6.300: Signal Processing

Circular Convolution and Impulse Trains

Circular convolution: $\frac{1}{N}(f \circledast g)[n] \iff F[k]G[k]$

- Compute the usual convolution $(f * g)[n]$ to start.
- Wrap $(f * g)[n]$ into $[0, N-1]$ and scale by $1/N$.

Impulse trains: The Fourier transform of an impulse train is another impulse train!

$$f[n] = \sum_m \delta[n - mL] \iff F[k] = \frac{1}{L}\sum_m \delta\left[k - m\frac{N}{L}\right]$$

$$f(t) = \sum_m \delta(t - mT) \iff F(\omega) = \frac{2\pi}{T}\sum_m \delta\left(\omega - m\frac{2\pi}{T}\right)$$

April 2, 2026

Agenda for Recitation

- Circular convolution
- Impulse trains

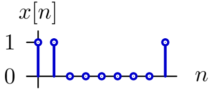
What questions do you have from lecture?

Agenda for Recitation

- Circular convolution
- Impulse trains

Convolution: Three Ways

The signal $x[n]$, defined below, is zero outside the indicated range.



Consider three ways to calculate the convolution of $x[n]$ with itself.

1. direct convolution:

$$y_1[n] = (x * x)[n] = \sum_{m=-\infty}^{\infty} x[m]x[n-m]$$

2. using DTFTs:

$$y_2[n] = \frac{1}{2\pi} \int_{2\pi} X^2(\Omega) e^{j\Omega n}\, d\Omega \quad \text{where} \quad X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

3. using DFTs of length $N=16$:

$$y_3[n] = 16 \sum_{k=0}^{15} X^2[k] e^{j\frac{2\pi k}{16}n} \quad \text{where} \quad X[k] = \frac{1}{16} \sum_{n=0}^{15} x[n] e^{-j\frac{2\pi k}{16}n}$$

Convolution: Three Ways

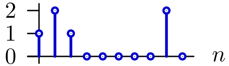
The plots on the right show the **first ten samples** of five signals.
Match the signals on the left with the corresponding plots on the right.

$y_1 = (x * x)$ ☐

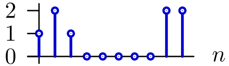
$y_2 = \mathrm{DTFT}^{-1}\big(X^2(\Omega)\big)$ ☐

$y_3 = N \times \mathrm{DFT}^{-1}\big(X^2[k]\big)$ ☐

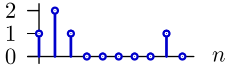
A



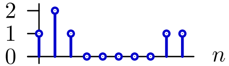
B



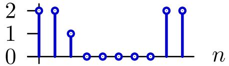
C



D



E



Circular Convolution

Multiplication of DFTs corresponds to **circular** convolution in time. Assume that $F[k]$ is the product of the DFTs of $f_a[n]$ and $f_b[n]$.

$$\begin{aligned} f[n] &= \sum_{k=0}^{N-1} F[k] e^{j\frac{2\pi k}{N}n} = \sum_{k=0}^{N-1} F_a[k] F_b[k] e^{j\frac{2\pi k}{N}n} \\ &= \sum_{k=0}^{N-1} F_a[k] \left(\frac{1}{N} \sum_{m=0}^{N-1} f_b[m] e^{-j\frac{2\pi k}{N}m} \right) e^{j\frac{2\pi k}{N}n} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] \sum_{k=0}^{N-1} F_a[k] e^{j\frac{2\pi k}{N}(n-m)} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] f_{ap}[n-m] \end{aligned}$$

where $f_{ap}[n] = f_a[n \bmod N]$ is a periodically extended version of $f_a[n]$.
We refer to this as **circular** or **periodic** convolution:

$$\frac{1}{N}(f_a \circledast f_b)[n] \quad \stackrel{\text{DFT}}{\Longrightarrow} \quad F_a[k] \times F_b[k]$$

Agenda for Recitation

- Circular convolution
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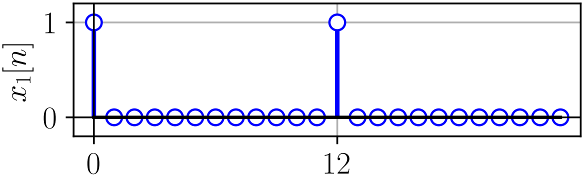
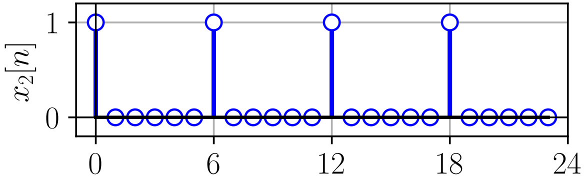
Impulse Trains: DFT



Let $N = 24$. Determine $X_1[k]$, the DFT of $x_1[n]$.

Impulse Trains: DFT



Let $N = 24$. Determine $X_2[k]$, the DFT of $x_2[n]$.

Impulse Trains: DFT

Suppose that N is an integer multiple of L. Using an analysis window of length N, determine the DFT of

$$f[n] = \sum_m \delta[n - mL].$$

Hint: Generalize the work we've done so far.

Impulse Trains: CTFT

Let $T > 0$. Determine $F(\omega)$, the Fourier transform of

$$f(t) = \sum_m \delta(t - mT).$$

Hint: $f(t)$ is periodic.

Impulse Trains: DTFT

Let $N > 0$. Determine $F(\Omega)$, the Fourier transform of

$$f[n] = \sum_m \delta[n - mN].$$

Hint: $f[n]$ is periodic.

Lessons Learned

Multiplication of DFTs corresponds to circular convolution in the time domain: $F[k]G[k] \iff \frac{1}{N}(f \circledast g)[n]$.

Circular convolution: $\frac{1}{N}(f \circledast g)[n] \iff F[k]G[k]$

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