6.003 Final Exam Spring 2022

Name: Solutions

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This quiz is closed book, but you may use three 8.5 × 11 sheets of notes (six sides).

You may NOT use any electronic devices (such as calculators and phones).

If you have questions, please **come to us** at the front of the room to ask.

Please enter all solutions in the boxes provided.

Work on other pages with QR codes will be considered for partial credit.
Please provide a note if you continue work on worksheets at the end of the exam.

Please do not write on the QR codes at the bottom of each page.

We use those codes to identify which pages belong to each student.

Trigonometric Identities Reference

$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$

$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b)$

$\cos(a+b)-\cos(a-b) = -2\sin(a)\sin(b)$

$\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$

$\sin(a+b)-\sin(a-b) = 2\cos(a)\sin(b)$

$2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b)$

$2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$

$2\sin(a)\cos(b) = \sin(a+b)+\sin(a-b)$

$2\cos(a)\sin(b) = \sin(a+b)-\sin(a-b)$

1 Frequency Responses (20 points)

Part a. Determine an expression for the frequency response of a discrete-time system with unit sample response

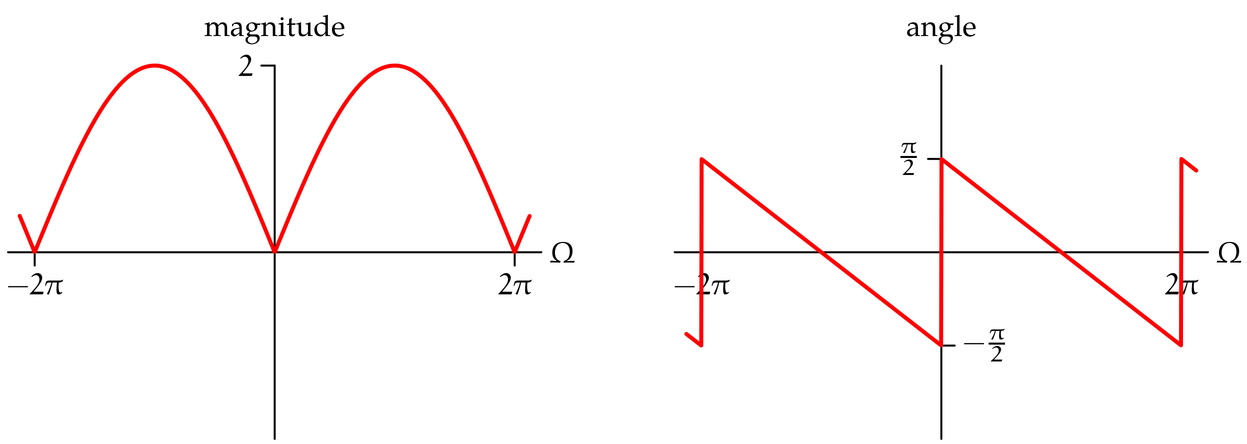
$h[n] = \delta[n] - \delta[n-1]$

Enter your expression in the box below.

$H(\Omega) =$ $1 - e^{-j\Omega} = 2je^{-j\Omega/2}\sin(\Omega/2)$

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} = 1 - e^{-j\Omega} = e^{-j\Omega/2}\left(e^{j\Omega/2} - e^{-j\Omega/2}\right) = 2je^{-j\Omega/2}\sin(\Omega/2)$$

Sketch the magnitude and angle of the frequency response on the axes below. Label the key points.



The magnitude of the frequency response is $2\left|\sin(\Omega/2)\right|$.

The angle of the frequency response is determined by three factors. First, the j in $2je^{-j\Omega/2}\sin(\Omega/2)$ contributes $+\pi/2$. Second, $\sin(\Omega/2)$ is negative when $-2\pi < \Omega < 0$. Third, the phase term $e^{-j\Omega/2}$ contributes a linear term that decreases in proportion to $\Omega/2$. When Ω is a small positive number, only the first factor contributes, so the angle of the frequency response is $\pi/2$. As Ω increases, the linear term decreases the angle of the frequency response until $\Omega = 2\pi$. At that point, the sign of $\sin(\Omega/2)$ becomes negative, so the phase jumps by π. Then the cycle repeats. Thus these three factors combine to give rise to the sawtooth function above.

Part b. Determine an expression for the frequency response of a continuous-time system with impulse response

$$h(t) = \begin{cases} 1 & \text{if } 0 \leqslant t < 1 \\ -1 & \text{if } 1 \leqslant t < 2 \\ 0 & \text{otherwise} \end{cases}$$

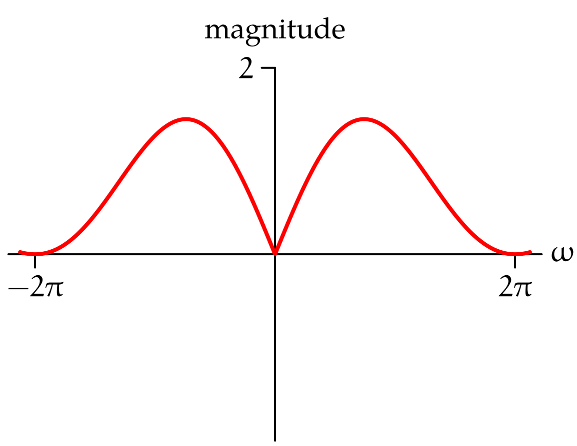
Enter your expression in the box below.

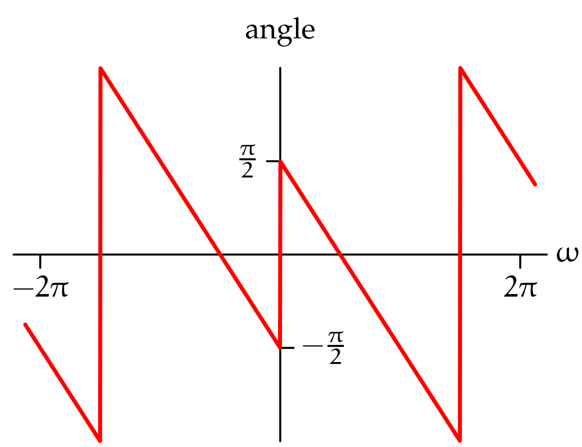
$$H(\omega) = \boxed{\frac{1}{j\omega}\left(1-2e^{-j\omega}+e^{-j2\omega}\right) = \frac{2}{j\omega}e^{-j\omega}\left(\cos(\omega)-1\right)}$$

$$H(\omega)\int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \int_0^1 e^{-j\omega t}dt - \int_1^2 e^{-j\omega t}dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_0^1 - \left.\frac{e^{-j\omega t}}{-j\omega}\right|_1^2$$

$$= \frac{1}{j\omega}\left(1-2e^{-j\omega}+e^{-j2\omega}\right) = \frac{2}{j\omega}e^{-j\omega}\left(\cos(\omega)-1\right)$$

Sketch the magnitude and angle of the frequency response on the axes below. Label the key points.





The magnitude of the frequency response is $2(1-\cos(\omega))/\omega$.

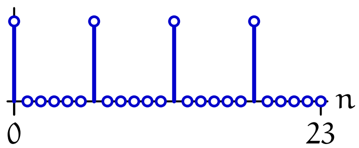
The angle of the frequency response is determined by four factors. First, $\cos(\omega)-1$ is never positive. Therefore, this term contributes π. Second, the j in the denominator contributes $-\pi/2$. Third, the phase term $e^{-j\omega}$ contributes a linear term that decreases in proportion to ω. Fourth, the ω in the denominator contributes π for $\omega < 0$.

When ω is a small positive number, only the first two factors contribute, and the net angle is $\pi/2$. As ω increases, the linear term decreases the angle of the frequency response until $\omega = 3\pi/2$. At that point, the angle is less than $-\pi$ and therefore wraps to $+\pi$. As ω increases above 2π, the sign of 1ω flips, and this changes the phase by π.

The angle function in part a was periodic in 2π. This angle function is not.

2 Discrete Fourier Transform Matching (21 points)

Each of the following plots shows the first 24 samples of a discrete-time signal. Find the plot on the following page that corresponds to the 24-point Discrete Fourier Transform (DFT) for each of these signals. Enter the letter of the plot (A-N) in the box provided.

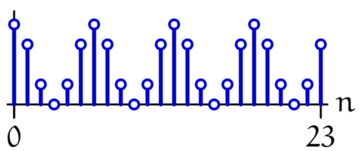


G

$x_1[n] = \delta[n] + \delta[n-6] + \delta[n-12] + \delta[n-18]$

$$X_1[k] = \frac{1}{24}\sum_{n=0}^{23} x[n]e^{-j2\pi kn/24} = 1 + e^{-j2\pi k/4} + e^{-j4\pi k/4} + e^{-j6\pi k/4} = 1 + (-j)^k + (-1)^k + j^k$$

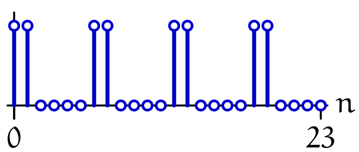
$$= \begin{cases} 1/6 & \text{if } k = 0, 4, 8, 12, 16, 20 \\ 0 & \text{otherwise} \end{cases}$$



I

$$x_2[n] = 1 + \cos(2\pi n/6) = 1 + \frac{1}{2}e^{j2\pi n/6} + \frac{1}{2}e^{-j2\pi n/6}$$

$$X_2[k] = \delta[k] + \frac{1}{2}\delta[k-4] + \frac{1}{2}\delta[k+4] = \delta[k] + \frac{1}{2}\delta[k-4] + \frac{1}{2}\delta[k-20]$$



J

$x_3[n] = x_1[n] + x_1[n-1]$

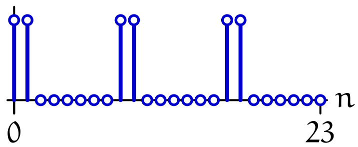
$$X_3[k] = \left(1 + e^{-j2\pi k/24}\right) X_1[k] = 2e^{-j\pi k/24}\cos(\pi k/24)X_1[k]$$

Similar to plot G except components near $k = 12$ are attenuated.

B

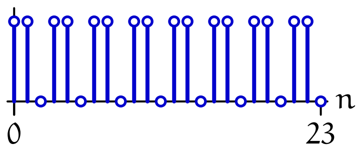
$$x_4[n] = \delta[n] + \frac{1}{2}\delta[n-4] + \frac{1}{2}\delta[n-20]$$

$$X_4[n] = \frac{1}{24} + \frac{1}{48}e^{-j2\pi k/6} + \frac{1}{48}e^{j2\pi k/6} = \frac{1}{24}\Big(1 + \cos(2\pi k/6)\Big)$$



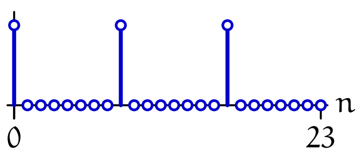
M

x_5 is imilar to x_3 except the period is 8 instead of 6. Therefore X_5 has non-zero components at $k = 0, 3, 6, \ldots$ and components near $k = 12$ are attenuated.



L

x_6 is similar to x_3 except the period is 3 instead of 6. Therefore X_6 has non-zero components at $k = 0, 8, 16$ and components near $k = 12$ are attenuated.



H

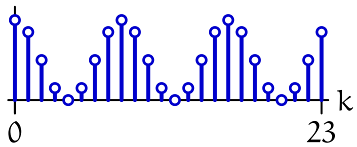
$$x_7[n] = \delta[n] + \delta[n-8] + \delta[n-16]$$

$$X_7[k] = \frac{1}{24}\sum_{n=0}^{23} x[n]e^{-j2\pi kn/24} = 1 + e^{-j2\pi k/3} + e^{-j4\pi k/3}$$

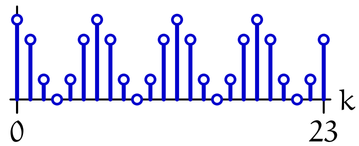
$$= \begin{cases} 1/6 & \text{if } k = 0, 3, 6, 9, 12, 15, 18, 21 \\ 0 & \text{otherwise} \end{cases}$$

Each of the following plots shows the magnitude of a DFT computed with an analysis window $N = 24$. The vertical scale for each plot is different: it has been normalized so that the peak value in each plot is 1.

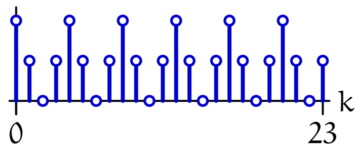
A



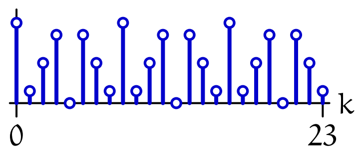
B



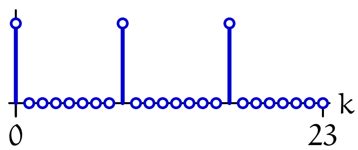
C



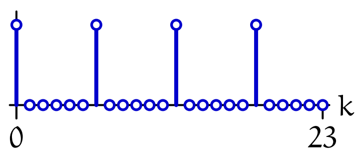
D



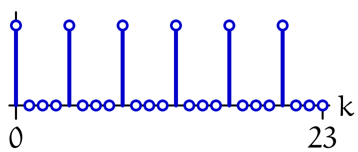
E



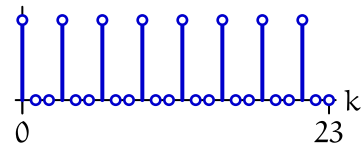
F



G

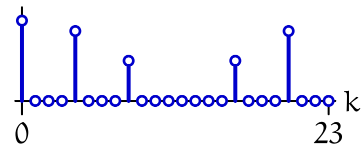


H

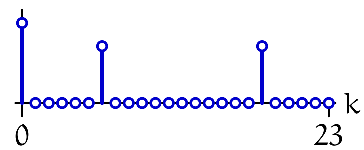


I

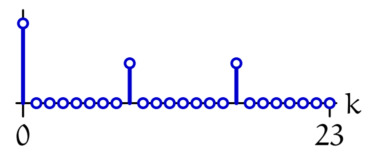
J



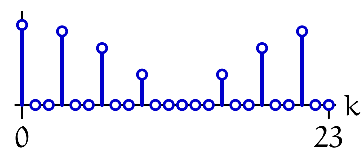
K



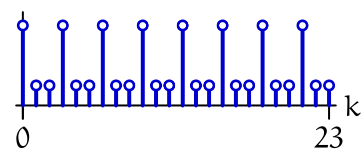
L



M



N

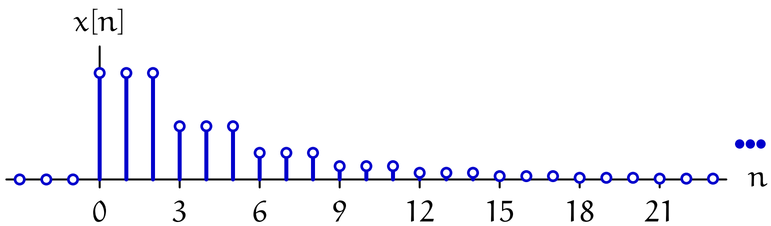


3 Steps (23 points)

Part 1. Let $x[n]$ represent the following discrete-time signal

$$x[n] = \begin{cases} 0 & \text{for } n < 0 \\ a^0 & \text{for } n = 0,1,2 \\ a^1 & \text{for } n = 3,4,5 \\ a^2 & \text{for } n = 6,7,8 \\ \cdots \end{cases}$$

where a is a real number between 0 and 1, as shown in the plot below.



Determine a closed form expression for $X(\Omega)$, which is the discrete-time Fourier transform of $x[n]$.

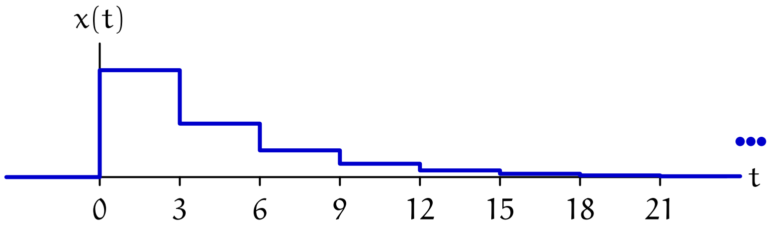
$$X(\Omega) = \boxed{\frac{1+e^{-j\Omega}+e^{-j2\Omega}}{1-ae^{-j3\Omega}}}$$

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \\ &= \sum_{m=0}^{\infty} a^m \left(e^{-j\Omega 3m} + e^{-j\Omega(3m+1)} + e^{-j\Omega(3m+2)}\right) \\ &= \sum_{m=0}^{\infty} a^m e^{-j\Omega 3m}\left(1+e^{-j\Omega}+e^{-j2\Omega}\right) \\ &= \frac{1+e^{-j\Omega}+e^{-j2\Omega}}{1-ae^{-j3\Omega}} \end{aligned}$$

Part 2. Let $x(t)$ represent the following continuous-time signal

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ a^0 & \text{for } 0 \leqslant t < 3 \\ a^1 & \text{for } 3 \leqslant t < 6 \\ a^2 & \text{for } 6 \leqslant t < 9 \\ \ldots \end{cases}$$

where a is a real number between 0 and 1, as shown in the plot below.



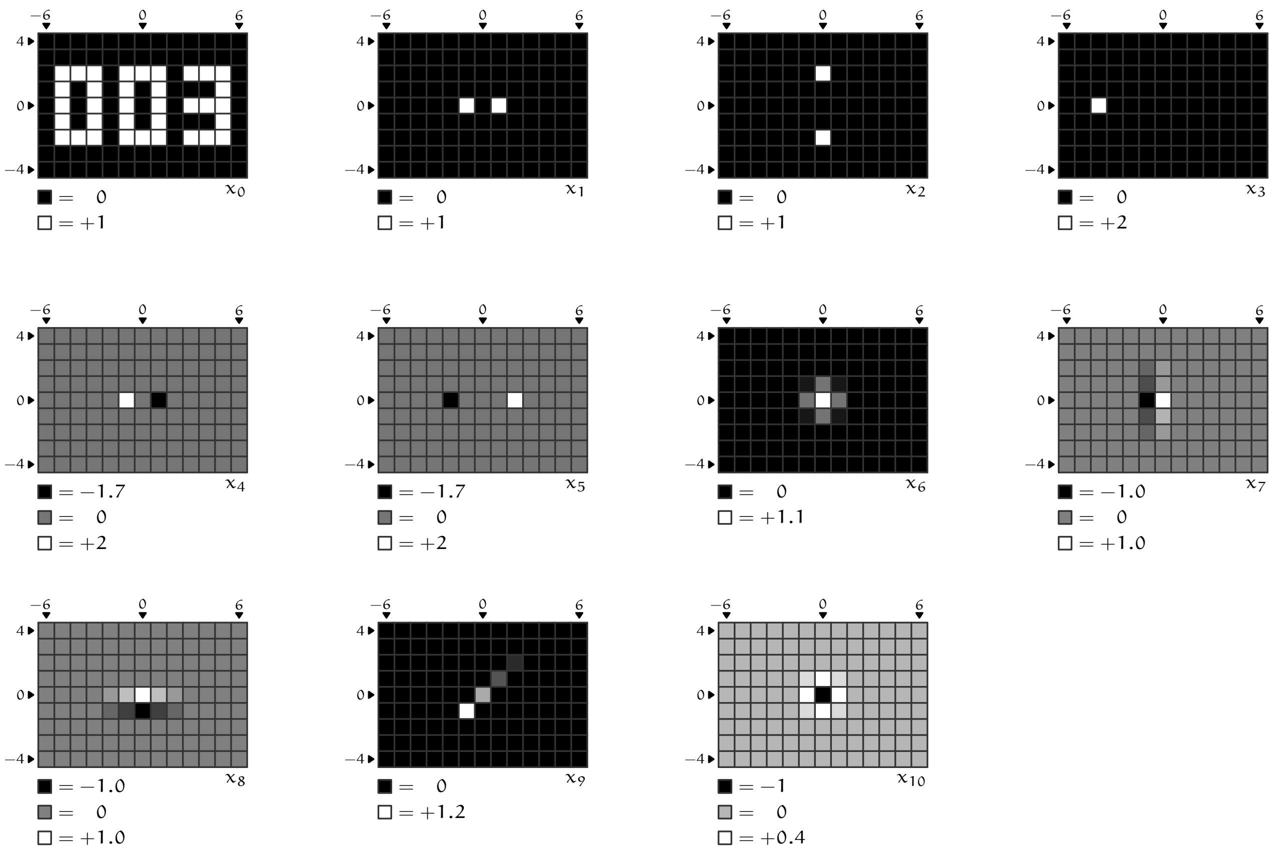
Determine a closed-form expression for $X(\omega)$, which is the continuous-time Fourier transform of $x(t)$.

$$X(\omega) = \boxed{\left(\frac{1}{j\omega}\right)\left(\frac{1-e^{-j3\omega}}{1-ae^{-j3\omega}}\right)}$$

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \\ &= \int_0^3 a^0 e^{-j\omega t}\,dt + \int_3^6 a^1 e^{-j\omega t}\,dt + \int_6^9 a^2 e^{-j\omega t}\,dt + \cdots \\ &= a^0\left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^3 + a^1\left[\frac{e^{-j\omega t}}{-j\omega}\right]_3^6 + a^2\left[\frac{e^{-j\omega t}}{-j\omega}\right]_6^9 + \cdots \\ &= -\frac{1}{j\omega}\left(a^0\left(e^{-j\omega 3} - e^{-j\omega 0}\right) + a^1 e^{-j\omega 3}\left(e^{-j\omega 3} - e^{-j\omega 0}\right) + a^2 e^{-j\omega 6}\left(e^{-j\omega 3} - e^{-j\omega 0}\right)\right) \\ &= \left(\frac{1-e^{-j3\omega}}{j\omega}\right)\left(\sum_{m=0}^{\infty}\left(ae^{-j3\omega}\right)^m\right) \\ &= \left(\frac{1-e^{-j3\omega}}{j\omega}\right)\left(\frac{1}{1-ae^{-j3\omega}}\right) \end{aligned}$$

4 2D Convolution (20 Points)

For this problem, we will consider the following 2D signals, labeled x_0 through x_{10}, each of which is 9 rows $\times$ 13 columns. Note that the color scale is different between some of the signals.



For each circular convolution below, indicate which of the graphs on the facing page most closely matches the result by entering a single letter in each box. Note that, for each graph on the facing page, black corresponds to the lowest value in the signal (not necessarily 0), and white corresponds to the highest value in the signal (not necessarily 1).

$x_1 \circledast x_0$ F

$x_2 \circledast x_0$ D

$x_3 \circledast x_0$ L

$x_4 \circledast x_0$ C

$x_5 \circledast x_0$ G

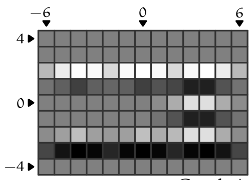
$x_6 \circledast x_0$ K

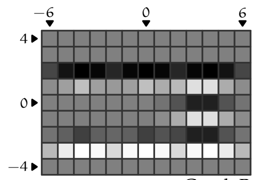
$x_7 \circledast x_0$ I

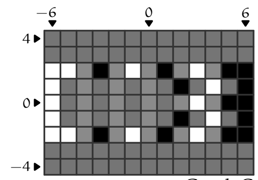
$x_8 \circledast x_0$ A

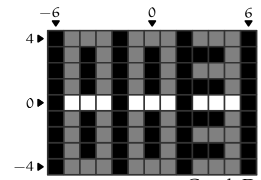
$x_9 \circledast x_0$ E

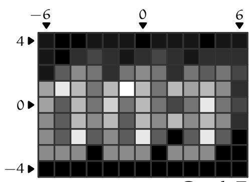
$x_{10} \circledast x_0$ J

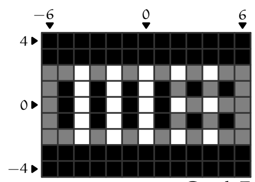

Graph A

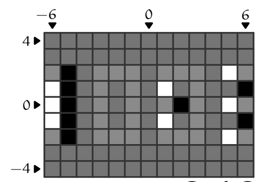

Graph B

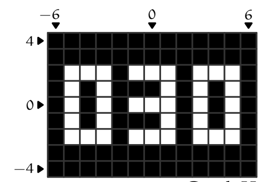

Graph C

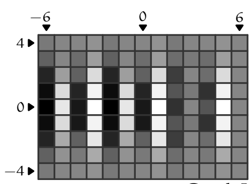

Graph D

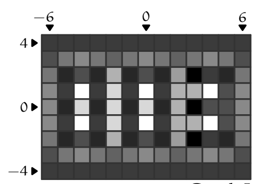

Graph E

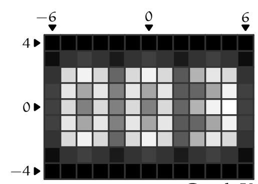

Graph F

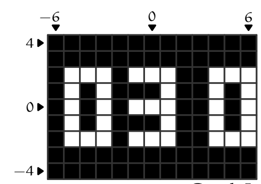

Graph G

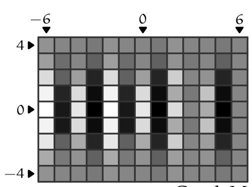

Graph H

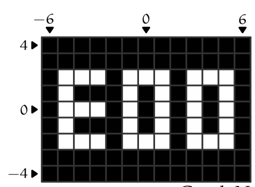

Graph I

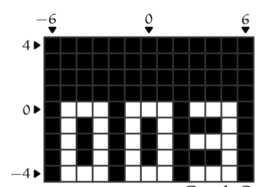

Graph J

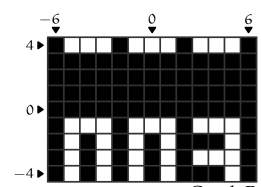

Graph K


Graph L


Graph M

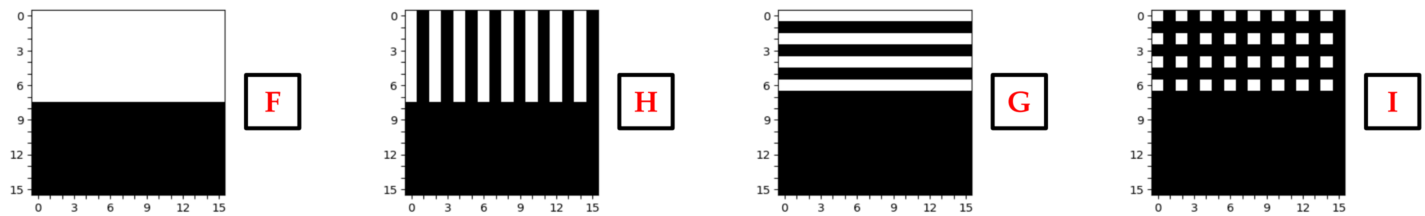

Graph N


Graph O

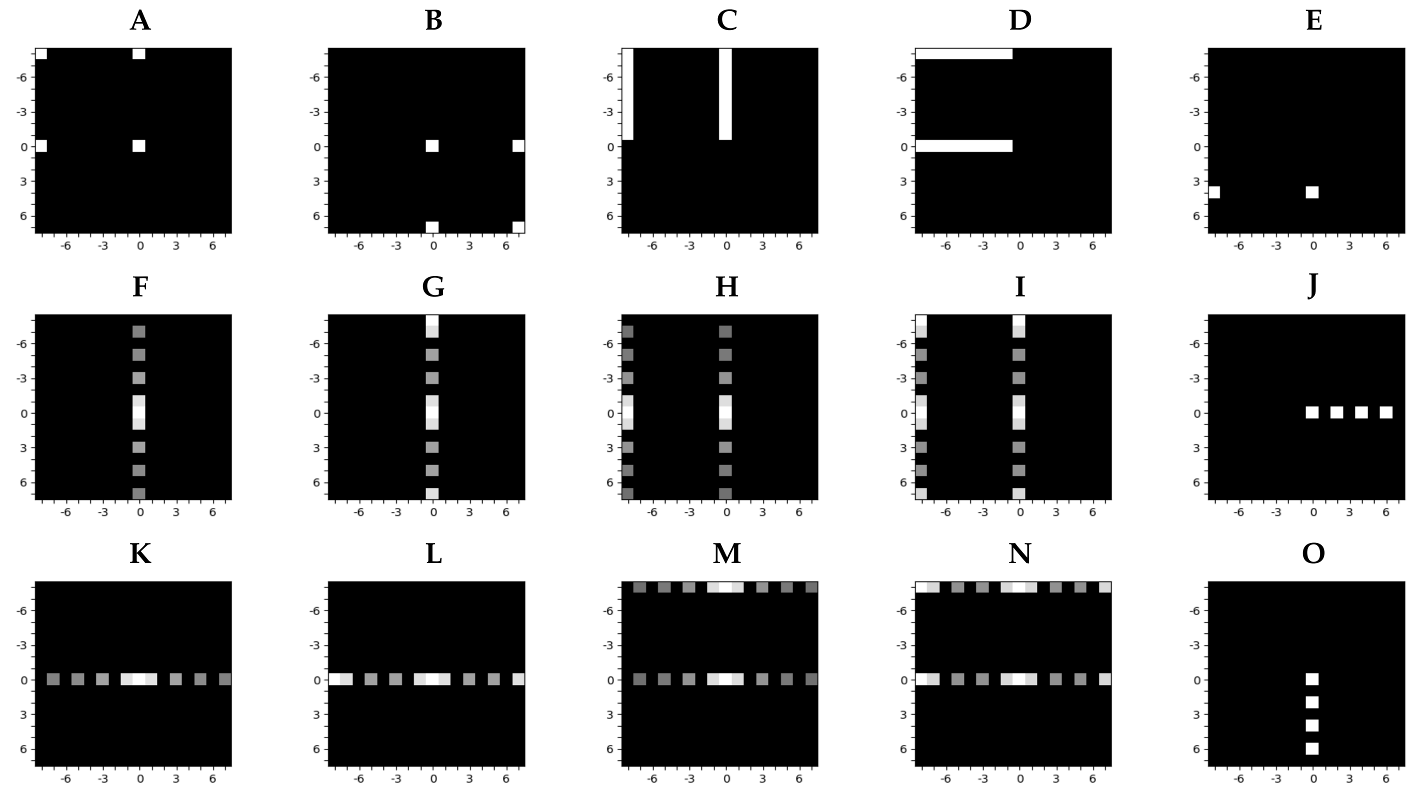

Graph P

5 2D Discrete Fourier Transforms (16 points)

Each of the following images contain 16×16 pixels that are either black (representing a value of 0) or white (representing a value of 1).



Determine which of the following images (**A-O**) shows the magnitude of the 2D DFT of each of the preceding images, and enter that letter in the corresponding box above.



In images A-O, white pixels represent the most positive magnitude in that image and black pixels a magnitude of 0. Notice that the zero-location inf images A-O is near the center of the image.

Left panel: Start by taking the 1D DFT of each row. For rows 0 through 7, the DFT is $\delta[k_c]$. For rows 8 through 15, the DFT is 0.

Now take the 1D DFT of each of the resulting columns. There is only one non-trivial column: $k_c = 0$. Values for this column are 1 if $r < 8$ and 0 otherwise.

$$F[k_r, 0] = \frac{1}{16}\sum_{r=0}^{7} e^{-j2\pi k_r r/16} = \frac{1}{16}\left(\frac{1-e^{-j2\pi k_r 8/16}}{1-e^{-j2\pi k_r/16}}\right) = \frac{1}{16}\left(\frac{1-(-1)^{k_r}}{1-e^{-j2\pi k_r/16}}\right)$$

Notice that the numerator is 0 for even values of k_r, and that the denominator is 0 at $k_r = 0$. The ratio is indeterminant at $k_r = 0$, but is easily seen to be 8/16 from the original sum over r.

The only non-zero values of the 2D DFT result with $k_c = 0$ and $|k_r| = 0, 1, 3, 5$, and 7. This corresponds to image F.

Second panel: Start by taking the 1D DFT of each row. For the first row, we get

$$F[k_c] = \sum_{\substack{c=0 \\ \text{even}}}^{15} e^{-j2\pi k_c c/16} = \sum_{m=0}^{7} e^{-j4\pi k_c m/16} = \frac{1-e^{-j4\pi k_c 8/16}}{1-e^{-j4\pi k_c/16}}$$

The numerator of this expression is 0 for all values of k_c. The denominator is 0 at $k_c = 0$ and at $k_c = 8$. The result is $\delta[k_c] + \delta[k_c - 8]$ for the top 8 rows.

Now take the 1D DFT of each column. The results for $k_c = 0$ are the same as the previous part. The results for $k_c = -8$ match those for $k_c = 0$, i.e.,

$$F[k_r, 0] = F[k_r, -8]$$

for all k_c. It follows that the answer is image H.

An even easier way to find this answer is to realize that the second panel can be generated by zeroing all of the odd numbered columns of the first panel. We discussed zeroing the odd numbered columns in the lecture and recitation on MRI, where we saw that multiplying by even numbered stripes is equivalent to convolving by $\delta[k_c] + \delta[k_c - C/2]$. By carrying out this operation on image F we get image H.

Third panel: Multiplying an image by horizontal, even-numbered stripes is equivalent to convolving in the frequency domain by $\delta[k_r] + \delta[k_r - R/2]$. By carrying out this operation on image F we get image G.

Right panel: Image I results by zeroing both the odd numbered rows and columns of the left panel. The resulting 2D DFT is therefore equal to panel F convolved with $\delta[k_r] + \delta[k_r - R/2]$ and by $\delta[k_c] + \delta[k_c - C/2]$. By carrying out this operation on image F we get image I.

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