6.3000 Quiz 1 Spring 2025

Name: Solutions

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This quiz is closed book, but you may use one 8.5×11 sheet of notes (both sides).

You may NOT use any electronic devices (such as calculators and phones).

If you have questions, please **come to us** at the front of the room to ask.

Please enter all solutions in the boxes provided.

Work on other pages with QR codes will be considered for partial credit.
Please provide a note if you continue work on worksheets at the end of the exam.

Please do not write on the QR codes at the bottom of each page.

We use those codes to identify which pages belong to each student.

Trigonometric Identities Reference

$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$

$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

$\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b)$

$\cos(a+b)-\cos(a-b) = -2\sin(a)\sin(b)$

$\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$

$\sin(a+b)-\sin(a-b) = 2\cos(a)\sin(b)$

$2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b)$

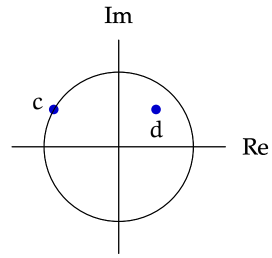
$2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$

$2\sin(a)\cos(b) = \sin(a+b)+\sin(a-b)$

$2\cos(a)\sin(b) = \sin(a+b)-\sin(a-b)$

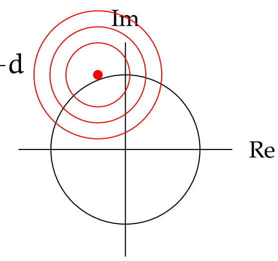
1 Complex Pair (24 points)

Let c and d represent the complex numbers shown by filled dots in the following diagram, where the real and imaginary parts of the complex numbers are shown on the horizontal and vertical axes, respectively, and the circle has a radius of 1.

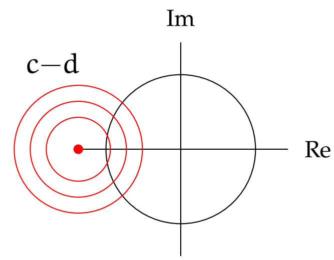


Below are eight complex-valued functions of c and d, each paired with a depiction of the complex plane demarked by the unit circle. Evaluate each expression and mark its value on the complex plane with a dot. Note that e represents Euler's number (2.71828...) and c^* represents the complex conjugate of c.

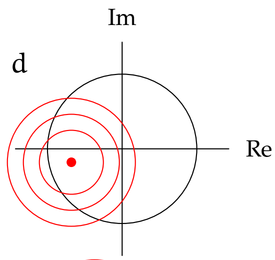
$c+d$



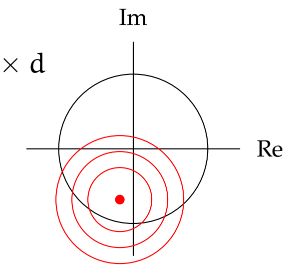
$c-d$



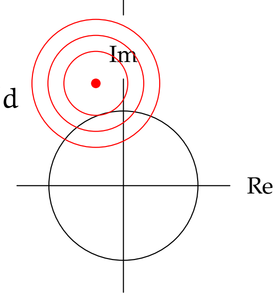
$c \times d$



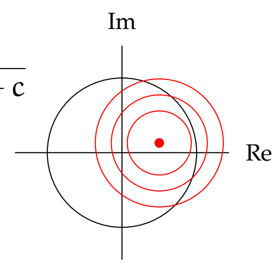
$c^* \times d$



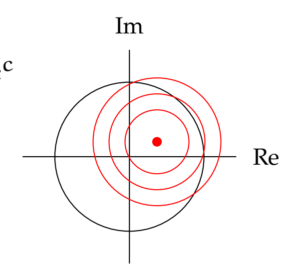
c/d



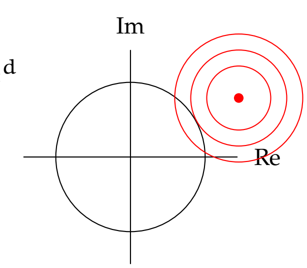
$\frac{1}{1-c}$



e^c



e^d



Worksheet (*intentionally blank*)

2 Complex Exponentials (18 points)

For each of the following parts, find a complex number that satisfies the given constraint. You need only provide one possible answer, even if multiple answers exist. If there are no possible answers, write **none** in the answer box.

Part a. Find a complex constant c_1 so that $\text{Re}\left(c_1 e^{j\omega t}\right) = \sin(\omega t)$ for all real numbers ω and t.

$c_1 =$ $-j$

$\text{Re}\left(-je^{j\omega t}\right) = \text{Re}\left(-j\cos(\omega t) + \sin(\omega t)\right) = \sin(\omega t)$

Part b. Find a complex constant c_2 so that $\text{Im}\left(c_2 e^{j\omega t}\right) = \cos(\omega t)$ for all real numbers ω and t.

$c_2 =$ j

$\text{Im}\left(je^{j\omega t}\right) = \text{Im}\left(j\cos(\omega t) - \sin(\omega t)\right) = \cos(\omega t)$

Part c. Find a complex constant c_3 so that $\text{Re}\left(c_3 e^{j\omega t}\right) = \cos(\omega t) + \sin(\omega t)$ for all real numbers ω and t.

$c_3 =$ $1-j$

$\text{Re}\left((1-j)e^{j\omega t}\right) = \text{Re}\left((1-j)\cos(\omega t) + j(1-j)\sin(\omega t)\right) = \cos(\omega t) + \sin(\omega t)$

Part d. Find a complex constant c_4 so that $\text{Re}\left(c_4 e^{j\omega t}\right) = A\cos(\omega t) + B\sin(\omega t)$ for all real numbers ω and t.

$c_4 =$ $A-jB$

$\text{Re}\left((A-jB)e^{j\omega t}\right) = \text{Re}\left((A-jB)\cos(\omega t) + j(A-jB)\sin(\omega t)\right) = A\cos(\omega t) + B\sin(\omega t)$

Part e. Find a complex constant c_5 so that $\text{Re}\left(c_5 e^{j\omega t}\right) = \cos(\omega t - \phi)$ for all real numbers ω and t.

$c_5 =$ $e^{-j\phi}$

$\text{Re}\left(e^{-j\phi}e^{j\omega t}\right) = \text{Re}\left(e^{j(\omega t-\phi)}\right) = \text{Re}\left(\cos(\omega t-\phi) + j\sin(\omega t-\phi)\right) = \cos(\omega t-\phi)$

Part f. Find a complex constant c_6 so that $\text{Re}\left(c_6 e^{j\omega t}\right) = \text{Im}\left(c_6^* e^{j\omega t}\right)$ for all real numbers ω and t.

$c_6 =$ $a(1-j)$ for any real number a

$\text{Re}\left(a(1-j)e^{j\omega t}\right) = a\cos(\omega t) + a\sin(\omega t) = \text{Im}\left(a(1+j)e^{j\omega t}\right)$

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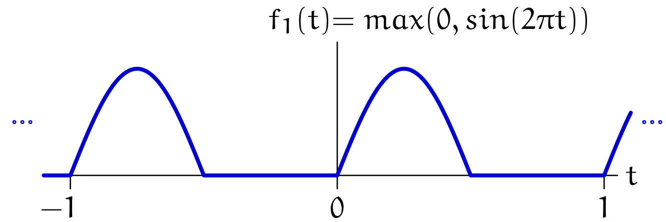
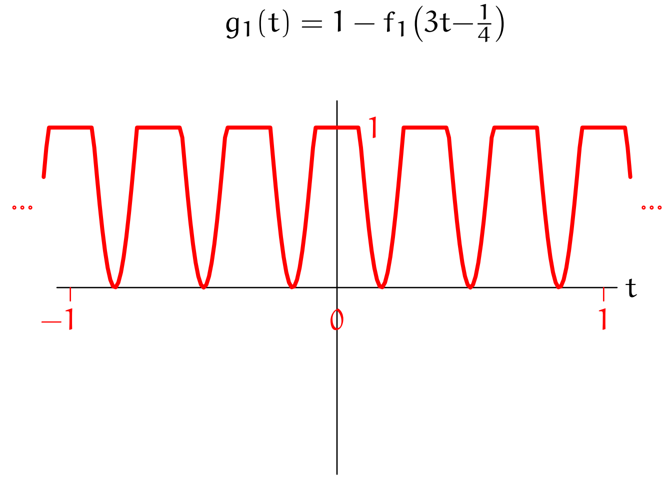
3 Peaks and Valleys (26 Points)

Let $f_1(t)$ represent the following periodic, continuous-time signal, with period $T = 1$:

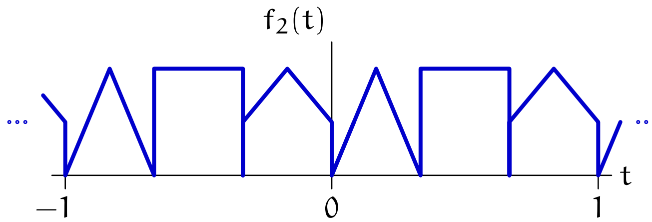


Let $g_1(t) = 1 - f_1\left(3t - \frac{1}{4}\right)$.

Sketch $g_1(t)$ on the following axes. Label the important parameters of your plot.



Let $f_2(t)$ represent the following periodic, continuous-time signal with period $T = 1$:



Let $g_2(t) = 1 - f_2\left(3t - \frac{1}{4}\right)$.

Let $F_2[k]$ and $G_2[k]$ represent the Fourier series coefficients for $f_2(t)$ and $g_2(t)$, respectively, where both series are computed **with the same period** $T = 1$. Determine expressions for each of $G_2[0]$ through $G_2[15]$ in terms of the Fourier coeeficients $F_2[k]$. Your table entries can contain real and/or imaginary numbers and constants such as e and π. Your entries should not contain integrals or infinite sums.

k	$G_2[k]$
0	$1 - F[0]$
1	0
2	0
3	$j\,F[1]$
4	0
5	0
6	$F[2]$
7	0

k	$G_2[k]$
8	0
9	$-j\,F[3]$
10	0
11	0
12	$-F[4]$
13	0
14	0
15	$j\,F[5]$

$$F_2[k] = \frac{1}{T}\int_T f(t)e^{-j\frac{2\pi k}{T}t}dt$$

$$G_2[k] = \frac{1}{T}\int_T g(t)e^{-j\frac{2\pi k}{T}t}dt = \frac{1}{T}\int_T \left(1 - f\left(3t - \frac{1}{4}\right)\right)e^{-j\frac{2\pi k}{T}t}dt$$

$$= \frac{1}{T}\int_T e^{-j\frac{2\pi k}{T}}\,dt - \frac{1}{T}\int_T f\left(3t - \frac{1}{4}\right)e^{-j\frac{2\pi k}{T}t}dt$$

Let $\tau = 3t - 1/4$. Then $d\tau = 3dt$.

$$G_2[k] = \delta[k] - \frac{1}{T}\int_{3T} f(\tau)e^{-j\frac{2\pi k}{T}(\frac{\tau}{3}+\frac{1}{12})}\frac{1}{3}d\tau$$

$$= \delta[k] - e^{-j\frac{2\pi k}{12T}}\frac{1}{3T}\int_{3T} f(\tau)e^{-j\frac{2\pi k}{T}(\frac{\tau}{3})}d\tau$$

$$= \delta[k] - e^{-j\frac{2\pi k}{12T}}\,F_2[k/3]$$

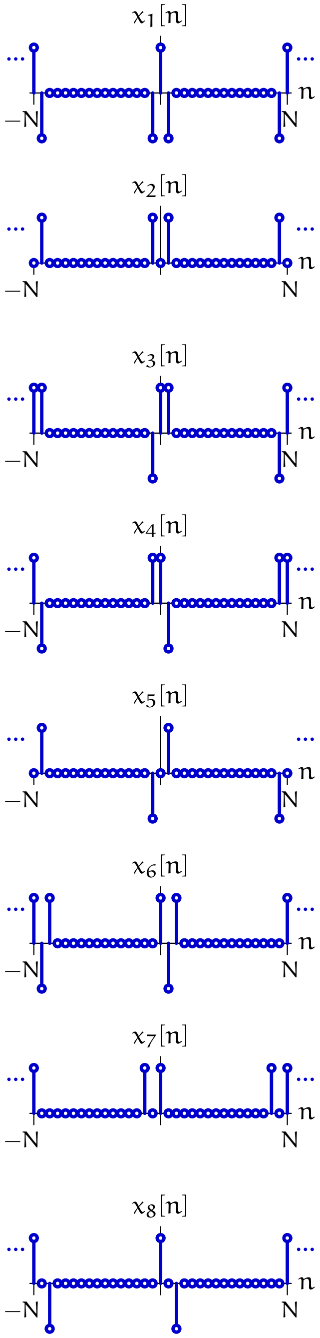
Notice that the $\delta[k]$ term contributes 1 if $k = 0$ and 0 otherwise. Also notice that $G_2[k] = 0$ unless $k \bmod 3 = 0$.

One way to think about this is that the period of $f(t)$ is 1 second, and therefore $f(t)$ can be expressed as a sum of harmonics that are integer multiples of 1 Hz. The $g(t)$ signal is a compressed version of $f(t)$, so the harmonics of $g(t)$ are spread out by a factor of three.

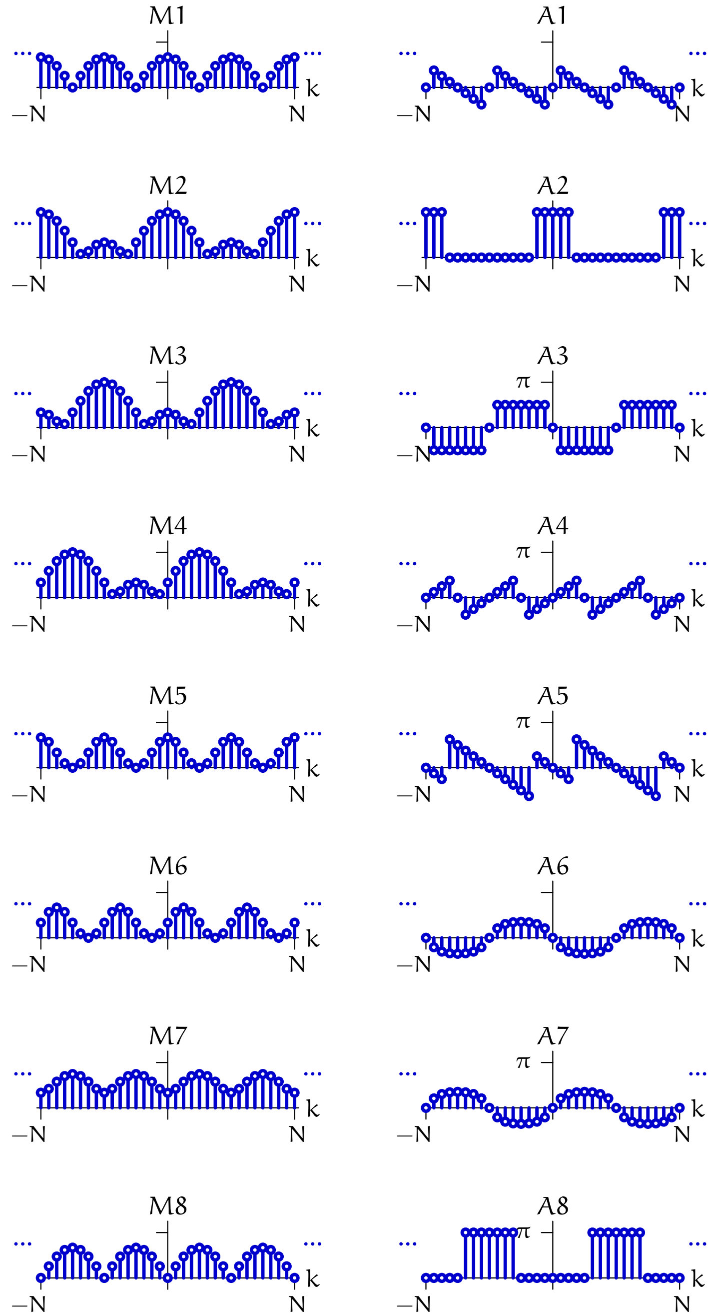
A second way to think about this is by looking at the $g(t)$ function itself. Since $g(t)$ is periodic in 1/3 second, we should be expecting that the Fourier series for $g(t)$ should only contain integer multiples of 3 Hz.

4 Fourier Series Matching (32 Points)

Each of the signals $x_i[n]$ in the left column below is periodic with period $N = 16$. Find the Fourier series coefficients $X_i[k]$ for each signal and then identify which of plots $M1 - M8$ shows the magnitude of $X_i[k]$ and which of plots $A1 - A8$ shows the angle of $X_i[k]$ as functions of k. Enter your answers in the boxes provided.



Signal	M:	A:
$x_1[n]$	M3	A2
$x_2[n]$	M1	A8
$x_3[n]$	M7	A6
$x_4[n]$	M7	A7
$x_5[n]$	M8	A3
$x_6[n]$	M3	A5
$x_7[n]$	M1	A4
$x_8[n]$	M8	A1



Worksheet (intentionally blank)

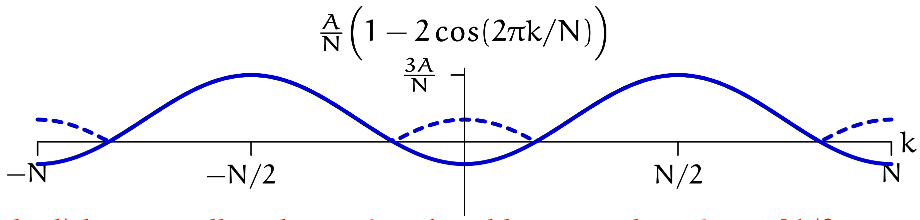
Find the Fourier series representation $X[k]$ for each of the given $x[n]$ using the analysis equation.

$$X[k] = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-j\frac{2\pi}{N}kn}$$

We will assume (arbitrarily) that the values of $x[n]$ are $-A$, 0, or A.

Since the signals are all real-valued, the corresponding magnitudes will be symmetric about $k = 0$. This eliminates M_4 and M_6.

Part 1. $X_1[k] = \frac{A}{N}(-e^{j\frac{2\pi}{N}k} + 1 - e^{-j\frac{2\pi}{N}k}) = \frac{A}{N}(1 - 2\cos(2\pi k/N))$



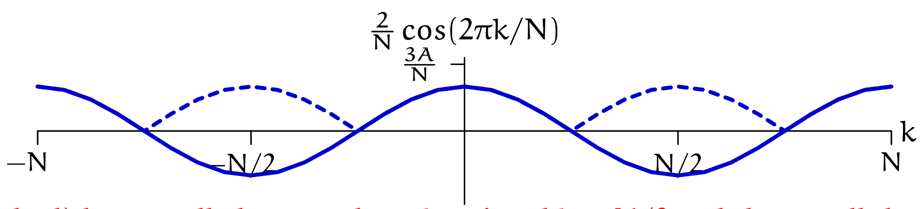
The magnitude (dashed) has a small peak near $k = 0$ and larger peaks at $k = \pm N/2$.

Answer = M3.

The angle is 0 for the range of k near $N/2$ (where there is no dashed line) and π for the range of k near 0, where the solid and dashed lines differ in sign).

Answer = A2.

Part 2. $X_2[k] = \frac{1}{N}(e^{j\frac{2\pi}{N}k} + e^{-j\frac{2\pi}{N}k}) = \frac{2}{N}\cos(2\pi k/N)$



The magnitude (dashed) has equally large peaks at $k = 0$ and $k = N/2$ and sharp nulls between.

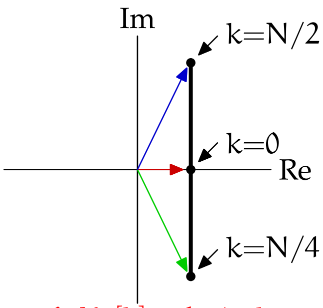
Answer = M1.

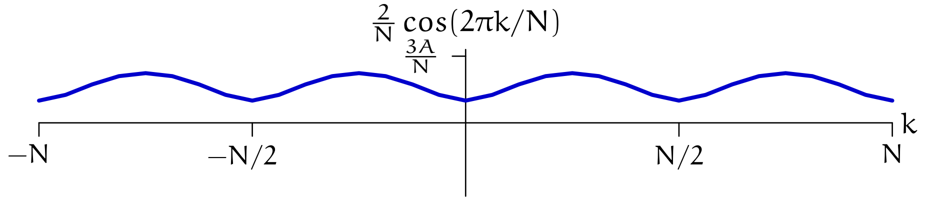
The angle is 0 for the range of k near $k = 0$ and π for the range of k near $N/2$ (where the dashed and solid curves differ in sign).

Answer = A8.

Part 3. $X_3[k] = \frac{1}{N}(-e^{j\frac{2\pi}{N}k} + 1 + e^{-j\frac{2\pi}{N}k}) = \frac{1}{N}\Big(1 - 2j\sin(2\pi k/N)\Big)$

This part is a bit trickier to plot since (unlike parts 1 and 2) this one has both real and imaginary parts.





When $k = 0$, $X_3[k]$ is 1. As k increases, the imaginary part of $X_3[k]$ gets increasingly negative – from 0 at $k = 0$ to -2 at $k = N/4$. Correspondingly, the magnitude increases from 1 at $k = 0$ to $\sqrt{5}$ at $k = N/4$.

As k increases from $N/4$ to $N/2$, the imaginary part of $X_3[k]$ change from -2 to 0 and the magnitude drops from $\sqrt{5}$ back to 1.

The plot of magnitude is not exactly sinusoidal, but it is smooth and does not have sharp notches.

Answer = M7.

The angle of $X_3[k]$ start at 0 for $k = 0$ and gradually decreases (going negative) for k between 0 and $N/4$. As k increases from $N/4$ to $N/2$, the angle decreases back to zero. The pattern from $N/2$ to N is similar to the pattern from 0 to $N/2$ except that the sign is now flipped to positive.

Answer = A6.

Part 4. $X_4[k] = \frac{1}{N}(e^{j\frac{2\pi}{N}k} + 1 - e^{-j\frac{2\pi}{N}k}) = 1 + 2j\sin(2\pi k/N)$

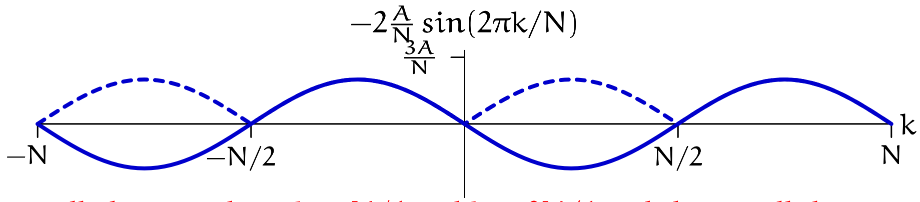
This part is similar to part 3, except that the imaginary part of $X_4[k]$ is the negative of that of $X_3[k]$.

Thus the magnitude is the same as part 3, i.e., M7.

The angle is the negative of that in part 3, i.e., A7.

Part 5. $X_5[k] = \dfrac{1}{N}(-e^{j\frac{2\pi}{N}k} + e^{-j\frac{2\pi}{N}k}) = -2j\sin(2\pi k/N)$

This part is purely imaginary.



The magnitude has equally large peaks at $k = N/4$ and $k = 3N/4$ and sharp nulls between.

Answer = M8.

The angle is $-\pi/2$ for $0 < k < N/2$ and $\pi/2$ for $N/2 < k < N$.

Answer = A3.

Part 6. $X_6[k] = \dfrac{1}{N}(1 - e^{-j\frac{2\pi}{N}k} + e^{-j\frac{2\pi}{N}2k}) = e^{-j2\pi k/N}(-1 + 2\cos(2\pi k/N))$

Notice that $x_6[n] = -x_1[n-1]$. Neither the delay nor the negation will affect the magnitude. Therefore the magnitude is given by M1.

To find the angle of $X_6[k]$, start with the angle of $X_1[k]$ (plot A2). Negating $x_1[n]$ adds π to all of the angles. As a result, the angle is 0 for k close to 0 and π otherwise.

Next consider the effect of the delay, which multiplies the Fourier transform by $e^{-j2\pi k/N}$. This delay adds an angle of $-2\pi k/N$ to each frequency point k. The resulting angle is shown in A5.

Part 7. $X_7[k] = \dfrac{1}{N}(e^{j\frac{2\pi}{N}2k} + 1) = e^{j2\pi k/N}(2\cos(2\pi k/N))$

$x_7[n]$ is a version of $x_2[n]$ that is shifted backwards in time by 1 sample. The time shift does not affect the magnitude. Therefore the magnitude is the same as part 2 – i.e., M1.

Without the delay, the angle would have been A8. The shift adds an angle of $2\pi k/N$ to each frequency point k, resulting in A4.

Part 8. $X_8[k] = \dfrac{1}{N}(1 - e^{-j\frac{2\pi}{N}2k}) = 2je^{-j2\pi k/N}\sin(2\pi k/N)$

This is a negated and delayed version of part 5. Neither the negation nor the delay affect the magnitude, which is therefore given by M8.

Without the delay, the angle would have been $\pi/2$ for $0 \leqslant k \leqslant N/2$ and $-\pi/2$ for $N/2 \leqslant k \leqslant N$. The delay adds a downward sloping phase, resulting in plot A1.

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