = \frac{2}{1+\omega^2}$$
$$X(t) = \frac{2}{1+t^2} \iff 2\pi x(-\omega) = 2\pi e^{-|\omega|}$$

# Time-Frequency Duality

Convolution ($*$) in the time domain corresponds to multiplication ($\times$) in the frequency domain.

$$(f * g)(t) \iff F(\omega)G(\omega)$$

Likewise, multiplication ($\times$) in the time domain corresponds to convolution ($*$) in the frequency domain.

$$f(t)g(t) \iff \frac{1}{2\pi}(F * G)(\omega)$$

**Convolution Theorem:** Convolution in one domain corresponds to multiplication in the other domain.

$$(f * g)(t) \iff F(\omega)G(\omega) \qquad f(t)g(t) \iff \tfrac{1}{2\pi}(F * G)(\omega)$$

# Time-Frequency Duality

The **frequency shift property** follows directly from **Fourier transform pairs** and the **convolution theorem**.

$$\begin{aligned} x(t) &\Longleftrightarrow X(\omega) \\ e^{j\omega_c t} &\Longleftrightarrow 2\pi\delta(\omega-\omega_c) \\ x(t)e^{j\omega_c t} &\Longleftrightarrow X(\omega-\omega_c) \end{aligned}$$

Of course, the frequency shift property also holds for multiplication by real-valued sinusoids, like $\cos(\omega_c t)$.

$$\begin{aligned} x(t) &\Longleftrightarrow X(\omega) \\ \cos(\omega_c t) = \tfrac{1}{2}e^{j\omega_c t} + \tfrac{1}{2}e^{-j\omega_c t} &\Longleftrightarrow \pi\delta(\omega-\omega_c) + \pi\delta(\omega+\omega_c) \\ x(t)\cos(\omega_c t) &\Longleftrightarrow \tfrac{1}{2}X(\omega-\omega_c) + \tfrac{1}{2}X(\omega+\omega_c) \end{aligned}$$

The frequency shift property arises all the time in the analysis and design of communication systems.

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# Amplitude Modulation: DSB-AM

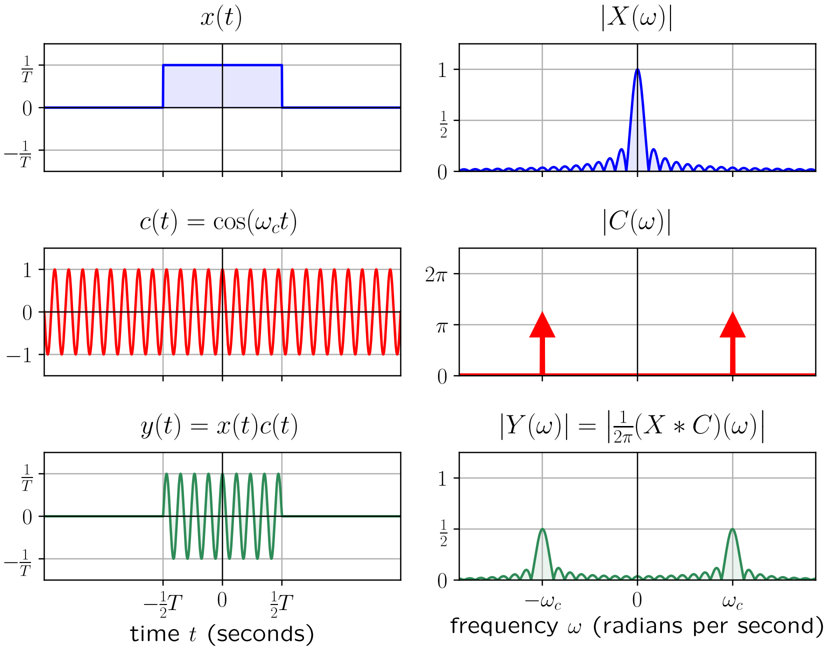
## Double Sideband Amplitude Modulation (DSB-AM)

Push the signal of interest to a higher band of frequencies ($|\omega| \gg 0$) before transmission over the air.

$$\begin{aligned} x(t) &\iff X(\omega) \\ \cos(\omega_c t) = \tfrac{1}{2}e^{j\omega_c t} + \tfrac{1}{2}e^{-j\omega_c t} &\iff \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c) \\ x(t)\cos(\omega_c t) &\iff \tfrac{1}{2}X(\omega - \omega_c) + \tfrac{1}{2}X(\omega + \omega_c) \end{aligned}$$

The signal $x(t)$ modulates the amplitude of $\cos(\omega_c t)$ — hence **amplitude modulation**. Also, the transform $X(\cdot)$ appears twice in the output — hence **double sideband**.

The fact that $X(\cdot)$ appears twice in the output reduces the number of messages that may be transmitted by a factor of two. Despite this, DSB-AM is still used in commercial AM radio. Simplicity often implies popularity.


$x(t)$
$|X(\omega)|$
$c(t) = \cos(\omega_c t)$
$|C(\omega)|$
$y(t) = x(t)c(t)$
$|Y(\omega)| = \left|\frac{1}{2\pi}(X * C)(\omega)\right|$
$\frac{1}{T}$
0
$-\frac{1}{T}$
1
$\frac{1}{2}$
$-1$
$2\pi$
$\pi$
$-\frac{1}{2}T$
$\frac{1}{2}T$
time $t$ (seconds)
$-\omega_c$
$\omega_c$
frequency $\omega$ (radians per second)

# Amplitude Modulation: DSB-AM

Is amplitude modulation linear? Is it time-invariant?

$$x(t) \rightarrow \boxed{\text{AM}} \rightarrow y(t) = x(t)\cos(\omega_c t)$$

# Amplitude Modulation: DSB-AM

Is amplitude modulation linear? Is it time-invariant?

$$x(t) \rightarrow \boxed{\text{AM}} \rightarrow y(t) = x(t)\cos(\omega_c t)$$

Amplitude modulation is linear.

$$\begin{aligned} y(t) &= x(t)\cos(\omega_c t) \\ &= (c_1 x_1(t) + c_2 x_2(t))\cos(\omega_c t) \\ &= c_1 x_1(t)\cos(\omega_c t) + c_2 x_2(t)\cos(\omega_c t) \\ &= c_1 y_1(t) + c_2 y_2(t) \end{aligned}$$

Amplitude modulation is not time-invariant. It generates new non-zero frequencies in the output.

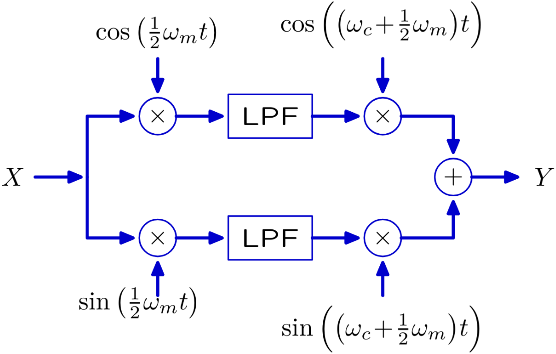
$$x(t-t_0)\cos(\omega_c t) \neq x(t-t_0)\cos(\omega_c (t-t_0))$$

# Amplitude Modulation: SSB-SC

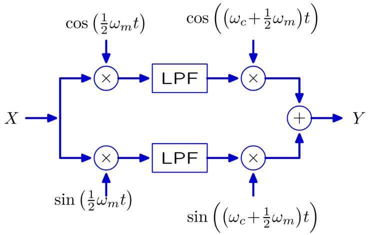
## Single Sideband – Suppressed Carrier (SSB-SC)

Real-valued time-domain signals have symmetric transforms. So, we can infer "the left" from "the right."

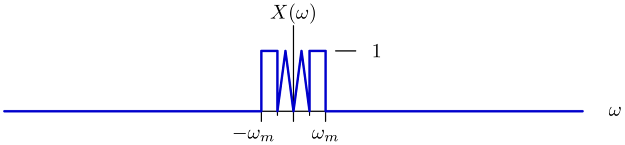
- Shift the left half of $X$ (lower sideband) to the left.
- Shift the right half of $X$ (upper sideband) to the right.



# Amplitude Modulation: SSB-SC

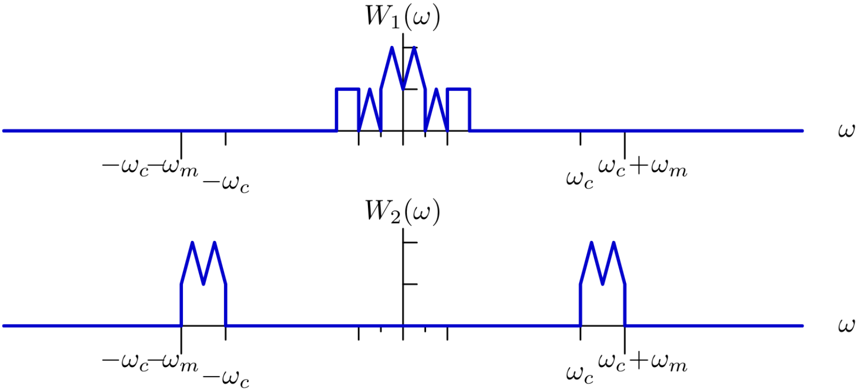



Let $\omega_c \gg \omega_m$. Assume that each low-pass filter (LPF) is ideal, has cut-off frequency $\frac{1}{2}\omega_m$, and has a DC gain of 2. $X(\omega)$ is shown below. Sketch $Y(\omega)$.



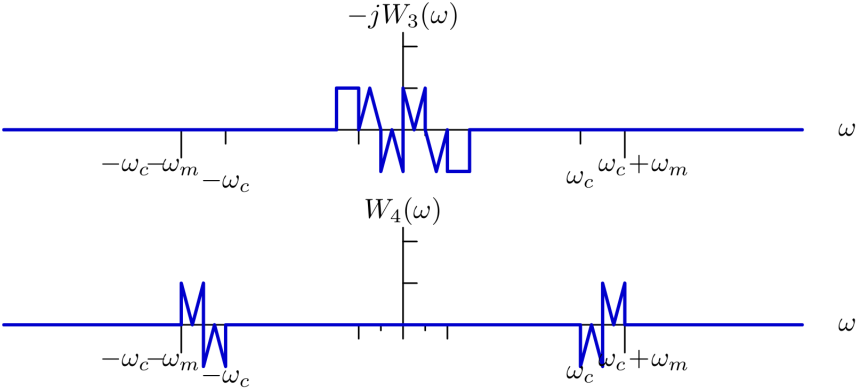
# Amplitude Modulation: SSB-SC

First, consider the top branch. Let $W_1$ denote the signal that results after multiplying $X$ by $\cos\left(\frac{1}{2}\omega_m t\right)$, and let $W_2$ denote the signal that results after low-pass filtering and multiplying $W_1$ by $\cos\left((\omega_c + \frac{1}{2}\omega_m)t\right)$.



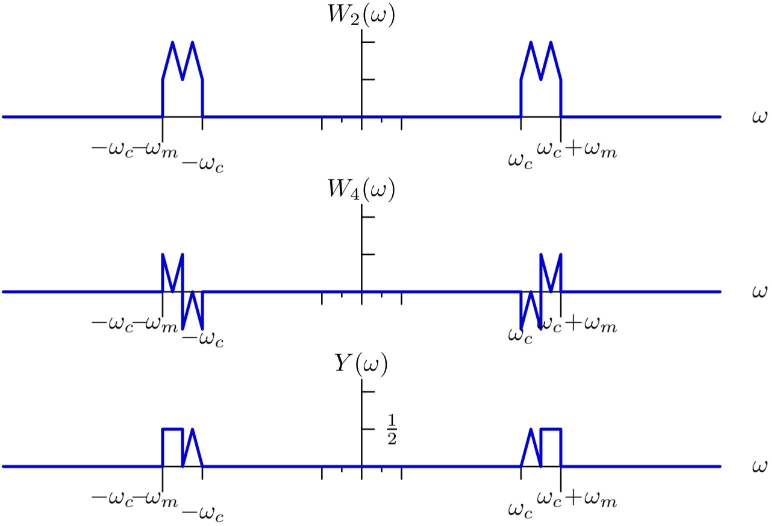
# Amplitude Modulation: SSB-SC

Next, consider the bottom branch. Let $W_3$ denote the signal that results after multiplying $X$ by $\sin\left(\frac{1}{2}\omega_m t\right)$, and let $W_4$ denote the signal that results after low-pass filtering and multiplying $W_3$ by $\sin\left((\omega_c + \frac{1}{2}\omega_m)t\right)$.



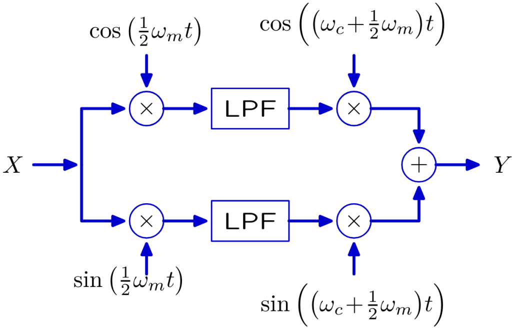
# Amplitude Modulation: SSB-SC

$Y(\omega)$ is the sum of $W_2(\omega)$ and $W_4(\omega)$.



# Amplitude Modulation: SSB-SC

## Single Sideband – Suppressed Carrier (SSB-SC)




**Takeaway:** The SSB-SC scheme conserves bandwidth. It lets us transmit twice as many messages over a given medium as naïve DSB-AM would.

# Sinusoidal Carrier Modulation

Amplitude modulation (AM) is one type of **sinusoidal carrier modulation**. There are other types, too.

Consider a sinusoidal carrier $s(t) = A\cos(\omega_c t - \phi)$.

- **amplitude modulation:** time-varying $A = A(t)$
- **frequency modulation:** time-varying $\omega_c = \omega_c(t)$
- **phase modulation:** time-varying $\phi = \phi(t)$

# Agenda for Recitation

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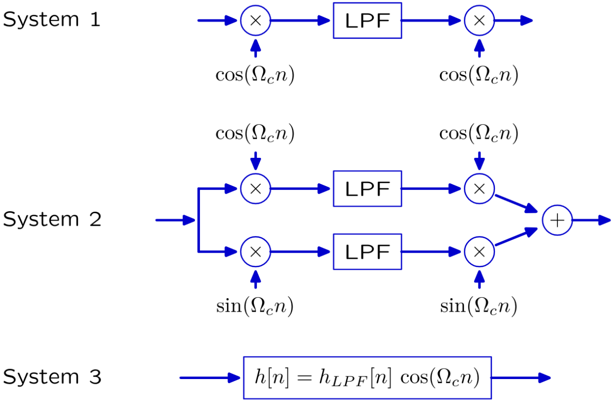
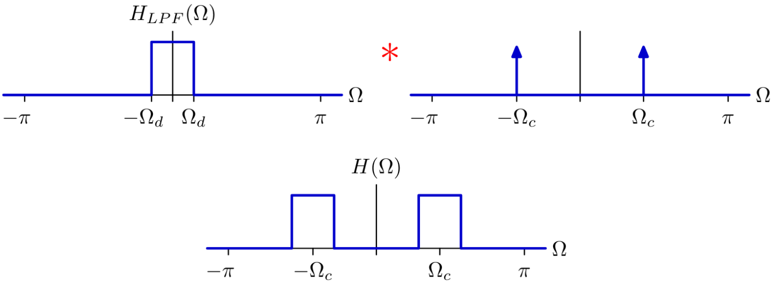
# Band-Pass Filter (BPF)

Band-pass filters play a role in many communication systems. Which of the following systems implement a band-pass filter?



# Band-Pass Filter (BPF)

Consider System #3. Since the unit-sample response $h[n]$ is the product of $h_{\text{LPF}}[n]$ and $\cos(\Omega_c n)$, the frequency response $H(\Omega)$ is the convolution of the Fourier transforms of $h_{\text{LPF}}[n]$ and $\cos(\Omega_c n)$.




The result is a band-pass filter. (Remember: All discrete-time Fourier transforms are periodic in $\Omega = 2\pi$.)

# Band-Pass Filter (BPF)

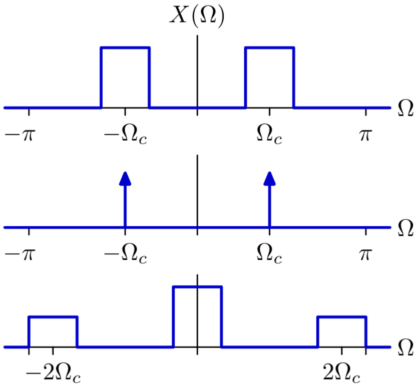
System #2 and System #3 are equivalent.

$$\begin{aligned} y[n] &= (x * h)[n] \\ &= \sum_m x[m] h_{\mathsf{LPF}}[n-m] \cos(\Omega_c (n-m)) \\ &= \sum_m x[m] h_{\mathsf{LPF}}[n-m] \big( \cos(\Omega_c n) \cos(\Omega_c m) + \sin(\Omega_c n) \sin(\Omega_c m) \big) \\ &= \left( \sum_m x[m] \cos(\Omega_c m) h_{\mathsf{LPF}}[n-m] \right) \cos(\Omega_c n) \\ &+ \left( \sum_m x[m] \sin(\Omega_c m) h_{\mathsf{LPF}}[n-m] \right) \sin(\Omega_c n) \\ &= \big( x[n] \cos(\Omega_c n) * h_{\mathsf{LPF}}[n] \big) \cos(\Omega_c n) \\ &+ \big( x[n] \sin(\Omega_c n) * h_{\mathsf{LPF}}[n] \big) \sin(\Omega_c n) \end{aligned}$$

System #3 is a band-pass filter, so System #2 is, too.

# Band-Pass Filter (BPF)

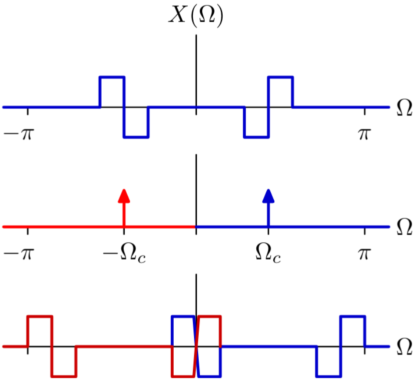
How about System #1? Do we really need both the cosine and sine paths in System #2? Consider this $X(\Omega)$.




Multiply by $\cos(\Omega_c n)$ and low-pass filter. Here, System #1 works like a band-pass filter. Is there a counterexample?

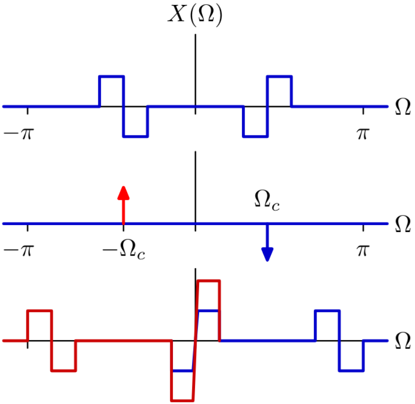
# Band-Pass Filter (BPF)

Yes, there is a counterexample.




When we multiply by $\cos(\Omega_c n)$, the red and blue spectra cancel out! A low-pass filter won't recover the signal about $\omega = 0$. This doesn't work like a band-pass filter.

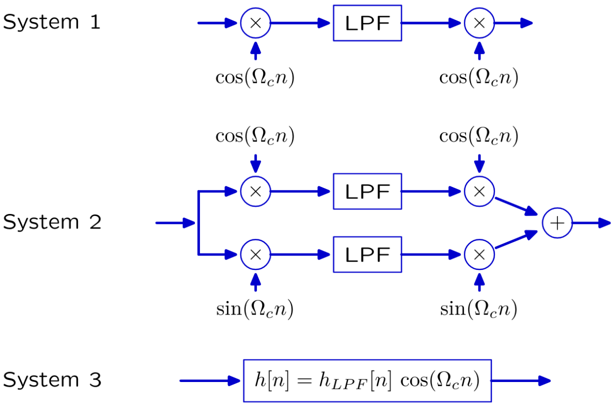
# Band-Pass Filter (BPF)




The cosine path preserves the symmetric part of $X(\Omega)$, while the sine path preserves the anti-symmetric part of $X(\Omega)$. Both are needed for a band-pass filter.

# Band-Pass Filter (BPF)

Band-pass filters play a role in many communication systems. Which of the following systems implement a band-pass filter? **System #2 and System #3**



# Lessons Learned

**Frequency Shift Property:** $x(t)e^{j\omega_0 t} \iff X(\omega - \omega_0)$

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# Question of the Day

Where does the term **AM radio** come from?