6.3000 Quiz 2 Spring 2026

Name: Solutions

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This quiz is closed book, but you may use two 8.5 × 11 sheets of notes (four sides).

You may NOT use any electronic devices (such as calculators and phones).

If you have questions, please **come to us** at the front of the room to ask.

Please enter all solutions in the boxes provided.

Work on other pages with QR codes will be considered for partial credit.
Please provide a note if you continue work on worksheets at the end of the exam.

Please do not write on the QR codes at the bottom of each page.

We use those codes to identify which pages belong to each student.

Trigonometric Identities Reference

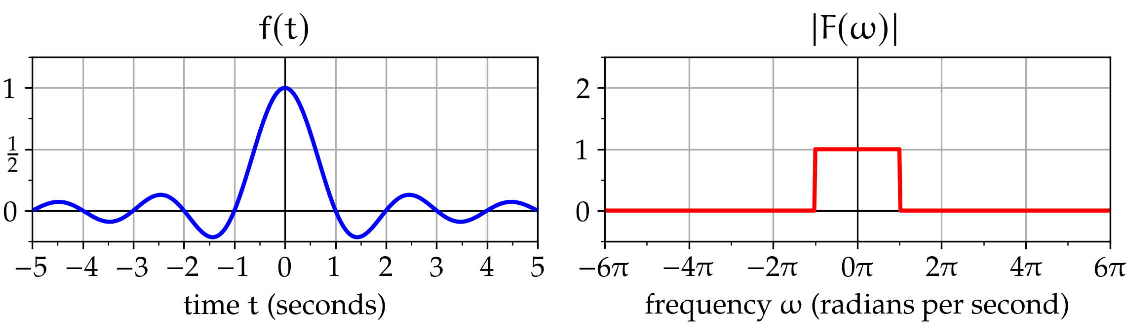
$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$
$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b)$$
$$\cos(a+b)-\cos(a-b) = -2\sin(a)\sin(b)$$
$$\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$$
$$\sin(a+b)-\sin(a-b) = 2\cos(a)\sin(b)$$
$$2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b)$$
$$2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$$
$$2\sin(a)\cos(b) = \sin(a+b)+\sin(a-b)$$
$$2\cos(a)\sin(b) = \sin(a+b)-\sin(a-b)$$

1 Sampling in Time and Frequency (24 points)

Part a. Consider a continuous-time LTI system with impulse response $f(t)$, as follows:

$$f(t) = \frac{\sin(\pi t)}{\pi t}$$

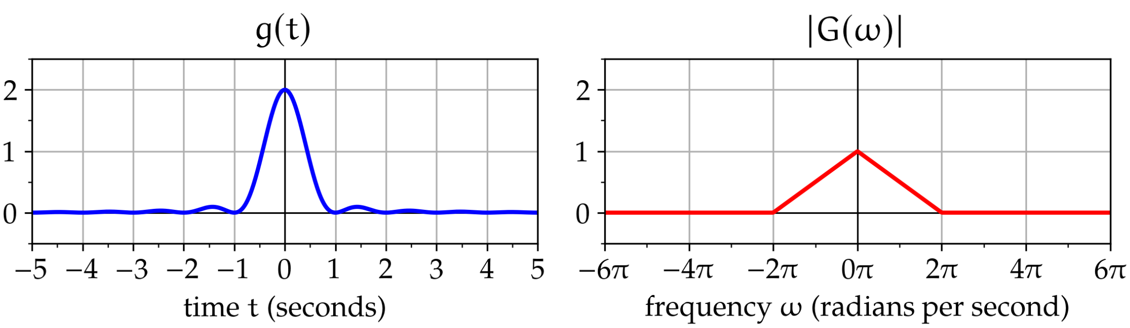
Let $F(\omega)$ denote the frequency response. Sketch $|F(\omega)|$ for $\omega \in [-6\pi, 6\pi]$. Label the vertical axis.



Part b. Consider a continuous-time LTI system with impulse response $g(t)$, as follows:

$$g(t) = f^2(t)$$

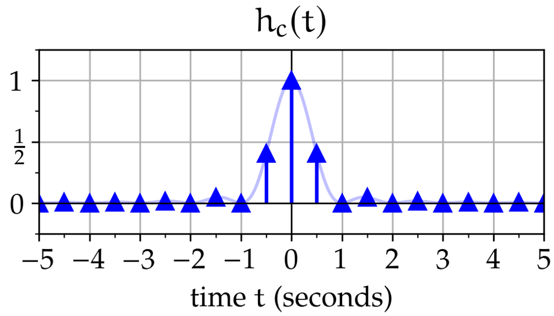
Let $G(\omega)$ denote the frequency response. Sketch $|G(\omega)|$ for $\omega \in [-6\pi, 6\pi]$. Label the vertical axis.

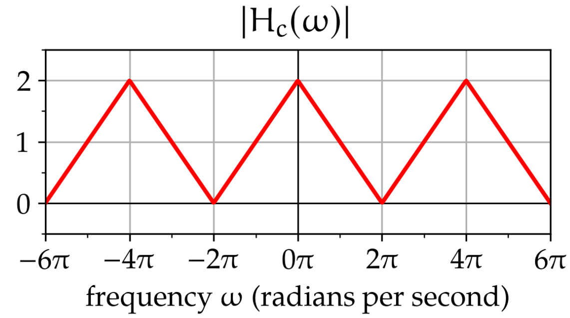


Part c. Consider a continuous-time LTI system with impulse response $h_c(t)$, as follows:

$$h_c(t) = g(t)\sum_{m=-\infty}^{\infty}\delta(t-0.5m)$$

Let $H_c(\omega)$ denote the frequency response. Sketch $\left|H_c(\omega)\right|$ for $\omega \in [-6\pi, 6\pi]$. Label the vertical axis.

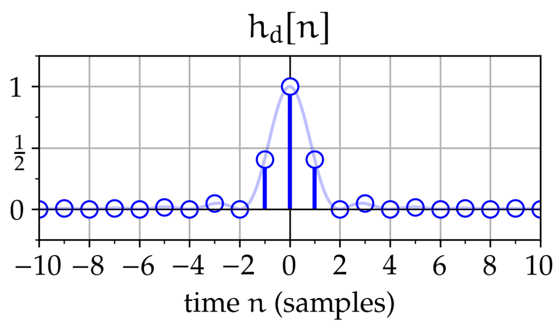


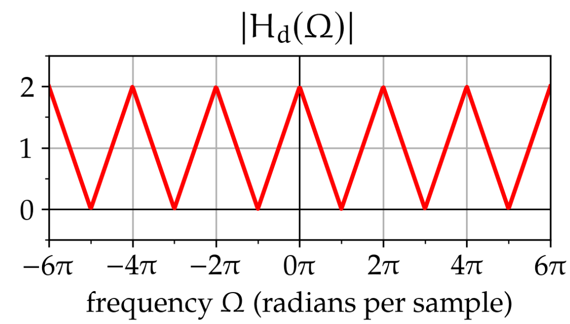


Part d. Consider a discrete-time LTI system with unit-sample response $h_d[n]$, as follows:

$h_d[n] = h_c(0.5n)$

Let $H_d(\Omega)$ denote the frequency response. Sketch $\left|H_d(\Omega)\right|$ for $\Omega \in [-6\pi, 6\pi]$. Label the vertical axis.





2 Difference Equations (28 points)

Let $\alpha>0$ denote a real-valued constant. Consider eight discrete-time LTI systems:

$x[n] \rightarrow \boxed{S_0} \rightarrow y[n] = x[n] + \alpha y[n-1]$

$x[n] \rightarrow \boxed{S_1} \rightarrow y[n] = x[n] + \alpha y[n-2]$

$x[n] \rightarrow \boxed{S_2} \rightarrow y[n] = x[n] - \alpha y[n-1]$

$x[n] \rightarrow \boxed{S_3} \rightarrow y[n] = x[n] - \alpha y[n-2]$

$x[n] \rightarrow \boxed{S_4} \rightarrow y[n] = x[n] + x[n-2] - \alpha y[n-4]$

$x[n] \rightarrow \boxed{S_5} \rightarrow y[n] = x[n] + x[n-1] + x[n-2] + x[n-3] - \alpha y[n-4]$

$x[n] \rightarrow \boxed{S_6} \rightarrow y[n] = x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]$

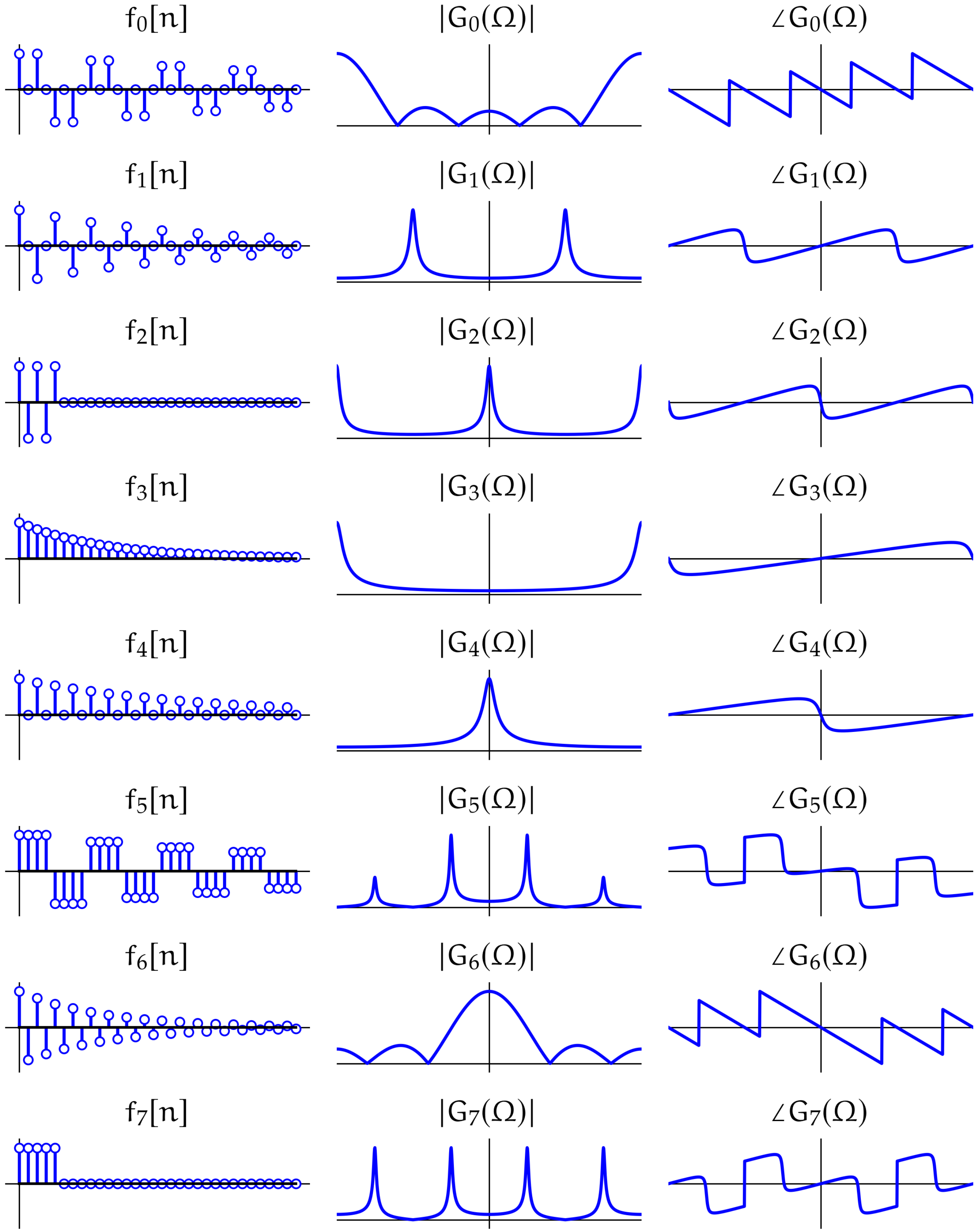
$x[n] \rightarrow \boxed{S_7} \rightarrow y[n] = x[n] - x[n-1] + x[n-2] - x[n-3] + x[n-4]$

Twenty-four plots are shown on the next page.

- Plots in the left column show (for $0 \leqslant n \leqslant 31$) a unit-sample response $f_k[n]$.
- Plots in the middle and right columns show (for $-\pi \leqslant \Omega \leqslant \pi$) the magnitude $|G_\ell(\Omega)|$ and angle $\angle G_\ell(\Omega)$, respectively, of a frequency response $G_\ell(\Omega)$.

For each system, determine which plots on the next page show the system's unit-sample response and frequency response. If the unit-sample response or frequency response of a system is not shown in any of the plots given, write an **X** in the corresponding box.

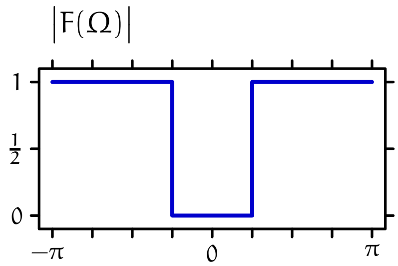
system	**unit-sample response** $(f_0, f_1, \ldots, f_7)$	**frequency response** $(G_0, G_1, \ldots, G_7)$
S_0	$f_3[n]$	$G_4(\Omega)$
S_1	$f_4[n]$	$G_2(\Omega)$
S_2	$f_6[n]$	$G_3(\Omega)$
S_3	$f_1[n]$	$G_1(\Omega)$
S_4	$f_0[n]$	$G_7(\Omega)$
S_5	$f_5[n]$	$G_5(\Omega)$
S_6	$f_7[n]$	$G_6(\Omega)$
S_7	$f_2[n]$	$G_0(\Omega)$

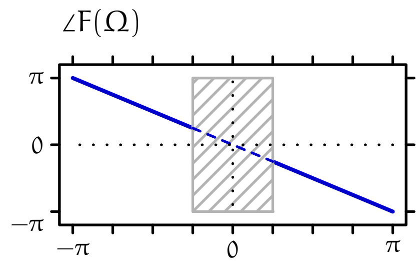


$f_0[n]$
$|G_0(\Omega)|$
$\angle G_0(\Omega)$
$f_1[n]$
$|G_1(\Omega)|$
$\angle G_1(\Omega)$
$f_2[n]$
$|G_2(\Omega)|$
$\angle G_2(\Omega)$
$f_3[n]$
$|G_3(\Omega)|$
$\angle G_3(\Omega)$
$f_4[n]$
$|G_4(\Omega)|$
$\angle G_4(\Omega)$
$f_5[n]$
$|G_5(\Omega)|$
$\angle G_5(\Omega)$
$f_6[n]$
$|G_6(\Omega)|$
$\angle G_6(\Omega)$
$f_7[n]$
$|G_7(\Omega)|$
$\angle G_7(\Omega)$

3 Fourier Matching (28 points)

The following plots show one period of the magnitude (left) and angle (right) of $F(\Omega)$, which is the Discrete-Time Fourier Transform of a discrete-time signal $f[n]$.



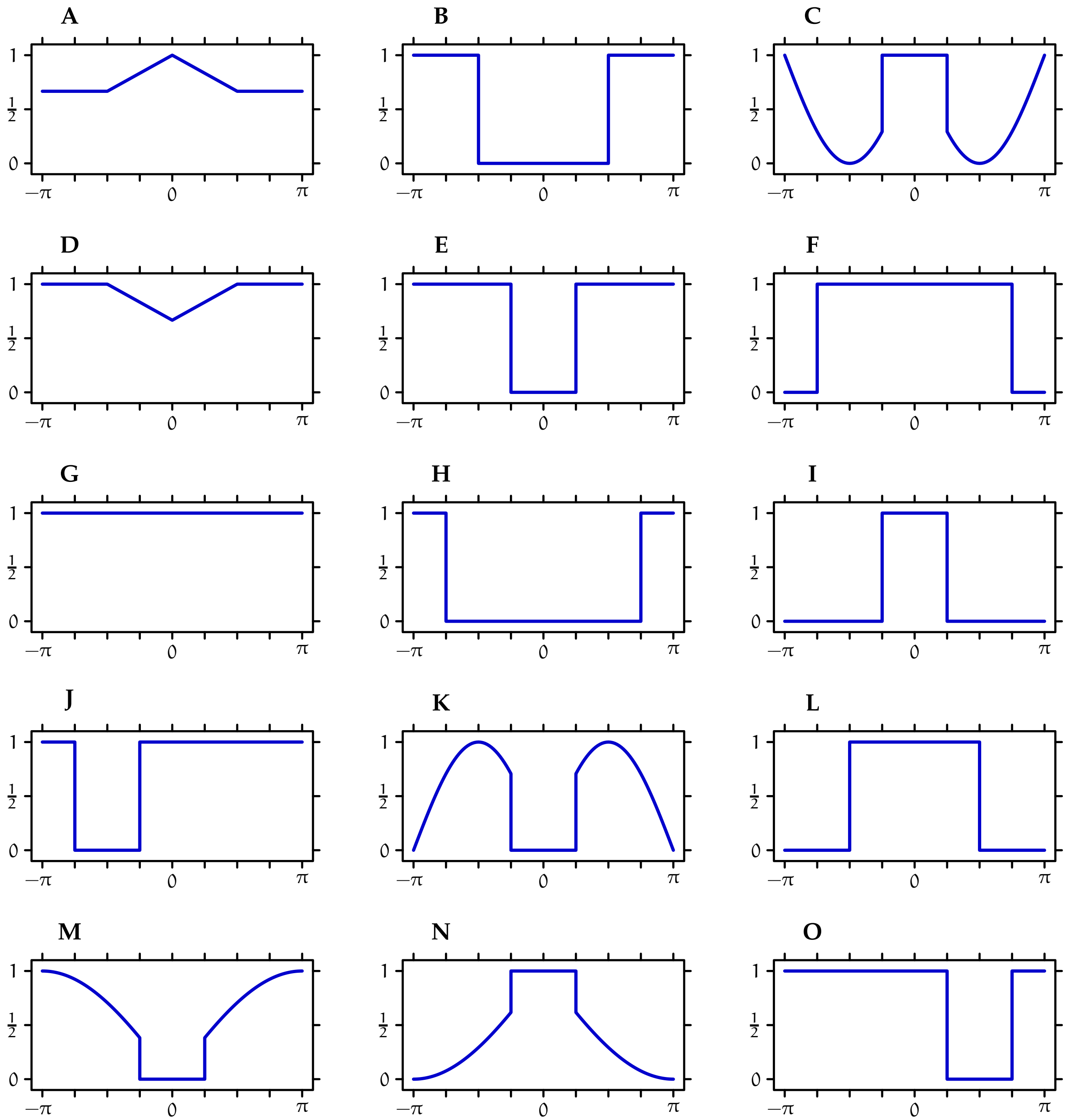


The crosshatched region of the angle plot indicates a region of Ω for which the angle is undefined (because $|F(\Omega)|$ is zero at those frequencies). The dashed line in the crosshatched region is shown as an aid to visualization of trends outside the crosshatched region.

The left column in the chart below shows seven signals that are derived from $f[n]$. Determine which (if any) of the magnitude plots on the following page and the angle plots on the subsequent page match your results for each of the **seven derived signals**.

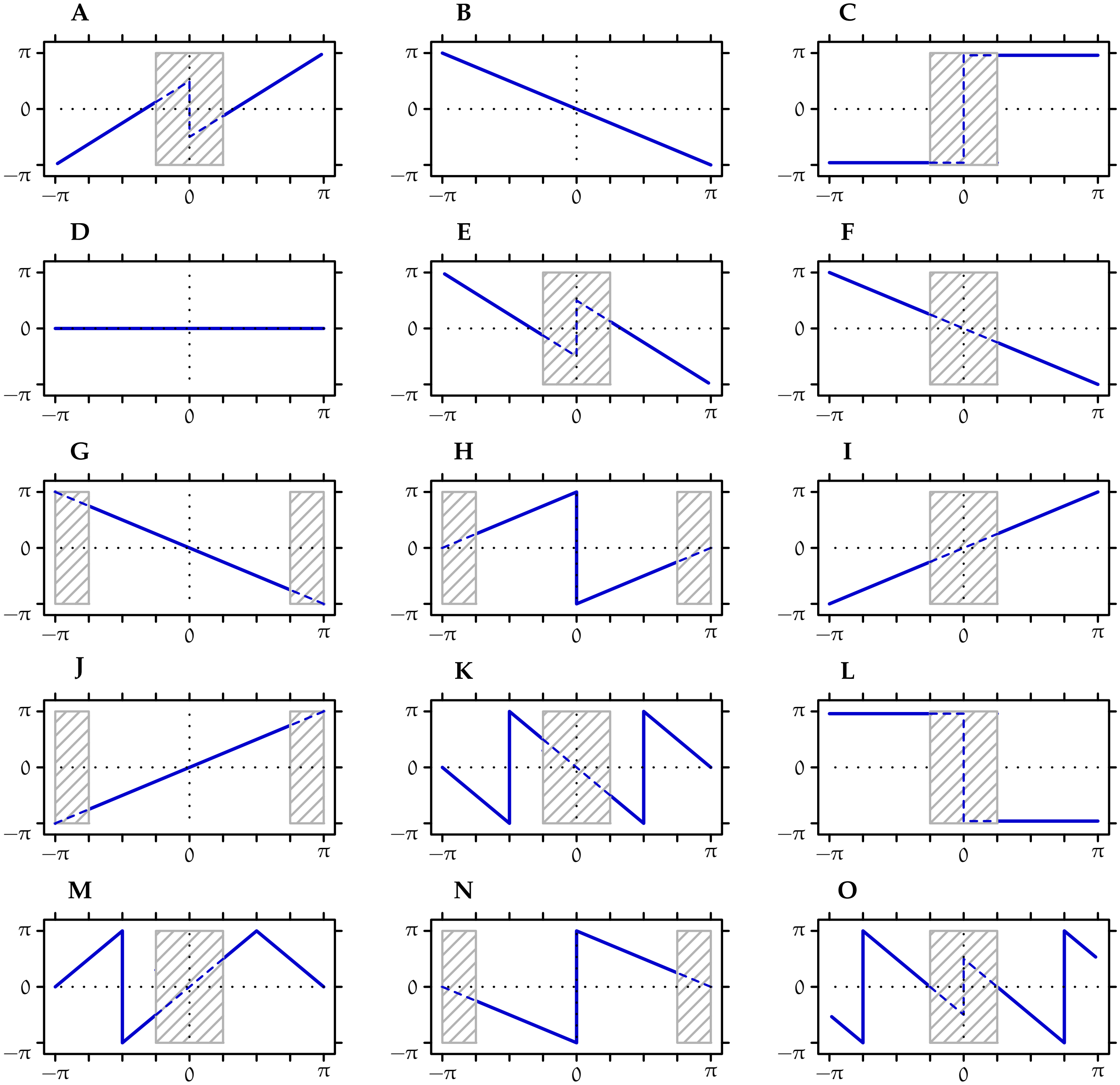
	magnitude (**A-O**) or **X** if none	**angle** (**A-O**) or **X** if none
$f[n-1]$:	**E**	**K**
$f[n]\times(-1)^n$:	**F**	**N**
$f[n]\times(-1)^{(n-1)}$:	**F**	**G**
$\big(f[n]-f[n-1]\big)/2$:	**M**	**E**
$\big(f[n]-f[n-2]\big)/2$:	**K**	**O**
$f^2[n]$:	**A**	**B**
$(f*f)[n]$:	**E**	**K**

Magnitude plots: Each of the following panels shows the magnitude of a DTFT as a function of radian frequency Ω. The magnitudes in each panel are normalized so that the maximum value in each panel is 1. Determine which (if any) of the following plots matches the magnitude of each of the **seven derived signals** shown on the previous page. Some of these plots may match more than one of the seven derived magnitudes. If none of these plots are a match, put X in the corresponding box(es).



X = none of the above.

Angle plots: Each of the following panels shows the angle of a DTFT as a function of radian frequency Ω. Determine which (if any) of the following plots matches the angle of each of the **seven derived signals**. The crosshatched regions indicate values of Ω for which the angle is undefined (because the corresponding magnitude function is zero at those frequencies). Angle plots should only be considered a match if the crosshatched regions match **AND** the plots match outside the crosshatched regions. The dashed lines in the crosshatched regions are provided to help visualize the trends outside the crosshatched region.



X = none of the above.

Part a.

$$G_1(\Omega) = \sum_n g_1[n]e^{-j\Omega n} = \sum_n f[n-1]e^{-j\Omega n} = \sum_m f[m]e^{-j\Omega(m+1)} = e^{-j\Omega}F(\Omega)$$

Multiplying $F(\Omega)$ by $e^{-j\Omega}$ has no effect on magnitude, and the result is shown in magnitude panel E. Multiplying $F(\Omega)$ by $e^{-j\Omega}$ subtracts Ω from the angle function, and the result is shown in angle panel K. The crosshatched region for this part is the same as that for $F(\Omega)$ since the region where the magnitude is zero is unchanged by the exra delay.

Part b.

$$G_2(\Omega) = \sum_n g_2[n]e^{-j\Omega n} = \sum_n f[n](-1)^n e^{-j\Omega n} = \sum_n f[n](e^{\pm j\pi})^n e^{-j\Omega n} = \sum_n f[n]e^{-j(\Omega\pm\pi)n} = F(\Omega \pm \pi)$$

Multiplying $f[n]$ by $(-1)^n$ shifts the dependence of the transform on Ω by $\pm\pi$. The resulting magnitude is shown in panel F, and the resulting angle is shown in panel N. Notice that multiplying by $(-1)^n$ alters the highpass nature of $F(\Omega)$ to lowpass. This shifts the cross-hatched region of the angle function from low frequencies to high frequencies.

Part c.

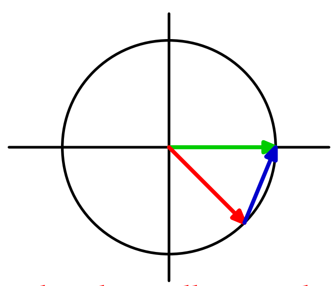
$$G_3(\Omega) = \sum_n g_3[n]e^{-j\Omega n} = \sum_n f[n](-1)^{(n-1)}e^{-j\Omega n} = -G_2(\Omega)$$

Multiplying by -1 has no effect on magnitude so the magnitude is shown in panel F (as it was in the previous part). Multiplying by -1 shifts the angles at all frequencies by π. Thus panel N (from the previous part) is shifted to panel G.

Part d.

$$G_4(\Omega) = \sum_n g_4[n]e^{-j\Omega n} = \frac{1}{2}\big(f[n] - f[n-1]\big) = \frac{1}{2}\big(1 - e^{-j\Omega}\big)F(\Omega)$$

The diagram below shows a vector representation of $(1 - e^{-j\Omega})$. The green arrow represents 1, the red arrow represents $e^{-j\Omega}$, and the blue arrow represents the difference $(1 - e^{-j\Omega})$.



As $\Omega \to 0$, the blue arrow approaches a vanishingly small arrow that points vertically. Thus at low frequencies, the magnitude goes to zero and the angle approaches $\pi/2$. As Ω increases, the magnitude increases monotonically to a maximum of 2 when $\Omega = \pi$. Similarly, as Ω increases from 0 to π, the angle decreases monotonically from $\pi/2$ to 0. These behaviors are consistent with magnitude plot M and angle plot E.

Part e.

$$G_5(\Omega) = \sum_n g_5[n]e^{-j2\Omega n} = \frac{1}{2}\big(f[n] - f[n-2]\big) = \frac{1}{2}\big(1 - e^{-j2\Omega}\big)F(\Omega)$$

The behavior of $G_5(\Omega)$ is similar to that of $G_4(\Omega)$ except that red arrow now represents a unit vector in the 2Ω direction. Instead of peaking in magnitude when $\Omega = \pi$, the peak now occurs when $\Omega = \pi/2$. Similarly, the angle of $G_5(\Omega)$ advances twice as fast with Ω as it did for G_4. These trends are consistent with magnitude K and angle O.

Part f.

$$G_6(\Omega) = \sum_n g_6[n]e^{-j\Omega n} = \sum_n f^2[n]e^{-j\Omega n}$$

Let $H(\Omega)$ represent the magnitude of $F(\Omega)$. Then (like $F(\Omega)$), $H(\Omega)$ is highpass, but (unlike $F(\Omega)$), $H(\Omega)$ has zero phase. It follows that $f[n] = h[n-1]$. Now we can compute the n^{th} sample of f^2 using the following

$$f^2[n] = f[n] \times f[n] = h[n-1] \times h[n-1]$$

which expresses a sample-by-sample view where the n^{th} sample of f^2 is equal to the the product of the $n-1$ sample of h times the $n-1$ sample of h. Alternatively, we can compute the entire sequence f^2 by first computing the entire sequence $h \times h$ and then equating the n^{th} sample of f^2 with the $n-1$ sample of $h \times h$:

$$f^2[n] = (h\times h)[n-1]$$

Multiplication in time is equivalent to convolution in frequency. Convolution of two high-pass magnitudes produces the triangular shape shown in magnitude panel A. The peak value of the convolution results when the shift is zero. The magnitude of the convolution decreases linearly as the convolution shift increases from 0 to $\pi/2$. The convolution is then constant again as the shift increases from $\pi/2$ to $3\pi/2$. The resulting magnitude is given in panel A.

Since $H[\Omega]$ has zero phase, the total phase of the answer is equivalent to a delay of one sample as in angle panel B. Notice that there is no crosshashed region of this result, since the magnitude function is never zero.

Part g.

$$G_7(\Omega) = \sum_n g_7[n]e^{-j\Omega n} = \sum_n (f * f)[n]e^{-j\Omega n}$$

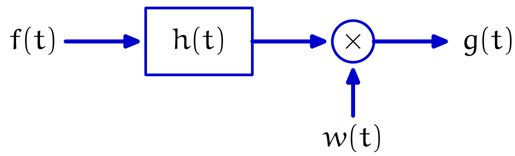
Convolution in time is multiplication in frequency.

$$G_7(\Omega) = F(\Omega) \times F(\Omega)$$

The magnitude is given in panel E and the angle is given in panel K.

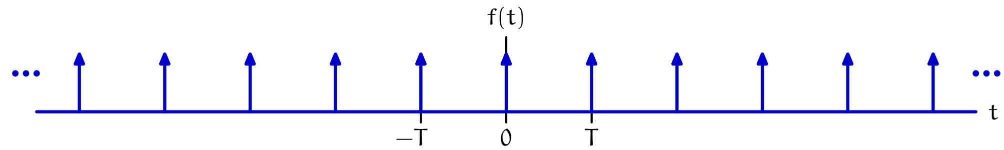
4 Continuous Processing (20 points)

In the following block diagram, $h(t)$ represents the impulse response of a linear, time-invariant subsystem whose input is $f(t)$, and whose output is multiplied by a window function $w(t)$ to generate the final output $g(t)$.



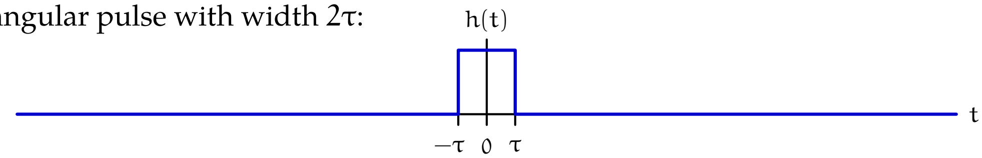
The input $f(t)$ is a periodic sequence of impulses with period T, as follows:

$$f(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT)$$



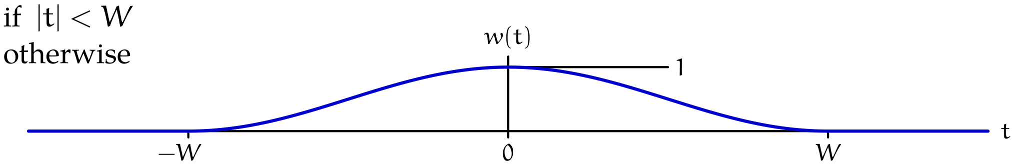
The impulse response $h(t)$ is a rectangular pulse with width 2τ:

$$h(t) = \begin{cases} 1 & \text{if } |t| < \tau \\ 0 & \text{otherwise} \end{cases}$$



The window function $w(t)$ is a Hann window with width $2W$:

$$w(t) = \begin{cases} \big(1+\cos(\pi t/W)\big)/2 & \text{if } |t| < W \\ 0 & \text{otherwise} \end{cases}$$



The top-left plot on the following page shows the magnitude of the Continuous-Time Fourier Transform (CTFT) of $g(t)$ for a "base case" in which $T = 8$ seconds, $\tau = 1$ second, and $W = 32$ seconds.

Determine which of plots (A-N) on the following page shows the CTFT of the output that results when a single change is made to the base case. Enter your answers in the boxes provided.

double W: **F**

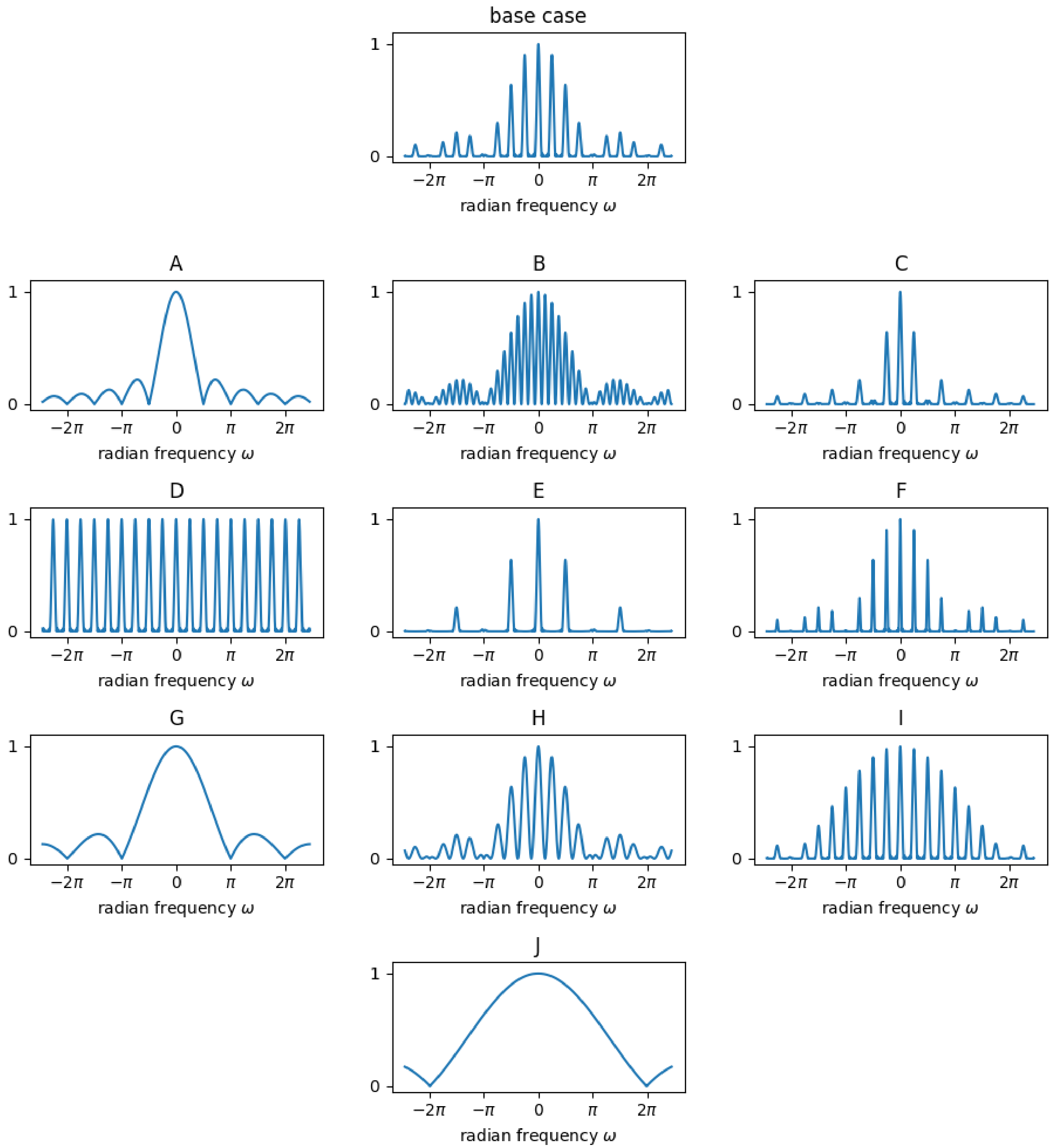
double T: **B**

halve τ: **I**

divide τ by 64: **D**

multiply T by 4: **G**

Each of the following panels shows the magnitude of a CTFT as a function of radian frequency ω. The magnitudes in each panel are normalized so that the maximum value in each panel is 1.



Each of the three important parameters (W, T, and τ) in this problem contributes to an important aspect of the basecase (where $W = 32$ seconds, $T = 8$ seconds, and $\tau = 1$ second).

Since $h(t)$ is a rectangular pulse, its Fourier transform has the form $sin(\omega)/\omega$. Because τ is the smallest of the parameters, it controls the largest feature(s). Thus $h(t)$ determines the envelope, which has a central lobe at frequencies $|\omega| < \pi$.

The parameter T controls the period of the impulses in the input signal $f(t)$. The Fourier transform of an impulse train in time with period T is an impulse train in frequency with period $\frac{2\pi}{T}$. Since T is larger than τ, there are multiple spikes (due to the impulse train) inside the central lobe of the envelope.

Finally, the largest parameter is W, so W controls the shape of the individual spikes, so that increasing W decreases the width of the individual spikes.

Part a: Doubling W will decrease the width of the individual spikes, as seen in panel F.

Part b: Doubling T will decrease the distance between individual spikes, as seen in panel B.

Part c: Halving τ will double the width of the envelope, as seen in panel I.

Part d: Dividing τ by 64 increases the width of the envelope by a factor of 64. Thus the envelope nearly a constant over the width of the panels. The result is shown in panel D.

Part e: Multiplying T by 4 yields $T = 32$ which is now equal to the half-width of W. For this condition, only the central impulse at $t = 0$ contributes to the output $g(t)$. Therefore there are no spikes, there is just the envelope function which is shown in panel G.

Worksheet (intentionally blank)

Worksheet (intentionally blank)

Worksheet (*intentionally blank*)

Worksheet (intentionally blank)

Worksheet (*intentionally blank*)