6.300: Signal Processing

Circular Convolution and Impulse Trains

Circular convolution: $\frac{1}{N}(f \circledast g)[n] \iff F[k]G[k]$

- Compute the usual convolution $(f * g)[n]$ to start.
- Wrap $(f * g)[n]$ into $[0, N-1]$ and scale by $1/N$.

Impulse trains: The Fourier transform of an impulse train is another impulse train!

$$f[n] = \sum_m \delta[n - mL] \iff F[k] = \frac{1}{L}\sum_m \delta\left[k - m\frac{N}{L}\right]$$

$$f(t) = \sum_m \delta(t - mT) \iff F(\omega) = \frac{2\pi}{T}\sum_m \delta\left(\omega - m\frac{2\pi}{T}\right)$$

April 2, 2026

Agenda for Recitation

- Circular convolution
- Impulse trains

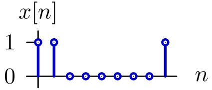
What questions do you have from lecture?

Agenda for Recitation

- Circular convolution
- Impulse trains

Convolution: Three Ways

The signal $x[n]$, defined below, is zero outside the indicated range.



Consider three ways to calculate the convolution of $x[n]$ with itself.

1. direct convolution:

$$y_1[n] = (x * x)[n] = \sum_{m=-\infty}^{\infty} x[m]x[n-m]$$

2. using DTFTs:

$$y_2[n] = \frac{1}{2\pi}\int_{2\pi} X^2(\Omega)e^{j\Omega n}\,d\Omega \quad \text{where} \quad X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

3. using DFTs of length $N=16$:

$$y_3[n] = 16\sum_{k=0}^{15} X^2[k]e^{j\frac{2\pi k}{16}n} \quad \text{where} \quad X[k] = \frac{1}{16}\sum_{n=0}^{15} x[n]e^{-j\frac{2\pi k}{16}n}$$

Convolution: Three Ways

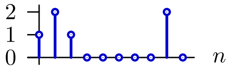
The plots on the right show the **first ten samples** of five signals.
Match the signals on the left with the corresponding plots on the right.

$y_1 = (x * x)$ ☐

$y_2 = \mathrm{DTFT}^{-1}\big(X^2(\Omega)\big)$ ☐

$y_3 = N \times \mathrm{DFT}^{-1}\big(X^2[k]\big)$ ☐

A



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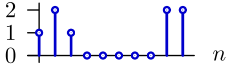

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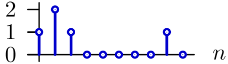

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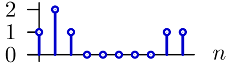

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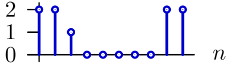
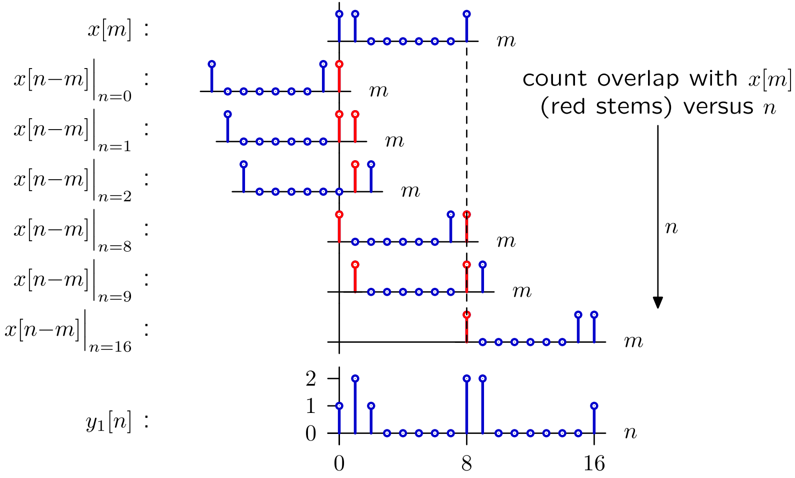


Convolution: Three Ways

Calculate $(x*x)[n]$ by direct convolution: flip and shift.

$$y_1[n] = (x*x)[n] = \sum_{m=-\infty}^{\infty} x[m]x[n-m]$$

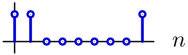


Convolution: Three Ways

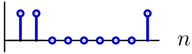
Calculate $(x*x)[n]$ by direct convolution: superposition.

$$y_1[n] = (x*x)[n] = \sum_{m=-\infty}^{\infty} x[m]x[n-m]$$

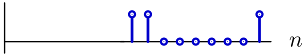
$x[0] \times x[n-0]$:



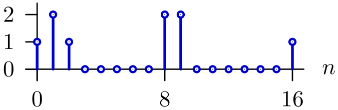
$x[1] \times x[n-1]$:



$x[8] \times x[n-8]$:



$y_1[n]$:



Note: Superposition and flip-and-shift are equivalent methods.
They always give the same answer.

Convolution: Three Ways

Plots on the left show the **first ten samples** of five signals.

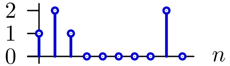
Match signals on the left with corresponding samples on the right.

$y_1 = (x * x)$ [B]

$y_2 = \mathrm{DTFT}^{-1}\big(X^2(\Omega)\big)$ [ ]

$y_3 = N \times \mathrm{DFT}^{-1}\big(X^2[k]\big)$ [ ]

A



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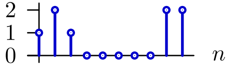

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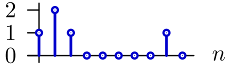

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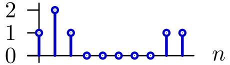

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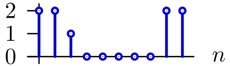
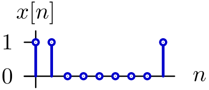


Convolution: Three Ways

Calculate $(x*x)[n]$ using DTFTs.



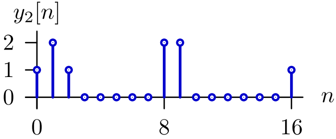
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = 1 + e^{-j\Omega} + e^{-j8\Omega}$$

$$X^2(\Omega) = \left(1 + e^{-j\Omega} + e^{-j8\Omega}\right)^2 = 1+2e^{-j\Omega}+e^{-j2\Omega}+2e^{-j8\Omega}+2e^{-j9\Omega}+e^{-j16\Omega}$$

$$y_2[n] = \frac{1}{2\pi}\int_{2\pi} X^2(\Omega)e^{j\Omega n}\,d\Omega$$

$$= \frac{1}{2\pi}\int_{2\pi} \left(1+2e^{-j\Omega}+e^{-j2\Omega}+2e^{-j8\Omega}+2e^{-j9\Omega}+e^{-j16\Omega}\right)e^{j\Omega n}\,d\Omega$$

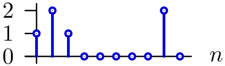
$$= \delta[n] + 2\delta[n-1] + \delta[n-2] + 2\delta[n-8] + 2\delta[n-9] + \delta[n-16]$$



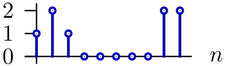
Multiplying DTFTs is always equivalent to direct convolution.

Convolution: Three Ways

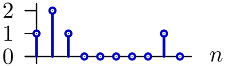
Plots on the left show the **first ten samples** of five signals.
Match signals on the left with corresponding samples on the right.

A


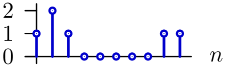
$y_1 = (x * x)$ B

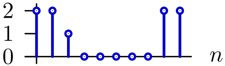
B


$y_2 = \mathrm{DTFT}^{-1}\big(X^2(\Omega)\big)$ B

C


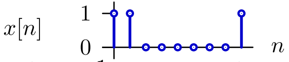
$y_3 = N \times \mathrm{DFT}^{-1}\big(X^2[k]\big)$ ☐

D


E


Convolution: Three Ways

Calculate $(x*x)[n]$ using DFTs ($N = 16$).



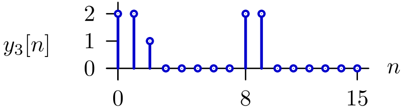
$$X[k] = \frac{1}{16}\sum_{n=0}^{15} x[n]e^{-j\frac{2\pi k}{16}n} = \frac{1}{16}\left(1 + e^{-j\frac{2\pi k}{16}} + e^{-j\frac{2\pi 8k}{16}}\right)$$

$$X^2[k] = \frac{1}{256}\left(1+2e^{-j\frac{2\pi k}{16}}+e^{-j2\frac{2\pi k}{16}}+2e^{-j8\frac{2\pi k}{16}}+2e^{-j9\frac{2\pi k}{16}}+\underbrace{e^{-j16\frac{2\pi k}{16}}}_{=1}\right)$$

$$y_3[n] = 16\sum_{k=0}^{15} X^2[k]e^{j\frac{2\pi k}{16}n}$$

$$= \frac{16}{256}\sum_{k=0}^{15}\left(2+2e^{-j\frac{2\pi k}{16}}+e^{-j2\frac{2\pi k}{16}}+2e^{-j8\frac{2\pi k}{16}}+2e^{-j9\frac{2\pi k}{16}}\right)e^{j\frac{2\pi k}{16}n}$$

$$= 2\delta[n] + 2\delta[n-1] + \delta[n-2] + 2\delta[n-8] + 2\delta[n-9]$$



Since N=16, the sample at n=16 in direct convolution **aliases** to n=0.

Convolution: Three Ways

Plots on the left show the **first ten samples** of five signals.

Match signals on the left with corresponding samples on the right.

$y_1 = (x * x)$	B
$y_2 = \mathrm{DTFT}^{-1}\big(X^2(\Omega)\big)$	B
$y_3 = N \times \mathrm{DFT}^{-1}\big(X^2[k]\big)$	E

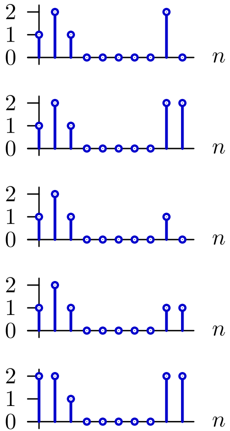
A

B

C

D

E



Circular Convolution

Multiplication of DFTs corresponds to **circular** convolution in time. Assume that $F[k]$ is the product of the DFTs of $f_a[n]$ and $f_b[n]$.

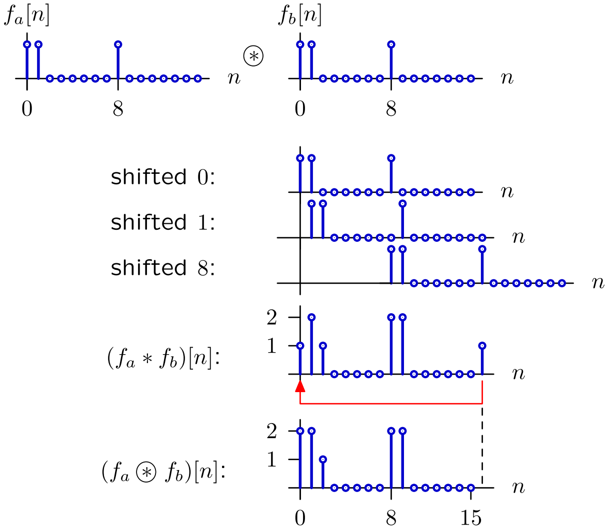
$$\begin{aligned}
f[n] &= \sum_{k=0}^{N-1} F[k] e^{j\frac{2\pi k}{N}n} = \sum_{k=0}^{N-1} F_a[k]F_b[k] e^{j\frac{2\pi k}{N}n} \\
&= \sum_{k=0}^{N-1} F_a[k] \left(\frac{1}{N} \sum_{m=0}^{N-1} f_b[m] e^{-j\frac{2\pi k}{N}m} \right) e^{j\frac{2\pi k}{N}n} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] \sum_{k=0}^{N-1} F_a[k] e^{j\frac{2\pi k}{N}(n-m)} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] f_{ap}[n-m]
\end{aligned}$$

where $f_{ap}[n] = f_a[n \bmod N]$ is a periodically extended version of $f_a[n]$.
We refer to this as **circular** or **periodic** convolution:

$$\frac{1}{N}(f_a \circledast f_b)[n] \quad \overset{\text{DFT}}{\Longrightarrow} \quad F_a[k] \times F_b[k]$$

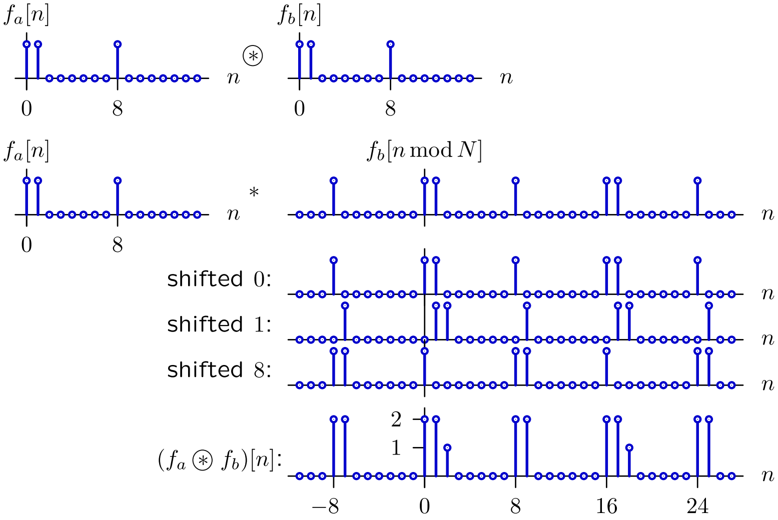
Circular Convolution

Circular convolution is equivalent to conventional convolution followed by periodic summation of results back into base period.



Circular Convolution

Circular convolution of two signals is equal to conventional convolution of one signal with a periodically extended version of the other.



Agenda for Recitation

- Circular convolution
- Impulse trains

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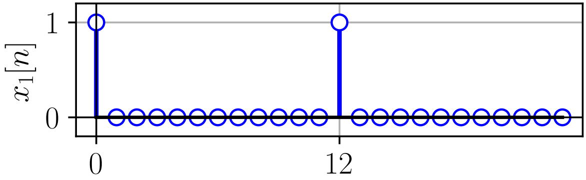
- Circular convolution
- Impulse trains

Impulse Trains: DFT



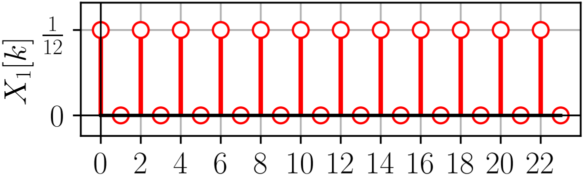
Let $N = 24$. Determine $X_1[k]$, the DFT of $x_1[n]$.

Impulse Trains: DFT

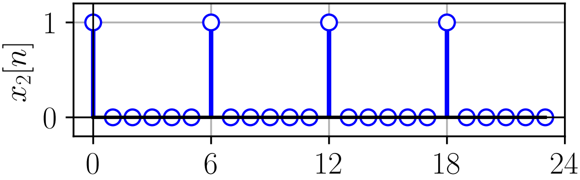
$$X_1[k] = \frac{1}{24}\sum_{n=0}^{23} x_1[n]e^{-jk\frac{2\pi}{24}n} = \frac{1}{24}e^{-jk\frac{2\pi}{24}0} + \frac{1}{24}e^{-jk\frac{2\pi}{24}12}$$

$$= \frac{1+(-1)^k}{24} = \begin{cases} \frac{1}{12} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

The fundamental period of $x_1[n]$ is 12 samples, while the fundamental period of $X_1[k]$ is $24/12 = 2$ samples.



Impulse Trains: DFT

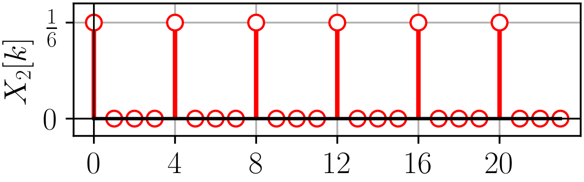


Let $N = 24$. Determine $X_2[k]$, the DFT of $x_2[n]$.

Impulse Trains: DFT

$$
\begin{aligned}
X_2[k] &= \frac{1}{24}\sum_{n=0}^{23} x_2[n] e^{-jk\frac{2\pi}{24}n} \\
&= \frac{1}{24}e^{-jk\frac{2\pi}{24}0} + \frac{1}{24}e^{-jk\frac{2\pi}{24}6} + \frac{1}{24}e^{-jk\frac{2\pi}{24}12} + \frac{1}{24}e^{-jk\frac{2\pi}{24}18} \\
&= \frac{1+(-j)^k+(-1)^k+j^k}{24} = \begin{cases} \frac{1}{6} & k \bmod 4 = 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

The fundamental period of $x_2[n]$ is 6 samples, while the fundamental period of $X_2[k]$ is $24/6 = 4$ samples.



Impulse Trains: DFT

Suppose that N is an integer multiple of L. Using an analysis window of length N, determine the DFT of

$$f[n] = \sum_{m} \delta[n - mL].$$

Hint: Generalize the work we've done so far.

Impulse Trains: DFT

Suppose that N is an integer multiple of L. Using an analysis window of length N, determine the DFT of

$$f[n] = \sum_m \delta[n - mL].$$

Hint: Generalize the work we've done so far.

$$F[k] = \frac{1}{L} \sum_m \delta\left[n - m\tfrac{N}{L}\right]$$

The fundamental period of $f[n]$ is L samples, while the fundamental period of $F[k]$ is N/L samples. Notice that the fundamental periods in time and frequency are inversely proportional: A "stretch" in one domain corresponds to a "compression" in the other domain.

Impulse Trains: CTFT

Let $T > 0$. Determine $F(\omega)$, the Fourier transform of

$$f(t) = \sum_m \delta(t - mT).$$

Hint: $f(t)$ is periodic.

Impulse Trains: CTFT

Let $T > 0$. Determine $F(\omega)$, the Fourier transform of

$$f(t) = \sum_m \delta(t - mT).$$

Hint: $f(t)$ is periodic.

Start by computing the Fourier series coefficients $F[k]$.
Relate the Fourier series to the Fourier transform $F(\omega)$.

$$F[k] = \frac{1}{T}\int_T \delta(t)dt = \frac{1}{T} \text{ (for all } k\text{)}$$

$$F(\omega) = \sum_k 2\pi F[k]\delta(\omega - k\omega_0) = \frac{2\pi}{T}\sum_k \delta\left(\omega - k\tfrac{2\pi}{T}\right)$$

The period in time is T. The period in frequency is $2\pi/T$.

Impulse Trains: DTFT

Let $N > 0$. Determine $F(\Omega)$, the Fourier transform of

$$f[n] = \sum_m \delta[n - mN].$$

Hint: $f[n]$ is periodic.

Impulse Trains: DTFT

Let $N > 0$. Determine $F(\Omega)$, the Fourier transform of

$$f[n] = \sum_m \delta[n - mN].$$

Hint: $f[n]$ is periodic.

Start by computing the Fourier series coefficients $F[k]$.
Relate the Fourier series to the Fourier transform $F(\Omega)$.

$$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} \delta[n] e^{-j\Omega n} = \frac{1}{N} \text{ (for all } k\text{)}$$

$$F(\Omega) = \sum_k 2\pi F[k] \delta(\Omega - k\Omega_0) = \frac{2\pi}{N} \sum_k \delta\big(\Omega - k\tfrac{2\pi}{N}\big)$$

The period in time is N. The period in frequency is $2\pi/N$.

Lessons Learned

Multiplication of DFTs corresponds to circular convolution in the time domain: $F[k]G[k] \iff \frac{1}{N}(f \circledast g)[n]$.

Circular convolution: $\frac{1}{N}(f \circledast g)[n] \iff F[k]G[k]$

- Compute the usual convolution $(f * g)[n]$ to start.
- Wrap $(f * g)[n]$ into $[0, N-1]$ and scale by $1/N$.

Impulse trains: The Fourier transform of an impulse train is another impulse train!

$$f[n] = \sum_m \delta[n - mL] \iff F[k] = \frac{1}{L}\sum_m \delta\left[k - m\frac{N}{L}\right]$$

$$f(t) = \sum_m \delta(t - mT) \iff F(\omega) = \frac{2\pi}{T}\sum_m \delta\left(\omega - m\frac{2\pi}{T}\right)$$

Question of the Day

Suppose that $f[n]$ is non-zero only over $0 \leq n \leq L-1$, and suppose $g[n]$ is non-zero only over $0 \leq n \leq M-1$. Now, zero-pad both $f[n]$ and $g[n]$ to a length $N \geq L, M$. Determine the smallest N such that $(f * g)[n] = (f \circledast g)[n]$ for $0 \leq n \leq N-1$.

Takeaway: You can **zero-pad** signals to a length $\geq N$ to prevent circular-convolution artifacts, i.e., "time-aliasing."