

Quiz #1 Review: Story Sheet

for 6.300 (Signal Processing) by Titus K. Roesler (tkr@mit.edu)

The Signal-Processing Story So Far

- **02/03/2026:** Signal Processing identifying, analyzing, manipulating signals
- **02/05/2026:** Fourier Series (Sinusoids) series representation for periodic CT signals
- **02/10/2026:** Fourier Series (Exponentials) series representation for periodic CT signals
- **02/12/2026:** Sampling and Aliasing discretization: from continuous time to discrete time
- **02/19/2026:** Discrete-Time Fourier Series series representation for periodic DT signals
- **02/24/2026:** Continuous-Time Fourier Transform frequency representation for aperiodic CT signals
- **02/26/2026:** Discrete-Time Fourier Transform frequency representation for aperiodic DT signals

Mathematics Review

Dimensional Analysis

$$T_0 \text{ (seconds)} \times f_s \text{ (samples / second)} = N_0 \text{ (samples)}$$

$$\omega_0 \text{ (radians / second)} \div f_s \text{ (samples / second)} = \Omega_0 \text{ (radians / sample)}$$

CT cyclical frequency $f_0 = 1/T_0$

cycles per second or hertz (Hz)

CT angular frequency $\omega_0 = 2\pi/T_0 = 2\pi f_0$

radians per second

DT cyclical frequency $F_0 = f_0/f_s = f_0 T_s = 1/N_0$

cycles per sample

DT angular frequency $\Omega_0 = \omega_0/f_s = 2\pi f_0/f_s = 2\pi f_0 T_s = 2\pi/N_0$

radians per sample

Geometric Series

$$\sum_{n=0}^{N-1} z^n = \frac{1 - z^N}{1 - z}$$

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z} \text{ for } |z| < 1$$

Binomial Theorem

$$(\alpha + \beta)^n = \binom{n}{0} \alpha^n + \binom{n}{1} \alpha^{n-1} \beta + \binom{n}{2} \alpha^{n-2} \beta^2 + \dots + \binom{n}{n-1} \alpha \beta^{n-1} + \binom{n}{n} \beta^n$$

$$\binom{n}{k} \equiv \frac{n!}{k!(n-k)!} \text{ where } n! \equiv (n)(n-1)(n-2) \dots (3)(2)(1)$$

Write down any other formulas you think are especially important to remember.

Continuous-Time Fourier Series

Continuous-Time Fourier Series

$f(t) = f(t + T)$ is a T -periodic function, and $\omega_0 = 2\pi/T$ denotes the fundamental angular frequency.

Continuous-Time Fourier Series in Trigonometric Form

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_0 t)$$

where $c_0 = \frac{1}{T} \int_T f(t) dt$ and $c_k = \frac{2}{T} \int_T f(t) \cos(k\omega_0 t) dt$ and $d_k = \frac{2}{T} \int_T f(t) \sin(k\omega_0 t) dt$

c_0 , the “direct current” (DC) term, represents the average value of $f(t)$ over a single period.

Continuous-Time Fourier Series in Complex Exponential Form

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \text{ where } a_k = \frac{1}{T} \int_T f(t) e^{-jk\omega_0 t} dt \quad \left(\text{e.g., } a_0 = \frac{1}{T} \int_T f(t) dt \right)$$

Frequency

Time and frequency are inversely proportional.

$$\text{cyclical: } f_0 = \frac{1}{T} \text{ (cycles per second, or hertz)} \quad \text{angular: } \omega_0 = 2\pi f_0 = \frac{2\pi}{T} \text{ (radians per second)}$$

Complex Variables

Think of complex variables geometrically — as points in the complex plane.

$$z = \underbrace{\text{Re}\{z\} + j \text{Im}\{z\}}_{\text{rectangular}} = \underbrace{r e^{j\phi}}_{\text{polar}} \text{ where } r = \underbrace{\sqrt{\text{Re}\{z\}^2 + \text{Im}\{z\}^2}}_{\text{magnitude of } z} \text{ and } \underbrace{\tan(\phi) = \frac{\text{Im}\{z\}}{\text{Re}\{z\}}}_{\text{angle or phase of } z}$$

Euler's Formula

Euler's formula relates the rectangular-coordinate and polar-coordinate descriptions of complex variables.

$$e^{j\theta} = \cos(\theta) + j \sin(\theta) \quad \cos(\theta) = \text{Re}\{e^{j\theta}\} = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \sin(\theta) = \text{Im}\{e^{j\theta}\} = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Sampling and Aliasing

From Continuous to Discrete

We refer to the process of discretizing time or space as *sampling*.

$$x[n] = x(n\Delta) \text{ where } n \in \mathbb{Z} \text{ and } \Delta \text{ denotes the sampling period, sampling interval, or time step}$$

We refer to the process of discretizing amplitude as *quantization*. (e.g., rounding)

Digital signals are discrete in both time and amplitude. Digital systems such as laptops process digital signals. *Discrete-time* signals are discrete in time — but not necessarily in amplitude. In 6.300, we won't study quantization in great depth. We'll focus on *discrete-time signal processing*.

Sampling and Aliasing

We *sample* a continuous-time signal $x(t)$ every Δ seconds to obtain a discrete-time signal $x[n]$.

$$x[n] = x(n\Delta) \text{ where } n \in \mathbb{Z} \text{ and } \Delta \text{ denotes the sampling period, sampling interval, or time step}$$

Note that $x[n]$ is a function of the integer n , which is enclosed in square brackets. In contrast, $x(t)$ is a function of the real variable t , which is enclosed in parentheses. With this notation, $x[n] \neq x(n)$ in general — when you write $x(n)$, you're implicitly saying that $\Delta = 1$.

Aliasing

Sampling involves throwing away information. If we don't sample frequently enough, the information within our signal will be distorted: Frequencies will “fold in” on each other.

Nyquist-Shannon sampling theorem: Let f_{\max} denote the highest frequency in $x(t)$. The minimum sampling rate that prevents aliasing is $2f_{\max}$ — twice the highest frequency in $x(t)$.

Frequencies

Always keep the dimensions of quantities in mind.

$$\text{CT cyclical: } f = \frac{1}{T} \quad \text{CT angular: } \omega = 2\pi f = \frac{2\pi}{T} \quad \text{DT angular: } \Omega = \frac{2\pi f}{f_s} = \frac{\omega}{f_s} = \frac{2\pi}{N}$$

The argument to a trigonometric or exponential function must be expressed in radians.

- $\text{units}\{2\pi ft\} = (\text{radians/cycle}) \times (\text{cycles/second}) \times (\text{seconds}) = \text{radians}$
- $\text{units}\{\omega t\} = (\text{radians/second}) \times (\text{seconds}) = \text{radians}$
- $\text{units}\{\Omega n\} = (\text{radians/sample}) \times (\text{samples}) = \text{radians}$

Discrete-Time Fourier Series

Fourier Series Formulæ

Continuous-Time Fourier Series (CTFS)

$f(t)$ is a T -periodic function with fundamental angular frequency $\omega_0 = 2\pi/T$.

$$\text{Synthesis: } f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \text{Analysis: } a_k = \frac{1}{T} \int_T f(t) e^{-jk\omega_0 t}$$

Discrete-Time Fourier Series (DTFS)

$f[n]$ is an N -periodic sequence with fundamental angular frequency $\Omega_0 = 2\pi/N$.

$$\text{Synthesis: } f[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} \qquad \text{Analysis: } a_k = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\Omega_0 n}$$

A Few Properties of Fourier Series

Here are a few properties of Fourier series that we'll use often in this class. We'll learn more over time.

Linearity

$$\begin{aligned} f_1(t) &\iff F_1[k] & f_2(t) &\iff F_2[k] & f(t) = \alpha f_1(t) + \beta f_2(t) &\iff F[k] = \alpha F_1[k] + \beta F_2[k] \\ f_1[n] &\iff F_1[k] & f_2[n] &\iff F_2[k] & f[n] = \alpha f_1[n] + \beta f_2[n] &\iff F[k] = \alpha F_1[k] + \beta F_2[k] \end{aligned}$$

Time Shift

$$\begin{aligned} f(t) &\iff F[k] & f(t - t_0) &\iff F[k] e^{-jk\omega_0 t_0} = |F[k]| e^{j(\angle F[k] - k\omega_0 t_0)} \\ f[n] &\iff F[k] & f[n - n_0] &\iff F[k] e^{-jk\Omega_0 n_0} = |F[k]| e^{j(\angle F[k] - k\Omega_0 n_0)} \end{aligned}$$

Time Flip

$$\begin{aligned} f(t) &\iff F[k] & f(-t) &\iff F[-k] \\ f[n] &\iff F[k] & f[-n] &\iff F[-k] \end{aligned}$$

Conjugate Symmetry (Hermitian Symmetry)

Real-valued signals have conjugate-symmetric Fourier series coefficients.

real-valued $f(t) \iff F[k]$ such that $F^*[k] = F[-k]$ where $*$ denotes complex conjugation

real-valued $f[n] \iff F[k]$ such that $F^*[k] = F[-k]$ where $*$ denotes complex conjugation

Continuous-Time Fourier Transform

Continuous-Time Fourier Transform (CTFT)

The continuous-time Fourier transform may be conceptualized as the continuum limit of a continuous-time Fourier series. Infinitely-many discrete harmonics $k\omega_0$ cluster infinitely-close together to form a continuous frequency spectrum: $k\omega_0$ (function of integer k) $\mapsto \omega$ (function of real-valued ω).

$$\text{Synthesis: } x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$\text{Analysis: } X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Transform Pairs

$$\delta(t) \iff 1 \text{ (for all } \omega)$$

$$1 \text{ (for all } t) \iff 2\pi\delta(\omega)$$

$$\delta(t - t_0) \iff e^{-j\omega t_0}$$

$$e^{j\omega_0 t} \iff 2\pi\delta(\omega - \omega_0)$$

Fourier Transform Properties

You fill this in! Many properties that we've seen in the context of Fourier series still hold true.

Fourier Series vs. Fourier Transform for Periodic Signals

$$\text{Series: } f(t) \iff F[k]$$

$$\text{Transform: } f(t) \iff \sum_k \underbrace{2\pi F[k] \delta(\omega - k\omega_0)}_{\text{impulses at harmonics}}$$

Discrete-Time Fourier Transform

Discrete-Time Fourier Transform (DTFT)

The discrete-time Fourier transform is the discrete-time analogue of the continuous-time Fourier transform — a Fourier transform for discrete-time signals. Infinitely-many discrete harmonics $k\Omega_0$ cluster infinitely-close together to form a continuous frequency spectrum: $k\Omega_0 \mapsto \Omega$.

$$\text{Synthesis: } x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega \qquad \text{Analysis: } X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

Discrete-Time Fourier Transform Pairs

$$\begin{aligned} \delta[n] &\iff 1 \text{ (for all } \Omega) & 1 \text{ (for all } n) &\iff 2\pi\delta(\Omega \bmod 2\pi) \\ \delta[n - n_0] &\iff e^{-j\Omega n_0} & e^{j\Omega_0 n} &\iff 2\pi\delta(\Omega - \Omega_0 \bmod 2\pi) \end{aligned}$$

Discrete-Time Fourier Transform Properties

You fill this in! Many properties that we've seen in the context of Fourier series still hold true.

Fourier Series vs. Fourier Transform for Periodic Signals

$$\text{Series: } f[n] \iff F[k] \qquad \text{Transform: } f[n] \iff \sum_k \underbrace{2\pi F[k] \delta(\Omega - k\Omega_0)}_{\text{impulses at harmonics}}$$