6.3000 Final Exam Spring 2026

Name: Solutions

Kerberos (Athena) Username:

Please WAIT until we tell you to begin.

This exam is closed book, but you may use three 8.5×11 sheets of notes (six sides).

You may NOT use any electronic devices (such as calculators and phones).

If you have questions, please **come to us** at the front of the room to ask.

Please enter all solutions in the boxes provided.

Work on other pages with QR codes will be considered for partial credit.
Please provide a note if you continue work on worksheets at the end of the exam.

Please do not write on the QR codes at the bottom of each page.

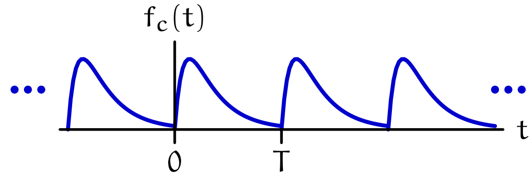
We use those codes to identify which pages belong to each student.

Trigonometric Identities Reference

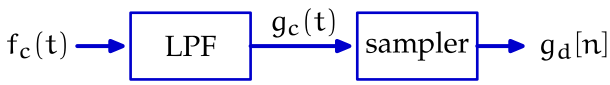
$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$
$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b)$$
$$\cos(a+b)-\cos(a-b) = -2\sin(a)\sin(b)$$
$$\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$$
$$\sin(a+b)-\sin(a-b) = 2\cos(a)\sin(b)$$
$$2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b)$$
$$2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$$
$$2\sin(a)\cos(b) = \sin(a+b)+\sin(a-b)$$
$$2\cos(a)\sin(b) = \sin(a+b)-\sin(a-b)$$

1 Sampling Harmonics (16 points)

Let $f_c(t)$ represent a periodic, continuous-time signal with period $T = 1/440$ seconds, as shown below.



Suppose $f_c(t)$ is the input to an ideal low-pass filter (LPF) with cut-off frequency $\omega_0 = 11\pi/T$ radians per second. The output of the low-pass filter, $g_c(t)$, is subsequently sampled at times $t = n\Delta$, where $\Delta = 1/1000$ seconds. The result is the sequence $g_d[n] = g_c(n\Delta)$.



Let $G_d(\Omega)$ represent the discrete-time Fourier transform (DTFT) of $g_d[n]$. $G_d(\Omega)$ is non-zero only for countably many frequencies $\Omega \in [0, \pi]$. Determine the number of frequencies Ω for which $G_d(\Omega)$ is non-zero, and write your answer in the box below.

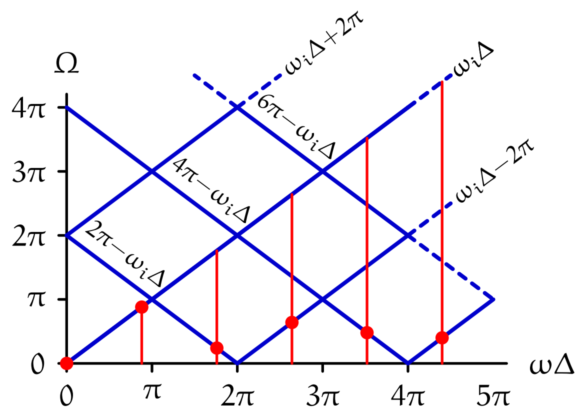
Number of frequencies Ω for which $G_d(\Omega)$ is non-zero: 6

List all these frequencies in the boxes provided below. You may list the frequencies as real numbers or as a fraction of π (such as 0.1π or $\pi/10$), and you may list the frequencies in any order. Use as many boxes as needed. Leave unneeded boxes blank.

0.00π	0.88π	0.24π	0.64π	0.48π	0.40π

Since $f_c(t)$ is periodic in time with period T it comprises only countably many non-negative frequencies: 0, $2\pi/T$, $4\pi/T$, $6\pi/T$, ... Low-pass filtering $f_c(t)$ with cut-off frequency $\omega_o = 11\pi/T$ results in a signal with just 6 non-negative frequencies: $\omega = 0, 2\pi/T, 4\pi/T, 6\pi/T, 8\pi/T$, and $10\pi/T$. These frequencies correspond to sinusoids of the following forms: $\cos(0\pi t/T)$, $\cos(2\pi t/T)$, $\cos(4\pi t/T)$, $\cos(6\pi t/T)$, $\cos(8\pi t/T)$, and $\cos(10\pi t/T)$.

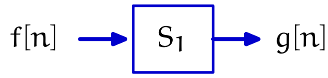
Sampling these sinusoids at $t = n\Delta$ results in the followin sequences of samples: $\cos(0\pi n\Delta/T)$, $\cos(2\pi n\Delta/T)$, $\cos(4\pi n\Delta/T)$, $\cos(6\pi n\Delta/T)$, $\cos(8\pi n\Delta/T)$, and $\cos(10\pi n\Delta/T)$. Substituting $T = 1/440$ and $\Delta = 1/1000$ yields the following discrete sinusoids: $\cos(0.00\pi n)$, $\cos(0.88\pi n)$, $\cos(1.76\pi n)$, $\cos(2.64\pi n)$, $\cos(3.52\pi n)$, and $\cos(4.40\pi n)$ as shown below:



Both $\omega\Delta = 0$ and $\omega\Delta = 0.88\pi$ fall in the base band, so the first two values of Ω are 0 and 0.88π. The next frequency $\omega\Delta = 1.76\pi$ is in the second band. It has an "alias" at $\Omega = 0.24\pi$. The next frequency $\omega\Delta = 2.64\pi$ is in the third band. It has an "alias" at $\Omega = 0.64\pi$. The next frequency $\omega\Delta = 3.52\pi$ is in the fourth band. It has an "alias" at $\Omega = 0.48\pi$. The next frequency $\omega\Delta = 4.40\pi$ is in the fifth band. It has an "alias" at $\Omega = 0.40\pi$.

2 System Identification and Inversion (22 points)

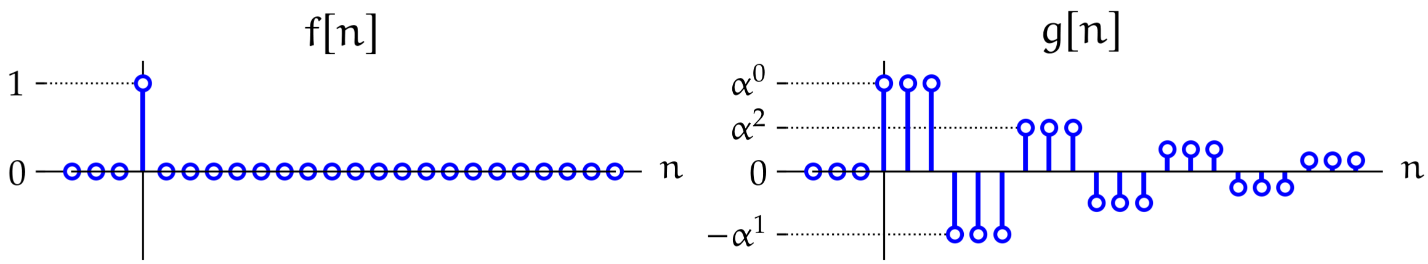
Part a. Let S_1 represent a linear, time-invariant (LTI) system.



Let α denote a real number in the interval $(0, 1)$. When the input is $f[n] = \delta[n]$, the response is

$$g[n] = \begin{cases} 0 & n < 0 \\ \alpha^0 & 0 \leqslant n \leqslant 2 \\ -\alpha^1 & 3 \leqslant n \leqslant 5 \\ \alpha^2 & 6 \leqslant n \leqslant 8 \\ -\alpha^3 & 9 \leqslant n \leqslant 11 \\ \dots & \end{cases}$$

as shown below for $n \in [-3, 20]$.



Determine a linear, constant-coefficient difference equation that relates $f[n]$ and $g[n]$. Assume initial rest conditions. Your answer may include α as well as familiar constants like $\pi \approx 3.14159$ and $e \approx 2.718$.

$$g[n] = f[n] + f[n-1] + f[n-2] - \alpha g[n-3]$$

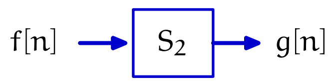
Part b. Let $H_1(\Omega)$ denote the frequency response of system S_1. Determine a closed-form expression for $H_1(\Omega)$. Your answer may include α as well as familiar constants like $\pi \approx 3.14159$ and $e \approx 2.718$.

$$H_1(\Omega) = \frac{G(\Omega)}{F(\Omega)} = \frac{1 + e^{-j\Omega} + e^{-j2\Omega}}{1 + \alpha e^{-j3\Omega}} = e^{-j\Omega}\frac{1 + 2\cos(\Omega)}{1 + \alpha e^{-j3\Omega}}$$

For which values of $\Omega \in [-\pi, \pi]$ (if any) is $H_1(\Omega) = 0$? Write **X** if there are no such values of Ω.

$$\Omega = \pm 2\pi/3$$

Part c. Let S_2 represent a linear, time-invariant (LTI) system.



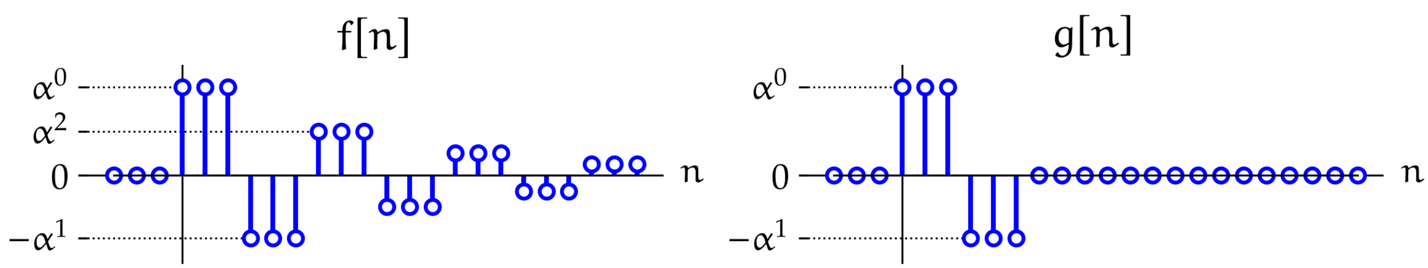
Let α denote a real number in the interval $(0, 1)$. When the input is

$$f[n] = \begin{cases} 0 & n < 0 \\ \alpha^0 & 0 \leqslant n \leqslant 2 \\ -\alpha^1 & 3 \leqslant n \leqslant 5 \\ \alpha^2 & 6 \leqslant n \leqslant 8 \\ -\alpha^3 & 9 \leqslant n \leqslant 11 \\ \ldots & \end{cases}$$

the response is

$$g[n] = \begin{cases} 0 & n < 0 \\ \alpha^0 & 0 \leqslant n \leqslant 2 \\ -\alpha^1 & 3 \leqslant n \leqslant 5 \\ 0 & 6 \leqslant n \end{cases}$$

as shown below for $n \in [-3, 20]$.



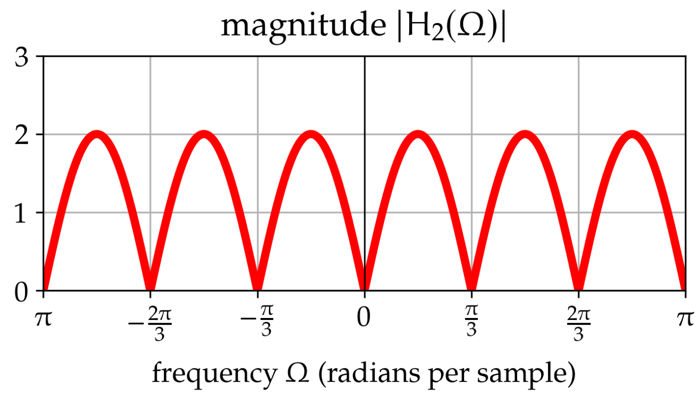
Let $h_2[n]$ denote the unit-sample response of system S_2. Determine a closed-form expression for $h_2[n]$. Your answer may include α as well as familiar constants like $\pi \approx 3.14159$ and $e \approx 2.718$.

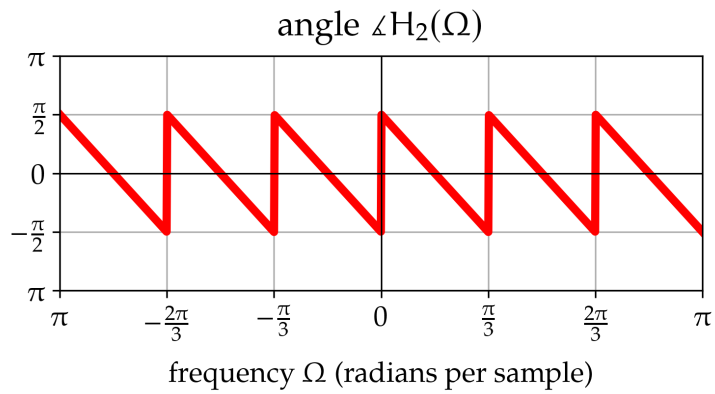
$$h_2[n] = \delta[n] - \alpha^2 \delta[n - 6]$$

Part d. Let $H_2(\Omega)$ denote the frequency response of system S_2. Determine a closed-form expression for $H_2(\Omega)$. Your answer may include α as well as familiar constants like $\pi \approx 3.14159$ and $e \approx 2.718$.

$$H_2(\Omega) = 1 - \alpha^2 e^{-j6\Omega}$$

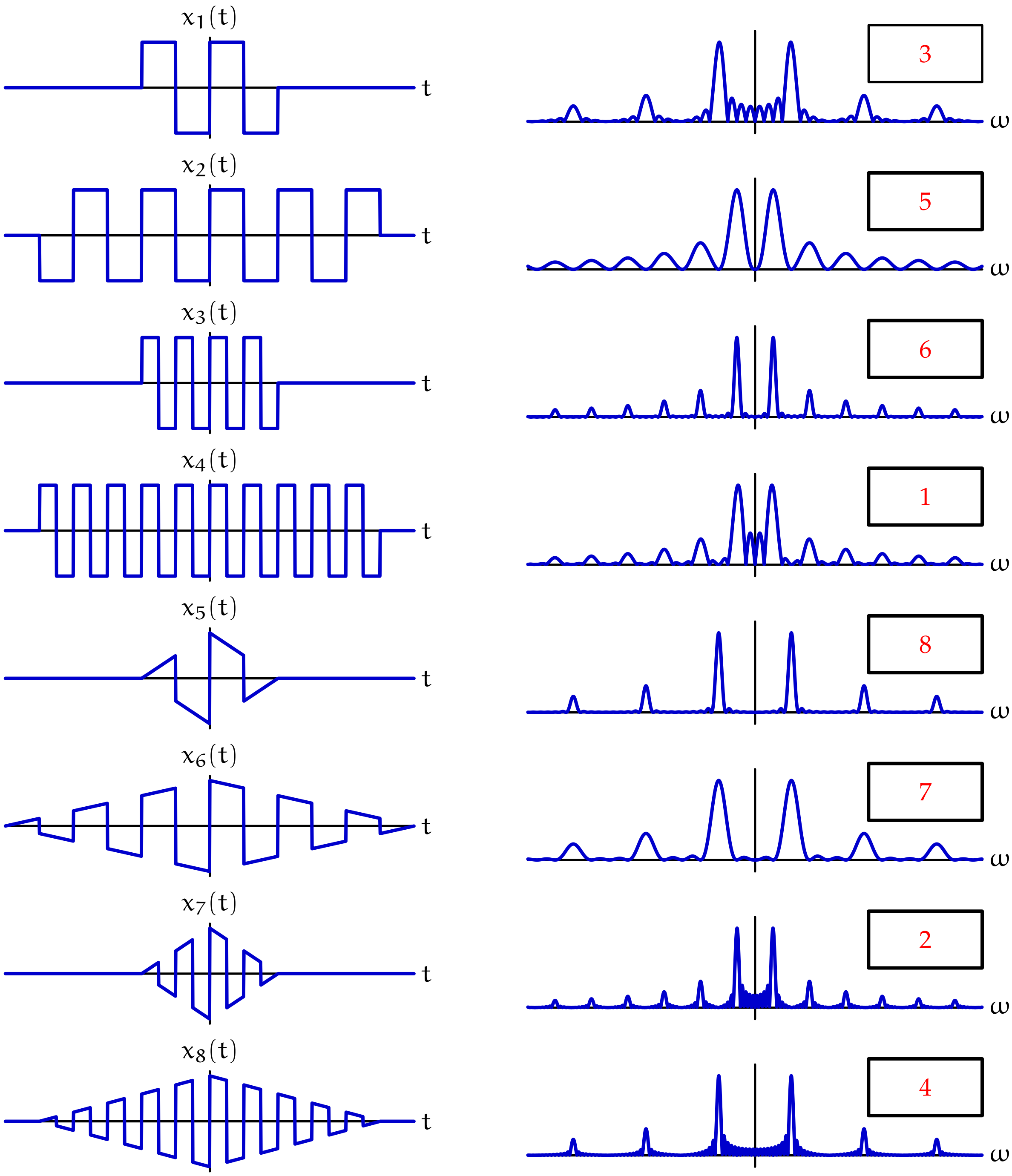
Sketch $|H_2(\Omega)|$ and $\measuredangle H_2(\Omega)$ on the axes below for $\alpha = 1$.





3 Time-Frequency Patterns (16 points)

Eight time-domain signals are shown below at left. Eight Fourier transform magnitudes are shown below at right. For each transform shown on the right, determine the corresponding time-domain signal on the left and enter its number in the box provided. Each time-domain signal is plotted on the same time scale, and each Fourier transform is plotted on the same frequency scale. Time-domain signals are zero outside the range shown.



The time-domain signals have three important parameters:

- Period: The periods of x_1, x_2, x_5, and x_6 are twice as long as those of the others.
- Shape: The envelope of the signal is either triangular or square, and can be thought of as multiplying an underlying periodic time-domain signal.
- Overall length: x_1, x_3, x_5, and x_7 are short. The others are long.

These parameters affect the magnitudes of the Fourier transforms in distinct ways. Let A, B, C, . . . , H represent the waveforms in the right column.

- The period in time is inversely related to the period in frequency. Thus A, E, F, and H (which have longer periods in frequency) correspond to x_3, x_4, x_7, and x_8.
- Since the shape multiplies the time waveform, it convolves with the frequency waveform. The shape is long compared to the periodicity of the time waveform, therefore the convolution affects the short timescale (i.e., between the periods). The square (in time) has more high frequencies than the triangle, so the square in frequency has larger overshoot. Thus A, D, G, and H correspond to squares (x_1, x_2, x_3, and x_4), and the others correspond to triangles.
- The overall length is long compared to the periodicity of the time waveform, therefore the convolution affects the short timescale (i.e., between the periods). The longer the shape, the shorter the spread around each lobe in the frequency domain. Therefore, the broad lobes (A, B, D, and F) correspond to the short overall lengths (x_1, x_3, x_5, and x_7).

The answer provided on the previous page is the only combination that satisfies all three of these constraints.

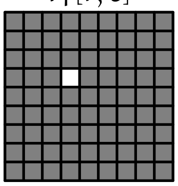
4 Two-Dimensional Patterns (16 points)

Eight two-dimensional, discrete-space signals ($f_1[r,c]$, $f_2[r,c]$,...,$f_8[r,c]$) are shown below, where pixels are indexed by row and column numbers (r and c) that are each in the range $[-4,4]$. White and black pixels represent values of 1 and -1 respectively, and grey pixels represent values of 0.

Determine which of the magnitude plots on the next page (A,B,...,H) shows the magnitude of the two-dimensional discrete Fourier transform of each of the discrete-space signals,a and record your answers by entering the appropriate labels in the corresponding boxes below.

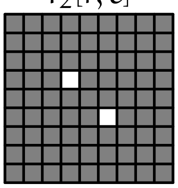
Similarly, determine which of the phase plots on the next page (a,b,...,h) shows the phase of the two-dimensional discrete Fourier transform of each of the discrete-space signals, and record your answers by entering the appropriate labels in the corresponding boxes below.

$f_1[r,c]$



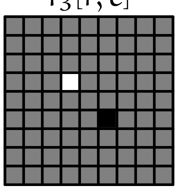
$|F_1[k_r,k_c]|$: B $\angle F_1[k_r,k_c]$: e

$f_2[r,c]$



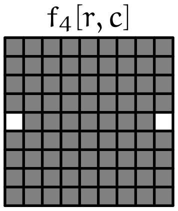
$|F_2[k_r,k_c]|$: D $\angle F_2[k_r,k_c]$: c

$f_3[r,c]$



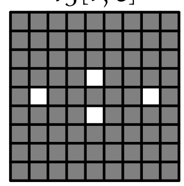
$|F_3[k_r,k_c]|$: C $\angle F_3[k_r,k_c]$: g

$f_4[r,c]$



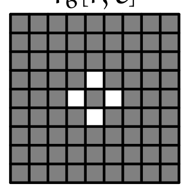
$|F_4[k_r,k_c]|$: E $\angle F_4[k_r,k_c]$: b

$f_5[r,c]$



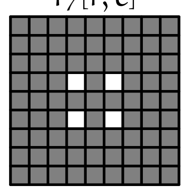
$|F_5[k_r,k_c]|$: H $\angle F_5[k_r,k_c]$: h

$f_6[r,c]$



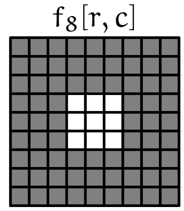
$|F_6[k_r,k_c]|$: F $\angle F_6[k_r,k_c]$: f

$f_7[r,c]$



$|F_7[k_r,k_c]|$: G $\angle F_7[k_r,k_c]$: a

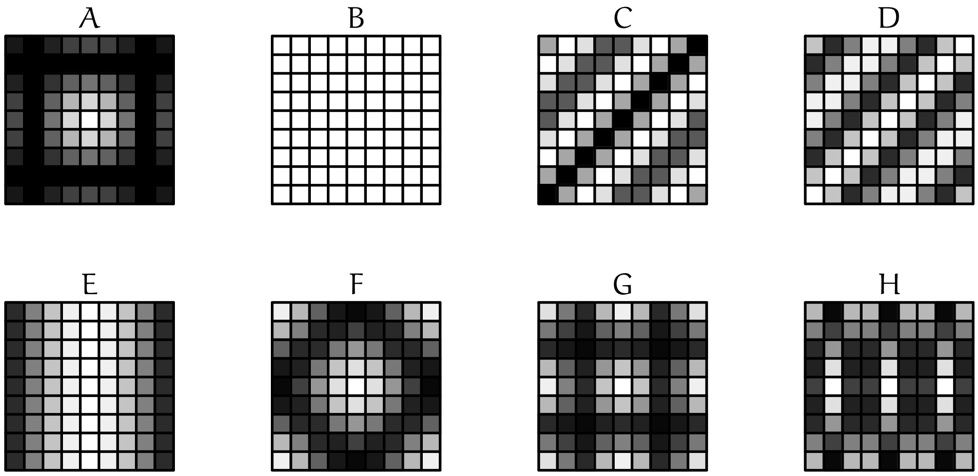
$f_8[r,c]$



$|F_8[k_r,k_c]|$: A $\angle F_8[k_r,k_c]$: d

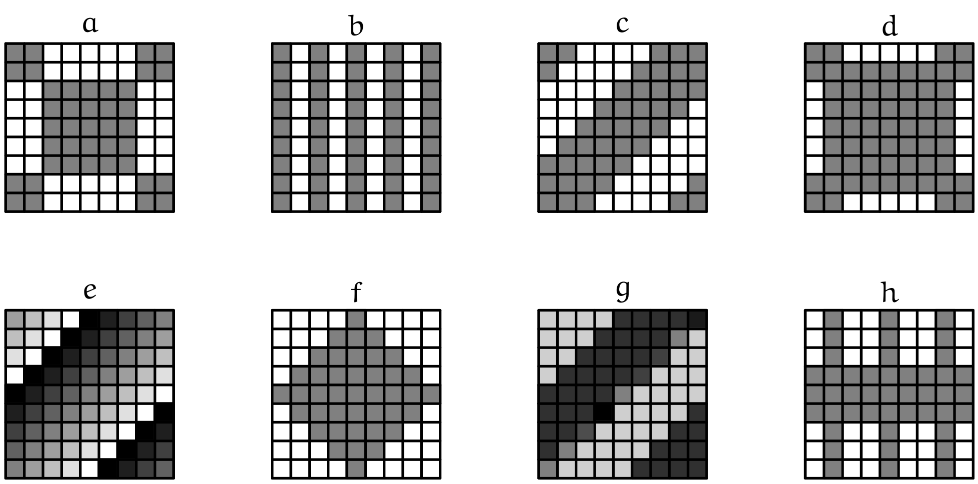
Magnitude plots

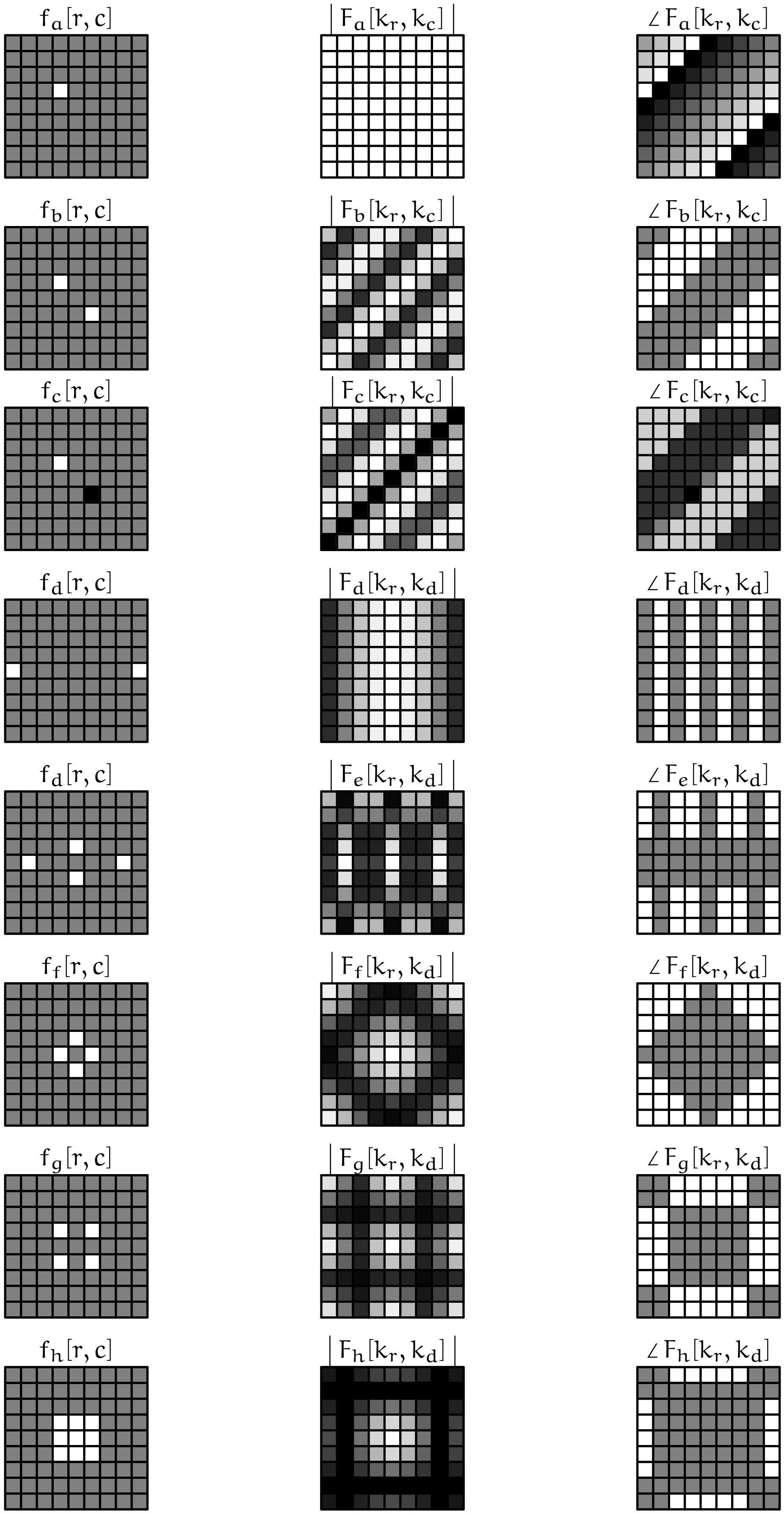
grayscale: from 0 (black) to the maximum magnitude in each image (white)



Phase plots

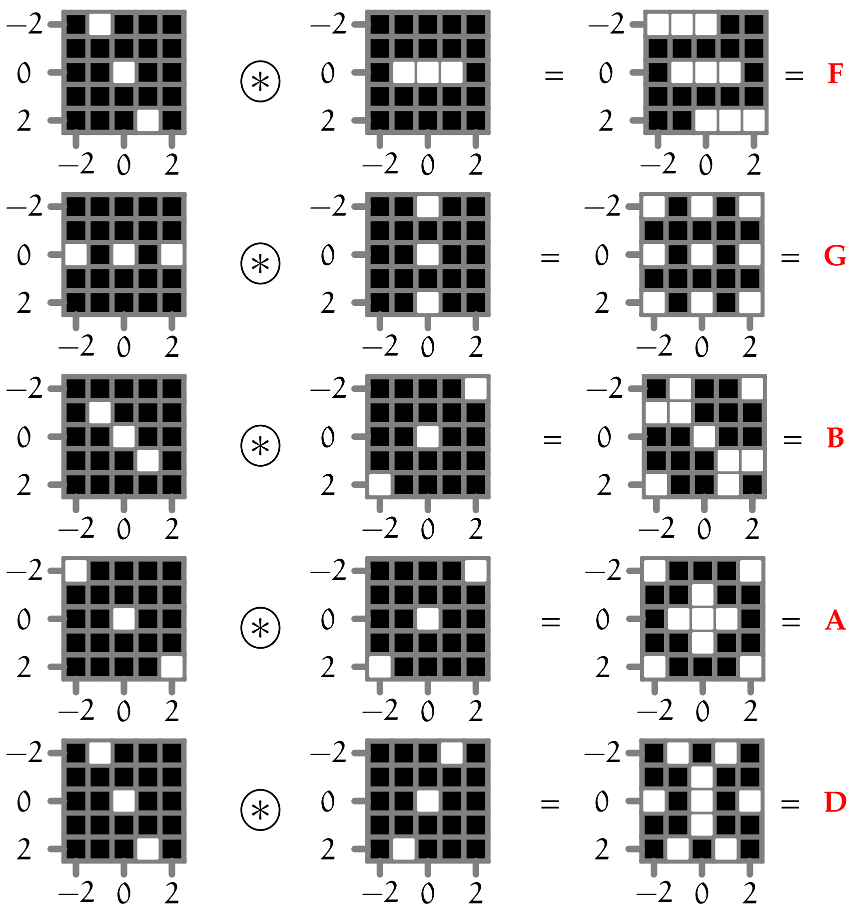
grayscale: from $-\pi$ (black) to π (white)



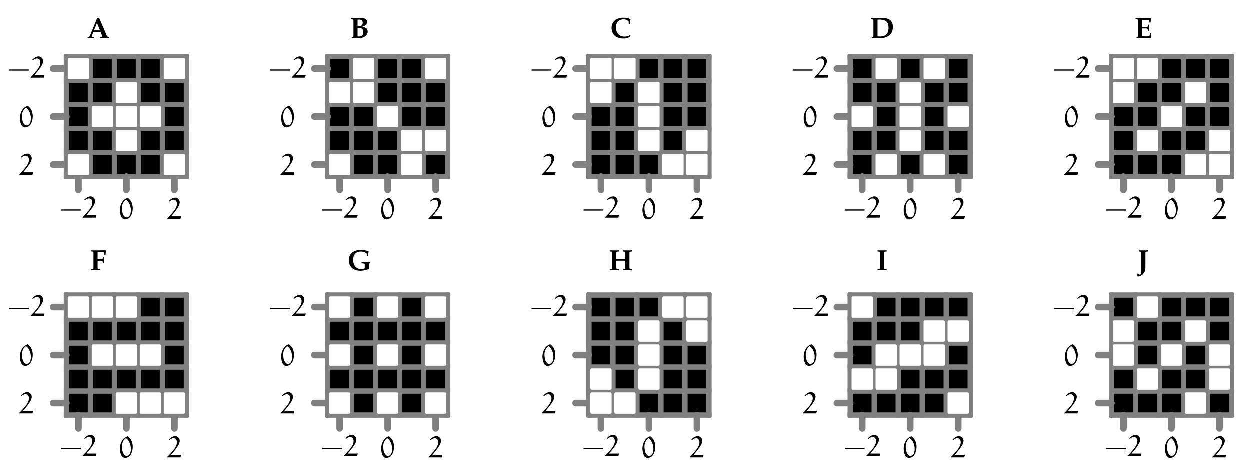

f_a[r, c]
|F_a[k_r, k_c]|
∠F_a[k_r, k_c]
f_b[r, c]
|F_b[k_r, k_c]|
∠F_b[k_r, k_c]
f_c[r, c]
|F_c[k_r, k_c]|
∠F_c[k_r, k_c]
f_d[r, c]
|F_d[k_r, k_d]|
∠F_d[k_r, k_d]
f_d[r, c]
|F_e[k_r, k_d]|
∠F_e[k_r, k_d]
f_f[r, c]
|F_f[k_r, k_d]|
∠F_f[k_r, k_d]
f_g[r, c]
|F_g[k_r, k_d]|
∠F_g[k_r, k_d]
f_h[r, c]
|F_h[k_r, k_d]|
∠F_h[k_r, k_d]

5 Convolving Signals (12 points)

Part a. Two 5×5 pixel images are circularly convolved (using 5×5 DFTs and a 5×5 inverse DFT) to produce a new 5×5 image, where black pixels represent values of 0 and white pixels represent values of 1.



Identify which of the following images (A–J) results for each of the previous circular convolutions, and enter that image label (A–J) in the corresponding box above. If none of the images match, enter **X**.



Part a. Convolving the left and right images produces 9 white dots. Each of those 9 white dots results from the convolution of one white dot in the left image with one white dot in the right image. and the (r, c) position of the resulting dot is the sum of the (r, c) positions of one dot from the left image with the (r, c) positions of one dot from the right image.

Part a1.

$$
\begin{array}{ccccc}
(-2,-1) & \circledast & (0,-1) & = & (0,-1) \\
(0,0) & \circledast & (0,0) & = & (0,0) \\
(2,1) & \circledast & (0,1) & = & (0,1) \\
(-2,-1) & \circledast & (0,-1) & = & (0,-1) \\
(0,0) & \circledast & (0,0) & = & (0,0) \\
(2,1) & \circledast & (0,1) & = & (0,1) \\
(-2,-1) & \circledast & (0,-1) & = & (0,-1) \\
(0,0) & \circledast & (0,0) & = & (0,0) \\
(2,1) & \circledast & (0,1) & = & (0,1)
\end{array}
$$

This result is consistent with panel F.

Part a2.

$$
\begin{array}{ccccc}
(0,-2) & \circledast & (-2,0) & = & (-2,-2) \\
(0,0) & \circledast & (-2,0) & = & (-2,0) \\
(0,2) & \circledast & (-2,0) & = & (-2,2) \\
(0,-2) & \circledast & (0,0) & = & (0,-2) \\
(0,0) & \circledast & (0,0) & = & (0,0) \\
(0,2) & \circledast & (0,0) & = & (0,2) \\
(0,-2) & \circledast & (2,0) & = & (2,-2) \\
(0,0) & \circledast & (2,0) & = & (2,0) \\
(0,2) & \circledast & (2,0) & = & (2,2)
\end{array}
$$

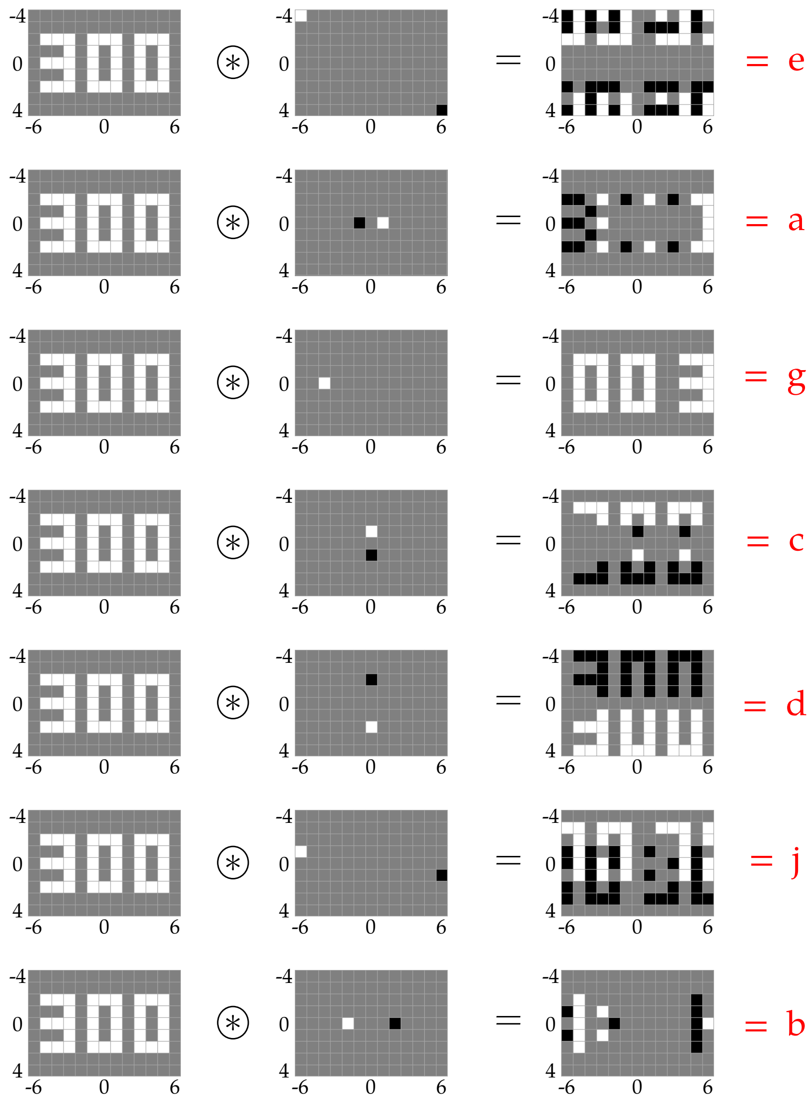
This result is consistent with panel F.

Part a3.

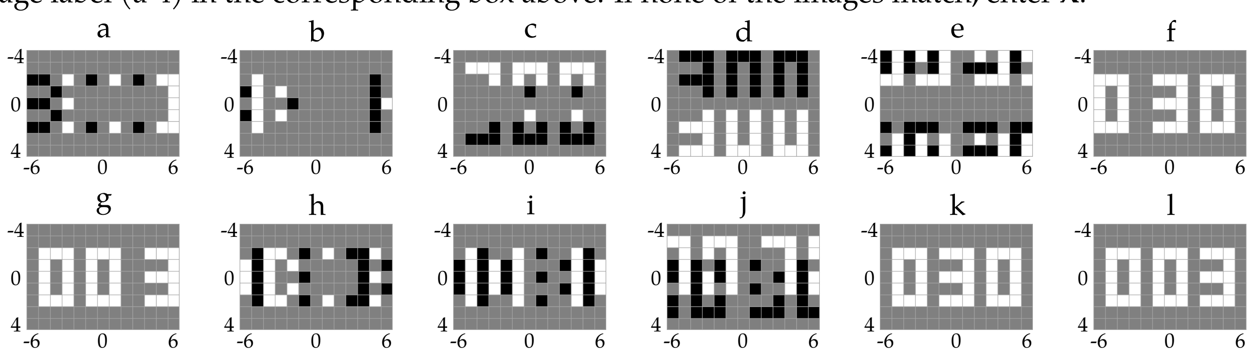
$$
\begin{array}{ccccc}
(-1,-1) & \circledast & (2,-2) & = & (1,-3) \\
(0,0) & \circledast & (2,-2) & = & (2,-2) \\
(1,1) & \circledast & (2,-2) & = & (3,-1) \\
(-1,-1) & \circledast & (0,0) & = & (-1,-1) \\
(0,0) & \circledast & (0,0) & = & (0,0) \\
(1,1) & \circledast & (0,0) & = & (1,1) \\
(-1,-1) & \circledast & (-2,2) & = & (-3,1) \\
(0,0) & \circledast & (-2,2) & = & (-2,2) \\
(1,1) & \circledast & (-2,2) & = & (-1,3)
\end{array}
$$

This result is consistent with panel F.

Part b. Two 13×9 pixel images are circularly convolved (using 13×9 DFTs and a 13×9 inverse DFT) to produce a new 13×9 image, where positive and negative values are represented by white and black pixels, respectively. Gray pixels represent values of 0.

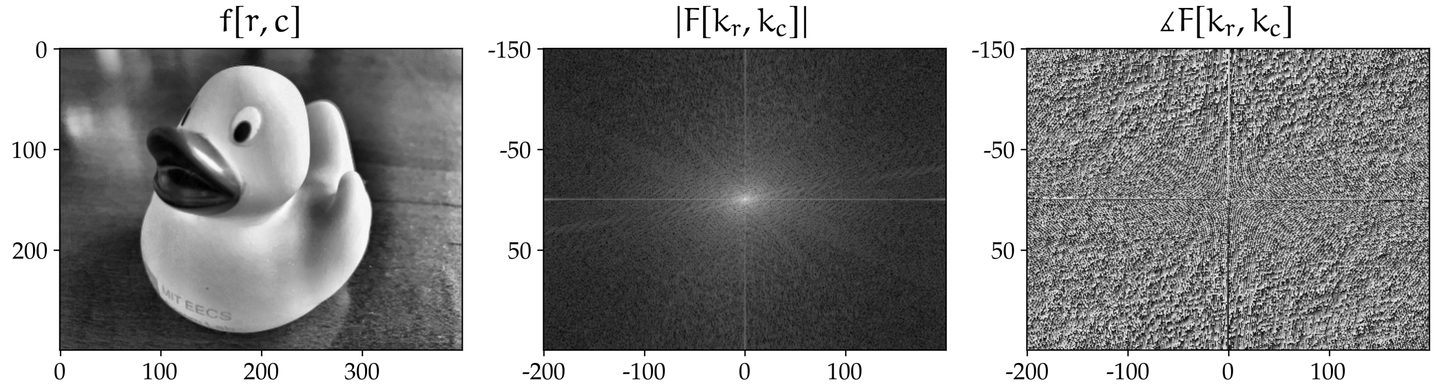


Identify which of the following images (a–l) results for each of the circular convolutions shown above, and enter that image label (a–l) in the corresponding box above. If none of the images match, enter **X**.



6 Ducks (18 points)

Let $f[r,c]$ represent the image of a rubber duck shown below at left. The pixels are indexed by row r and column c, where $0 \leqslant r < 300$ and $0 \leqslant c < 400$. Furthermore, let $F[k_r,k_c]$ denote the two-dimensional discrete Fourier transform (DFT) of $f[r,c]$ computed with $R = 300$ and $C = 400$. $|F[k_r,k_c]|$ and $\angle F[k_r,k_c]$ denote the magnitude and phase, respectively, of $F[k_r,k_c]$ and are shown below. In each image, black represents the minimum value, and white represents the maximum value. Note that the minima and maxima may differ between images.

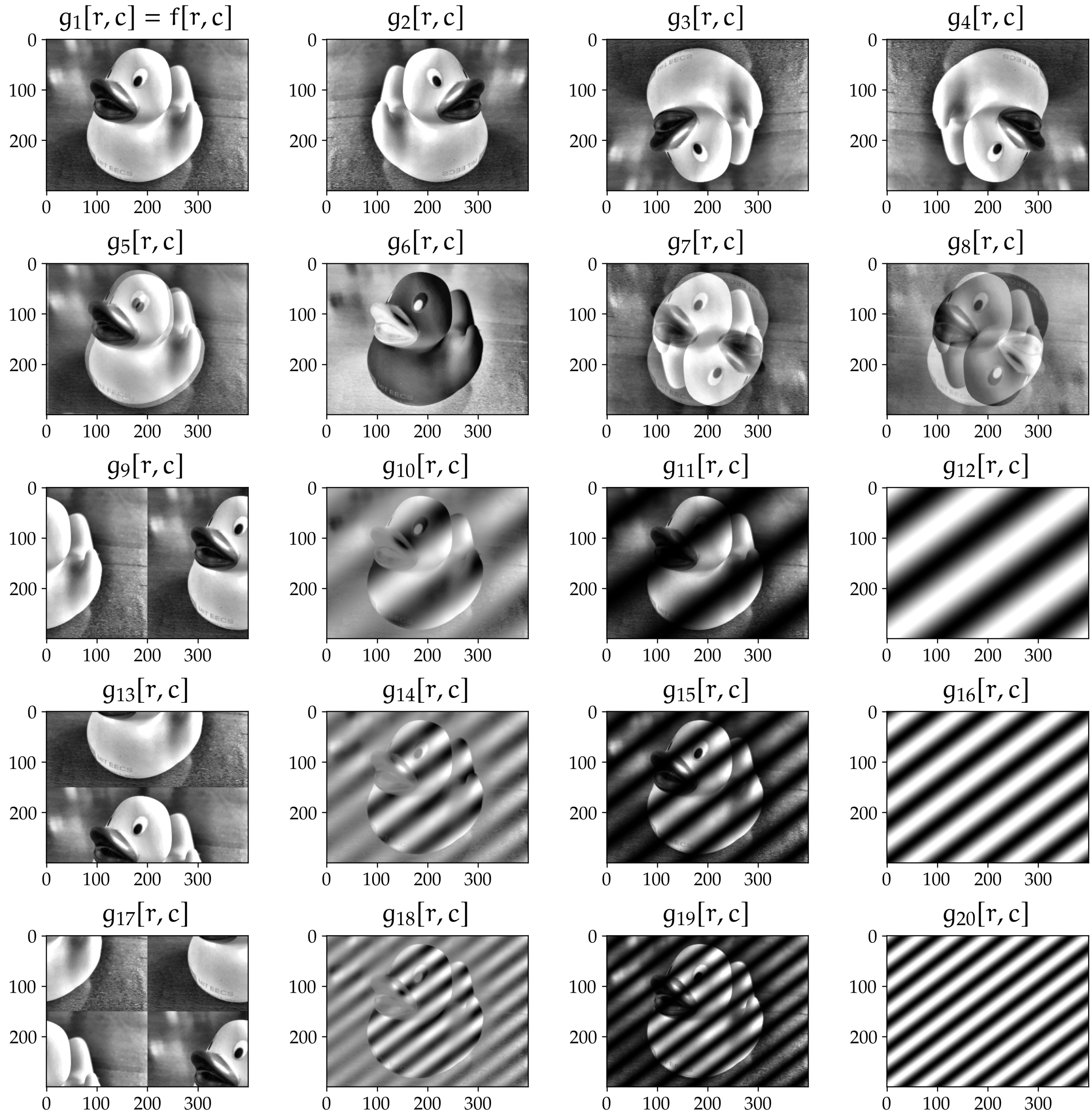


The table below lists operations which are performed either on $f[r,c]$ or on $F[k_r,k_c]$. Determine which (if any) of the images on the following page shows the image that results from the operation specified. In each image, black represents the minimum value and white represents the maximum value. The minima and maxima may differ between images. If there is no match, write **X** in the corresponding box.

operation	**resulting image** ($g_1, g_2, \ldots, g_{20}$ or **X**)
multiply $f[r,c]$ by $\cos\left(\frac{8\pi}{300}r + \frac{8\pi}{400}c\right)$	$g_{14}[r,c]$
convolve $f[r,c]$ with $\cos\left(\frac{8\pi}{300}r + \frac{8\pi}{400}c\right)$	$g_{16}[r,c]$
multiply $F[k_r,k_c]$ by $\cos\left(\frac{8\pi}{300}k_r + \frac{8\pi}{400}k_c\right)$	$g_5[r,c]$
convolve $F[k_r,k_c]$ with $\cos\left(\frac{8\pi}{300}k_r + \frac{8\pi}{400}k_c\right)$	X
multiply $F[k_r,k_c]$ by $\cos(\pi k_r)\cos(\pi k_c)$	$g_{17}[r,c]$
convolve $F[k_r,k_c]$ with $\cos(\pi k_r)\cos(\pi k_c)$	X
compute the inverse DFT of $\text{Re}\{F[k_r,k_c]\}$	$g_7[r,c]$
compute the inverse DFT of $\text{Im}\{F[k_r,k_c]\}$	X
compute the inverse DFT of $|F[k_r,k_c]|e^{-j\angle F[k_r,k_c]}$	$g_4[r,c]$

Images for "Ducks" Problem

In each image, black represents the minimum value and white represents the maximum value. The minima and maxima may differ between images. If there is no match, write **X** in the corresponding box on the previous page.



Worksheet (intentionally blank)

Worksheet (*intentionally blank*)

Worksheet (intentionally blank)

Worksheet (*intentionally blank*)