6.3000 Final Exam Fall 2022

Name: Solutions

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This quiz is closed book, but you may use three 8.5 × 11 sheets of notes (six sides).

You may NOT use any electronic devices (such as calculators and phones).

If you have questions, please **come to us** at the front of the room to ask.

Please enter all solutions in the boxes provided.

Work on other pages with QR codes will be considered for partial credit.
Please provide a note if you continue work on worksheets at the end of the exam.

Please do not write on the QR codes at the bottom of each page.

We use those codes to identify which pages belong to each student.

Trigonometric Identities Reference

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$
$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b)$$
$$\cos(a+b)-\cos(a-b) = -2\sin(a)\sin(b)$$
$$\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$$
$$\sin(a+b)-\sin(a-b) = 2\cos(a)\sin(b)$$
$$2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b)$$
$$2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$$
$$2\sin(a)\cos(b) = \sin(a+b)+\sin(a-b)$$
$$2\cos(a)\sin(b) = \sin(a+b)-\sin(a-b)$$

1 The Third Element (30 points)

Each part of this problem describes a different discrete-time signal $f_i[n]$ and then asks you to determine the $k = 3$ component of the DFT of that signal, where the DFT is computed with analysis window $N = 16$:

$$F_i[3] = \frac{1}{16}\sum_{n=0}^{15} f_i[n]e^{-j2\pi 3n/16}$$

Part a. Let $f_1[n] = (-1)^n$ Enter a closed form expression for $F_1[3]$ in the box below.

$F_1[3] =$ 0

The basis functions for DFTs of length N=16 are of the form $e^{-j2\pi kn/16}$. Therefore $f_1[n] = (-1)^n = e^{-j2\pi 8n/16}$ contains a single basis function, and that basis function is at index k=8. Since the k=8 and k=3 basis functions are orthogonal, $F_1[3]$ must be zero.

Mathematically:

$$F_1[k] = \frac{1}{N}\sum_{n=0}^{N-1} f_1[n]e^{-j2\pi kn/N} = \frac{1}{N}\sum_{n=0}^{N-1}(-1)^n e^{-j2\pi kn/N} = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\pi n}e^{-j2\pi kn/N}$$
$$= \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\pi Nn/N}e^{-j2\pi kn/N} = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\pi(2k+N)n/N} = \frac{1}{16}\left(\frac{1-e^{-j\pi 22}}{1-e^{-j\pi 22/16}}\right) = 0$$

Part b. Let $f_2[n] = \cos(3\pi n/8)$. Enter a closed form expression for $F_2[3]$ in the box below.

$F_2[3] =$ $\frac{1}{2}$

$$f_2[n] = \cos(3\pi n/8) = \frac{e^{j6\pi n/16} + e^{-j6\pi n/16}}{2} = \sum_{k=0}^{15} F_2[k]e^{j2\pi kn/16}$$

Since the basis functions for the DFT are orthogonal, it follows that

$$F_2[k] = \begin{cases} \frac{1}{2} & \text{if } k = 3 \text{ or } k = 13 \\ 0 & \text{otherwise} \end{cases}$$

Part c. Let $f_3[n] = \cos(3\pi n/8 - 9\pi/8)$. Enter a closed form expression for $F_3[3]$ in the box below.

$F_3[3] =$ $\frac{1}{2}e^{-j2\pi 9/16}$

Since

$$f_3[n] = f_2[n-3]$$

it follows that

$$F_3[k] = e^{-j2\pi k3/16}F_2[k]$$

Therefore $F_3[3] = \frac{1}{2}e^{-j2\pi 9/16}$.

Part d. Determine a closed-form expression for $F_4[3]$ where

$$f_4[n] = (\delta[n+1] + \delta[n-2]) \times (\delta[n+1] - \delta[n-1])$$

$F_4[3] =$ $\frac{1}{16}e^{j3\pi/8}$

Multiplying $(\delta[n+1] + \delta[n-2]) \times (\delta[n+1] - \delta[n-1])$ results in a single delta function:

$$f_4[n] = \delta[n+1]$$

Therefore $F_4[k] = \frac{1}{16}e^{-j2\pi 2k/16} = \frac{1}{16}e^{-j\pi k/4}$ and $F_4[3] = \frac{1}{16}e^{-j3\pi/4}$.

Part e. Determine a closed form expression for $F_5[3]$ where

$$f_5[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$F_5[3] = \boxed{\frac{1}{16}\left(\frac{1-(\frac{1}{2})^{16}}{1-\frac{1}{2}e^{-j2\pi 3/16}}\right)}$$

$$F_5[k] = \frac{1}{N}\sum_{n=0}^{N-1}\left(\frac{1}{2}\right)^n e^{-j2\pi kn/N} = \frac{1}{N}\left(\frac{1-(\frac{1}{2})^N e^{-j2\pi kN/N}}{1-\frac{1}{2}e^{-j2\pi k/N}}\right) = \frac{1}{N}\left(\frac{1-(\frac{1}{2})^N}{1-\frac{1}{2}e^{-j2\pi k/N}}\right)$$

Substituting $k = 3$ and $N = 16$ yields

$$F_5[3] = \frac{1}{16}\left(\frac{1-(\frac{1}{2})^{16}}{1-\frac{1}{2}e^{-j2\pi 3/16}}\right)$$

Part f. Let $f_6[n]$ represent a discrete time signal whose Discrete Cosine Transform (DCT)[1] coefficients $F_{C6}[k]$ are given as follows:

$$F_{C6}[k] = \frac{1}{16}\cos\left(\frac{5\pi k}{32}\right)$$

Let $F_6[k]$ represent the Discrete Fourier Transform (DFT) of $f_6[n]$. Determine a closed-form expression for $F_6[3]$.

$$F_6[3] = \boxed{\frac{1}{16}e^{-j3\pi/4}}$$

Notice that $F_{C6}[k]$ is equal to the k^{th} basis function (as a function of k) when n=2. It follows from the DCT analysis equation that $f[n] = \delta[n-2]$ The DFT of $f[n] = \delta[n-2]$ is

$$F_6[k] = \frac{1}{16}\sum_{n=0}^{15}\delta[n-2]e^{-j2\pi kn/16} = \frac{1}{16}e^{-12\pi/16}$$

[1] **Discrete Cosine Transform (DCT)**

$$F_C[k] = \frac{1}{N}\sum_{n=0}^{N-1} f[n]\cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \qquad \text{(analysis)}$$

$$f[n] = F_C[0] + 2\sum_{k=1}^{N-1} F_C[k]\cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \qquad \text{(synthesis)}$$

Worksheet (intentionally blank)

2 Frequency Response (30 points)

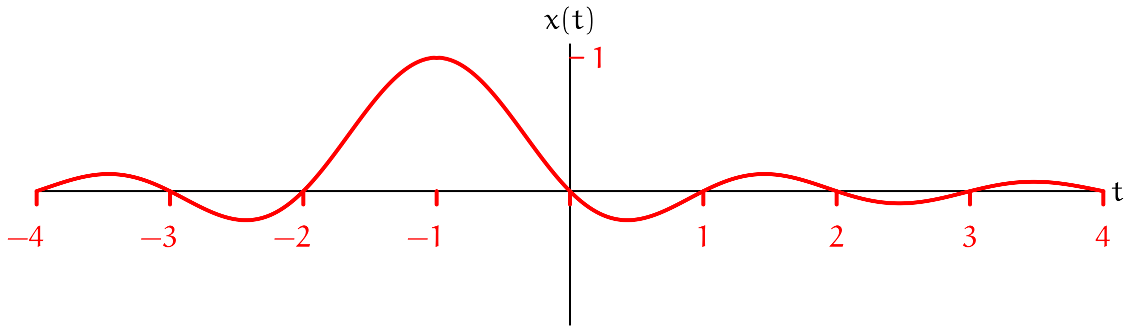
Let $X(\omega)$ represent the following Fourier Transform (CTFT) of the continuous-time signal $x(t)$.

$$X(\omega) = \begin{cases} e^{j\omega} & \text{if } -\pi \leqslant \omega \leqslant \pi \\ 0 & \text{otherwise} \end{cases}$$

Part a. Determine a closed-form expression for $x(t)$.

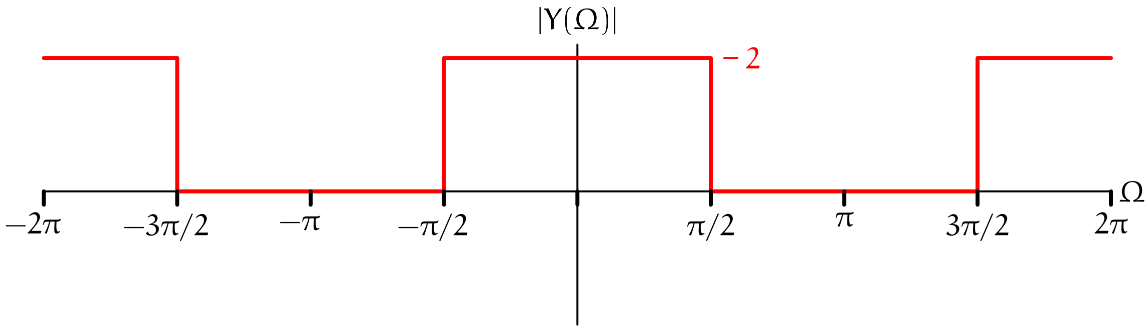
$$x(t) = \boxed{\frac{\sin(\pi(t+1))}{\pi(t+1)}}$$

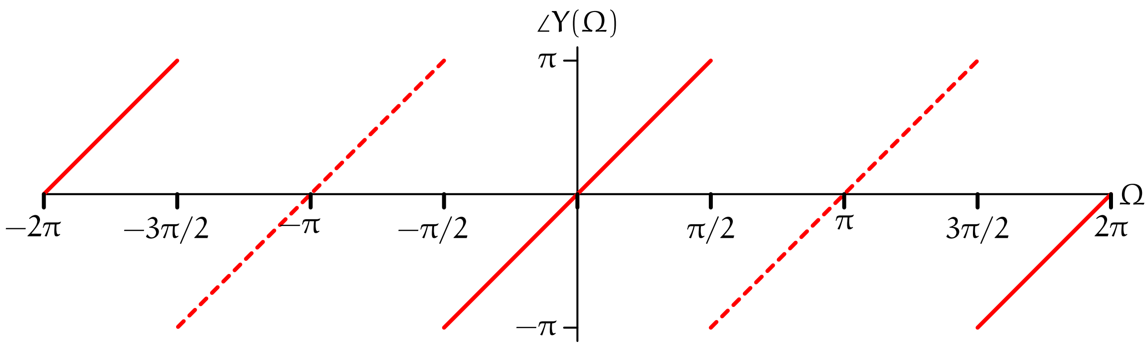
Plot $x(t)$ on the axes below. Indicate the times and values of all key features of your plot.



$$x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega}e^{j\omega t}d\omega = \frac{1}{2\pi}\left.\frac{e^{j\omega(t+1)}}{j(t+1)}\right|_{-\pi}^{\pi} = \frac{2j\sin(\pi(t+1))}{2\pi j(t+1)} = \frac{\sin(\pi(t+1))}{\pi(t+1)}$$

Part b. Let $y[n]$ represent the discrete-time signal that results from sampling $x(t)$ once every $\Delta = \frac{1}{2}$ second. Determine $Y(\Omega)$, which represents the discrete-time Fourier transform (DTFT) of $y[n]$. Plot the magnitude and angle of $Y(\Omega)$ on the axes below.





Determine an expression for $y[n]$.

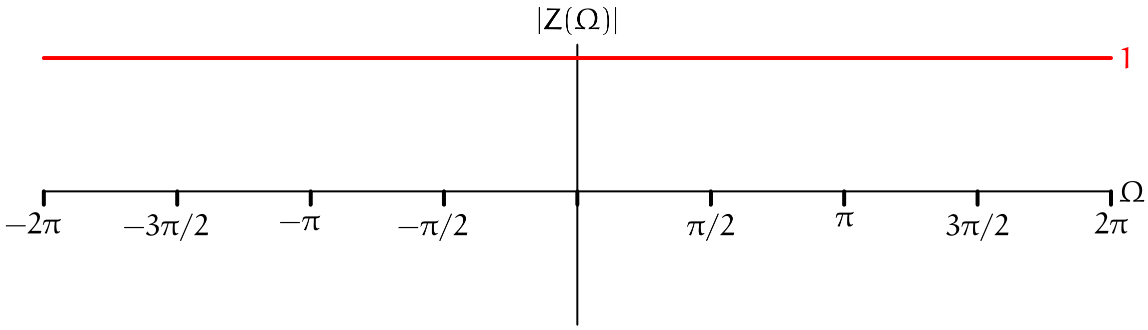
$$y[n] = \frac{\sin\frac{\pi(n+2)}{2}}{\frac{\pi(n+2)}{2}}$$

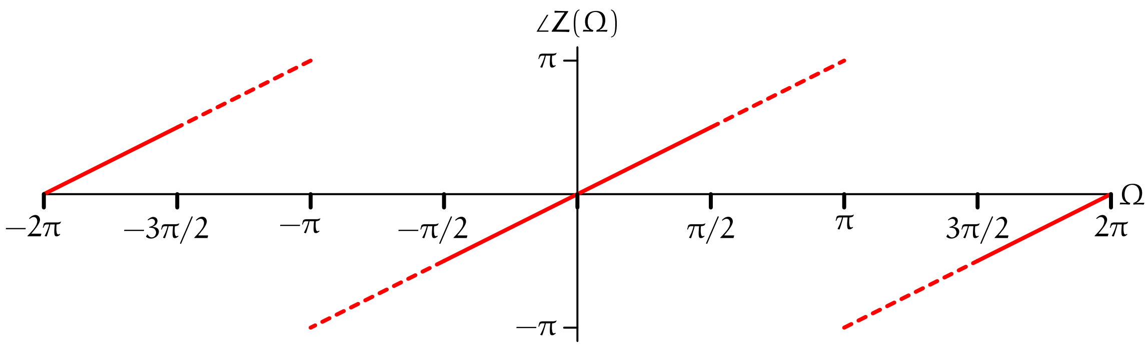
$$y[n] = x(n\Delta) = \left.\frac{\sin(\pi(t+1))}{\pi(t+1)}\right|_{t=n\Delta} = \frac{\sin(\pi(\frac{n}{2}+1))}{\pi(\frac{n}{2}+1)} = 2\frac{\sin(\frac{\pi}{2}(n+2))}{\pi(n+2)}$$

Since $y[n]$ is a sinc function of n, $Y(\Omega)$ is a lowpass filter with cutoff frequency $\frac{\pi}{2}$ and DC value 2. There is a also a time shift of 2 since n has been replaced by $n+2$. This time shift introduces a phase lead of 2Ω. The final answer is

$$Y(\Omega) = \begin{cases} 2e^{j2\Omega} & \text{if } -\pi < (\Omega+2\pi m) < \pi \text{ for some integer } m \\ 0 & \text{otherwise} \end{cases}$$

Part c. Let $z[n]$ represent the discrete-time signal that results from sampling $x(t)$ once every $\Delta = 1$ second. Determine $Z(\Omega)$, which represents the discrete-time Fourier transform (DTFT) of $z[n]$. Plot the magnitude and angle of $Z(\Omega)$ on the axes below.





Determine an expression for $z[n]$.

$$z[n] = \boxed{\frac{\sin(\pi(n+1))}{\pi(n+1)} = \delta[n+1]}$$

$$z[n] = x(n\Delta) = \left.\frac{\sin(\pi(t+1))}{\pi(t+1)}\right|_{t=nT} = \frac{\sin(\pi(n+1))}{\pi(n+1)} = \delta[n+1]$$

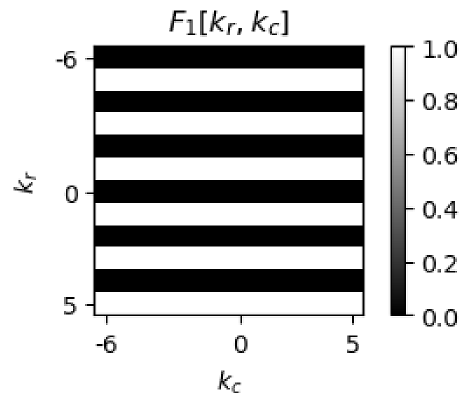
The DTFT of $\delta[n]$ is 1. The time shift introduces a phase lead of Ω. Thus

$$Z(\Omega) = e^{j\Omega}$$

3 Plaids (32 points)

Each part of this problem shows a 2D (12×12) Discrete Fourier Transform (DFT) $F_i[k_r, k_c]$ for which you must find the corresponding 2D signal $f_i[r, c]$. The values of the DFT $F_i[k_r, k_c]$ are real numbers that are displayed as brightnesses defined by the associated color bar.

Enter each non-zero value of $f_i[r, c]$ in the grid shown below the DFT. Empty cells in the grid will be taken as 0.



$f_1[r, c]$

row r \ column c	−6						0					5
−6							−72					
0							72					
5												

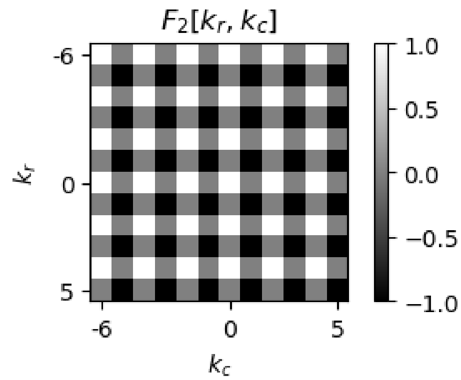
$F_1[k_r, k_c]$ can be written as $(1-\cos(\pi k_r))/2 = (1-e^{j\pi k_r})/2$. Substituting this into the synthesis equation yields

$$f_1[r,c] = \sum_{k_r=0}^{R-1}\sum_{k_c=0}^{C-1}\frac{1}{2}(1-e^{j\pi k_r})e^{j2\pi k_r r/R}e^{j2\pi k_c c/C}$$

$$= \frac{1}{2}\left(\sum_{k_r=0}^{R-1} e^{j2\pi k_r r/R}\right)\left(\sum_{k_c=0}^{C-1} e^{j2\pi k_c c/C}\right) - \frac{1}{2}\left(\sum_{k_r=0}^{R-1} e^{j2\pi k_r (r+R/2)/R}\right)\left(\sum_{k_c=0}^{C-1} e^{j2\pi k_c c/C}\right)$$

$$= \frac{1}{2}RC\,\delta[r]\delta[c] - \frac{1}{2}RC\,\delta[r+R/2]\delta[c] = \begin{cases} 72 & \text{if } k_r = k_c = 0 \\ -72 & \text{if } k_r = -6 \text{ and } k_c = 0 \\ 0 & \text{otherwise} \end{cases}$$

In the following image, white, gray, and black represent pixel values of 1, 0, and −1, respectively.



$f_2[r, c]$

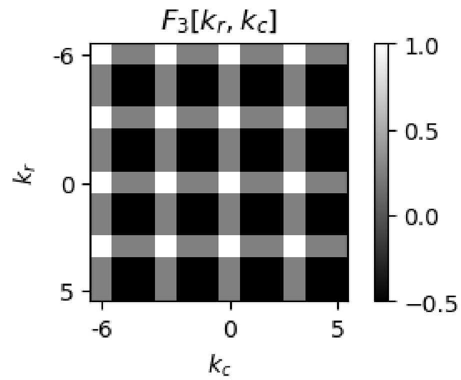
row r \ column c	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5
−6							72					
−5												
−4												
−3												
−2												
−1												
0	72											
1												
2												
3												
4												
5												

$F_2[k_r, k_c]$ can be written as $\frac{1}{2}(-1)^{k_r} + \frac{1}{2}(-1)^{k_c} = \frac{1}{2}e^{j\pi k_r} + \frac{1}{2}e^{j\pi k_c}$. Substituting this into the synthesis equation yields

$$f_2[r, c] = \sum_{k_r=0}^{R-1} \sum_{k_c=0}^{C-1} \left(\frac{1}{2}e^{j\pi k_r} + \frac{1}{2}e^{j\pi k_c}\right) e^{j2\pi k_r r/R} e^{j2\pi k_c c/C}$$

$$= \frac{RC}{2}\delta[r+R/2]\delta[c] + \frac{RC}{4}\delta[r]\delta[c+C/2] = \begin{cases} 72 & \text{if } r = -6 \text{ and } c = 0 \\ 72 & \text{if } r = 0 \text{ and } c = -6 \\ 0 & \text{otherwise} \end{cases}$$

In the following image, white, gray, and black represent pixel values of 1, 1/4, and −1/2, respectively.



$f_3[r, c]$

row r												
−6												
							36					
0			36								36	
							36					
5												
	−6						0					5

column c

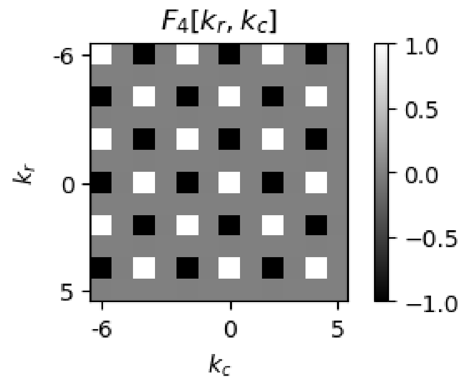
$F_3[k_r, k_c]$ can be written as $\frac{1}{2}\cos(2\pi k_r/3) + \frac{1}{2}\cos(2\pi k_c/3)$. Substituting this into the synthesis equation yields

$$f_3[r,c] = \sum_{k_r=0}^{R-1}\sum_{k_c=0}^{C-1}\left(\frac{1}{2}\cos(2\pi k_r/3) + \frac{1}{2}\cos(2\pi k_c/3)\right)e^{j2\pi k_r r/R}e^{j2\pi k_c c/C}$$

$$= \sum_{k_r=0}^{R-1}\sum_{k_c=0}^{C-1}\left(\frac{1}{4}e^{j2\pi k_r/3} + \frac{1}{4}e^{-j2\pi k_r/3} + \frac{1}{4}e^{j2\pi k_c/3} + \frac{1}{4}e^{-j2\pi k_c/3}\right)e^{j2\pi k_r r/R}e^{j2\pi k_c c/C}$$

$$= \frac{RC}{4}\delta[r+R/3]\delta[c] + \frac{RC}{4}\delta[r-R/3]\delta[c] + \frac{RC}{4}\delta[r]\delta[c+C/3] + \frac{RC}{4}\delta[r]\delta[c-C/3] + = \begin{cases} 36 & \text{if } r=-4 \text{ and } c=0 \\ 36 & \text{if } r=4 \text{ and } c=0 \\ 36 & \text{if } r=0 \text{ and } c=-4 \\ 36 & \text{if } r=0 \text{ and } c=4 \\ 0 & \text{otherwise} \end{cases}$$

In the following image, white, gray, and black represent pixel values of 1, 0, and −1, respectively.



$f_4[r, c]$

row r												
−6												
				36						36		
0												
				36						36		
5												
column c	−6						0					5

$F_4[k_r, k_c]$ can be written as $(\cos(2\pi k_r/4) * \cos(2\pi k_c/4))/2$. Substituting this into the synthesis equation yields

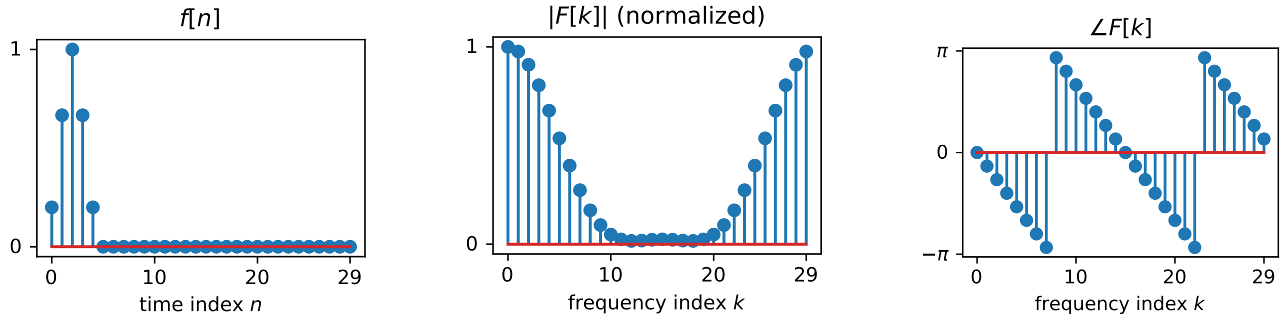
$$f_4[r,c] = \sum_{k_r=0}^{R-1}\sum_{k_c=0}^{C-1} \cos(2\pi k_c/4)\cos(2\pi k_r/4) e^{j2\pi k_r r/R} e^{j2\pi k_c c/C}$$

$$= \sum_{k_r=0}^{R-1}\sum_{k_c=0}^{C-1} \left(\frac{1}{2}e^{j2\pi k_r/4} + \frac{1}{2}e^{-j2\pi k_r/4}\right)\left(\frac{1}{2}e^{j2\pi k_c/3} + \frac{1}{2}e^{-j2\pi k_c/3}\right) e^{j2\pi k_r r/R} e^{j2\pi k_c c/C}$$

$$= \frac{RC}{4}\delta[r+R/4]\delta[c] + \frac{RC}{4}\delta[r-R/4]\delta[c] + \frac{RC}{4}\delta[r]\delta[c+C/4] + \frac{RC}{4}\delta[r]\delta[c-C/4] = \begin{cases} 36 & \text{if } r=-3 \text{ and } c=0 \\ 36 & \text{if } r=3 \text{ and } c=0 \\ 36 & \text{if } r=0 \text{ and } c=-3 \\ 36 & \text{if } r=0 \text{ and } c=3 \\ 0 & \text{otherwise} \end{cases}$$

4 Related DFTs (28 points)

Let $f[n]$ represent a periodic, discrete-time signal with period N=30 so that $f[n] = f[n+30]$ for all n. The following plots show one period of $f[n]$ along with the magnitude and angle of its Discrete Fourier Transform (DFT) computed with an analysis window of length N=30. The magnitude plot has been scaled so that the maximum magnitude is 1.



Seven signals ($g_1[n]$ through $g_7[n]$) are derived from $f[n]$ as shown below.

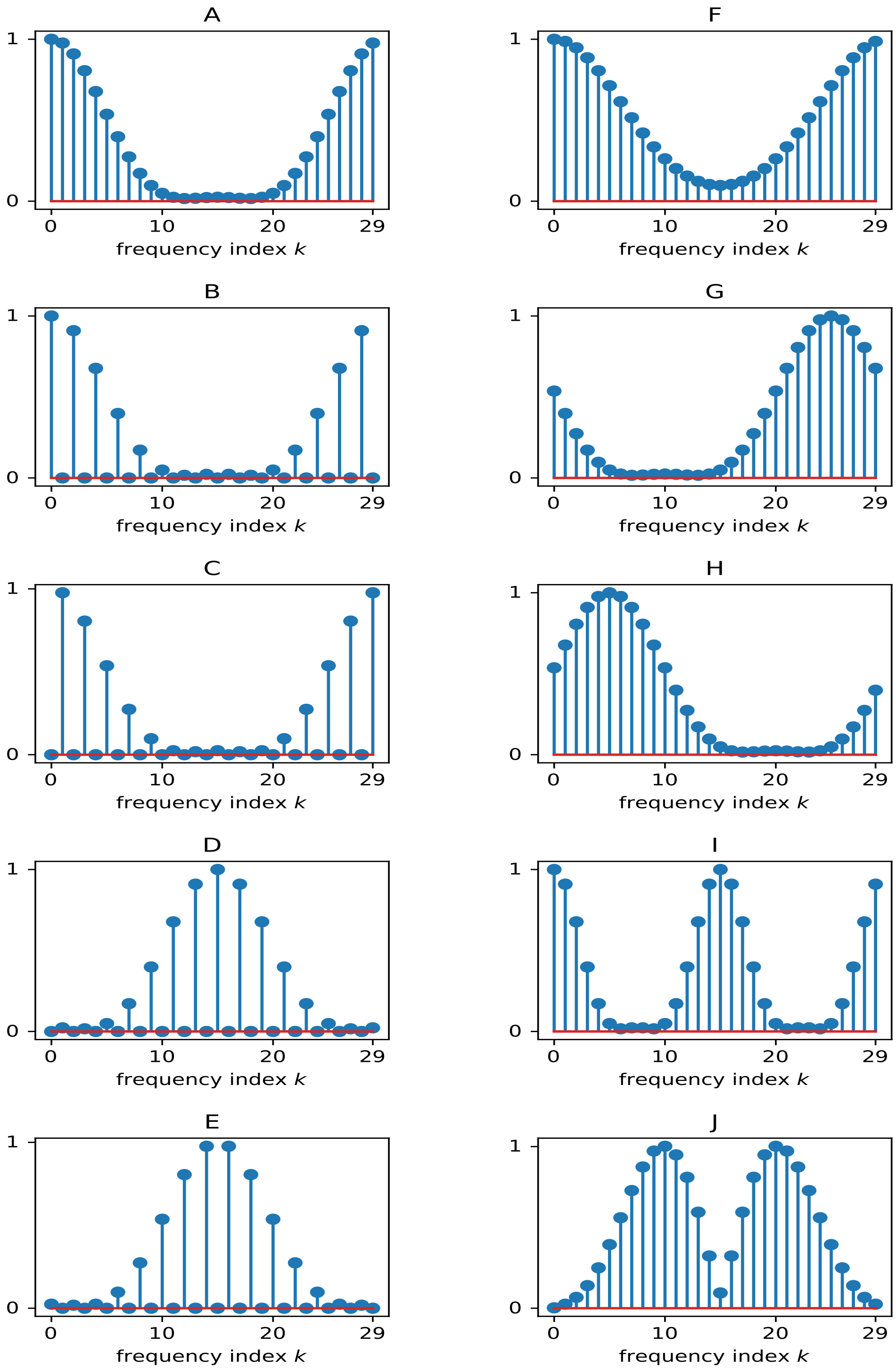
	normalized magnitude	angle
$g_1[n] = f^2[n]$	F	c
$g_2[n] = e^{-j\pi n/3}f[n]$	G	e
$g_3[n] = e^{j\pi n/3}f[n]$	H	d
$g_4[n] = \Big(f\big[n\big]+f\big[(n-15)\big]\Big)/2$	B	f
$g_5[n] = \Big(f\big[n\big]-f\big[(n-15)\big]\Big)/2$	C	g
$g_6[n] = \begin{cases} f\big[n/2\big] & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$	I	i
$g_7[n] = f[n+2]$	A	a

Determine which of the panels (A-J) on the following page shows the magnitude of the DFT of each of the derived signals and write the letter for that panel in the corresponding box in the magnitude column above.

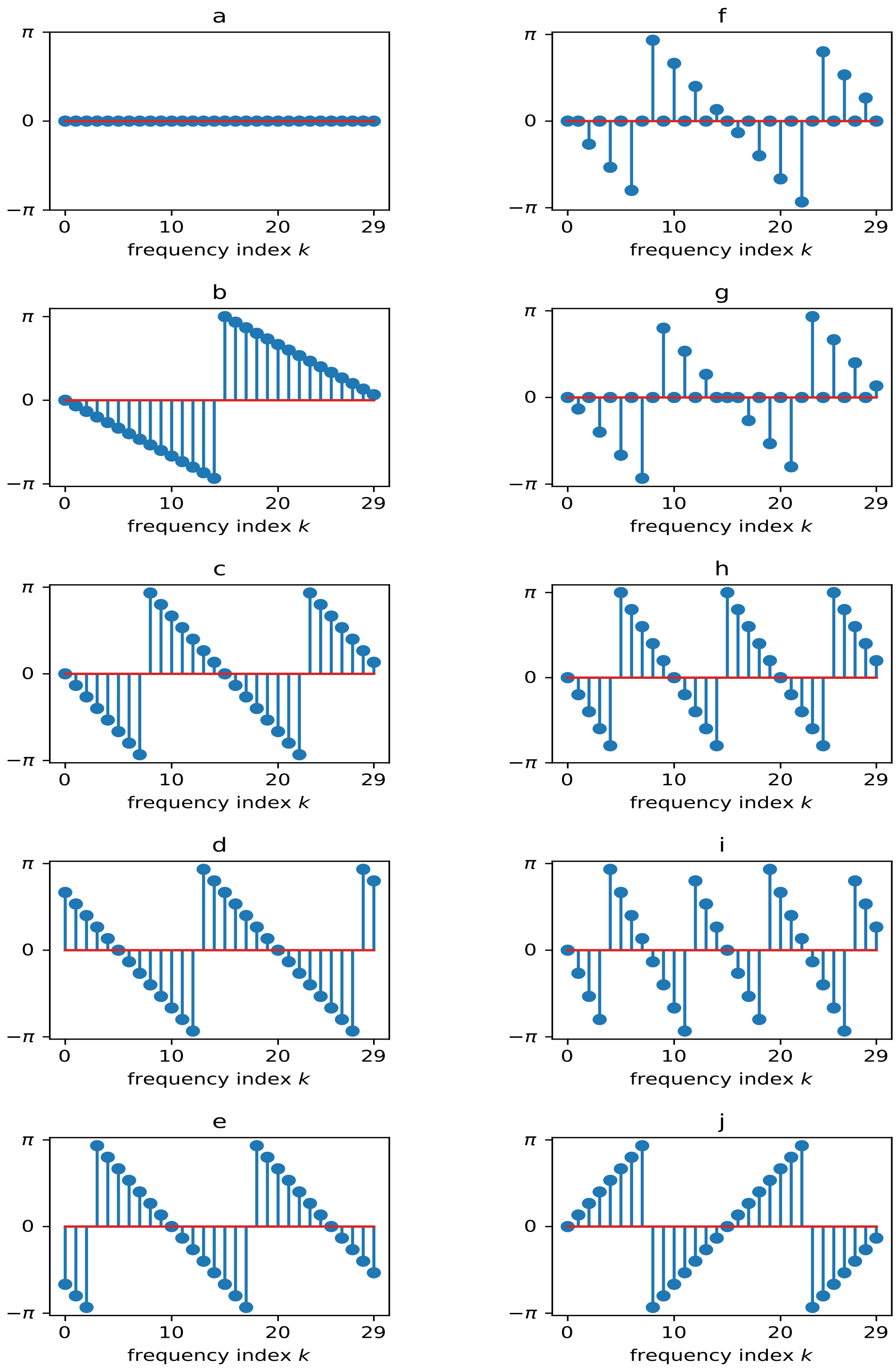
Similarly, determine which of the panels (a-j) on the subsequent page shows the angle of the DFT of each of the derived signals and write the letter for that panel in the corresponding box in the angle column above.

The same panel may be used more than once.

Magnitude Plots for Problem 4: the magnitudes in each plot have been scaled so that the largest magnitude in each plot is 1.



Angle Plots for Problem 4.

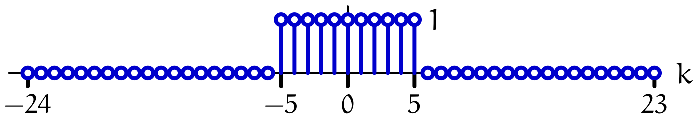


5 Highpass and Lowpass Filtering (30 points)

In this problem, we will determine the 1D and 2D unit-sample responses for high-pass and low-pass filters based on the Discrete Fourier Transform (DFT).

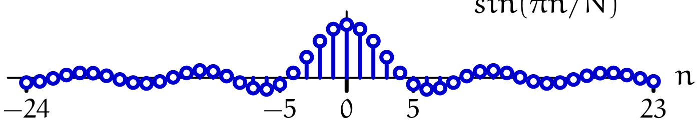
Part a. Start with the 1D case. The following plot shows one period of $G[k]$, which is a DFT of length $N = 48$ that passes low frequencies with $|k| \leqslant W$, where W is a positive integer constant ($W = 5$ in this example).

$$G[k] = \begin{cases} 1 & \text{if } |k| \leqslant W \\ 0 & \text{otherwise} \end{cases}$$



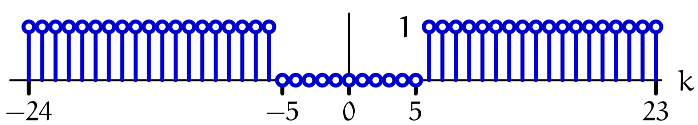
The inverse DFT of $G[k]$ is given by the following equation and plotted below:

$$g[n] = \frac{\sin(\pi(2W+1)n/N)}{\sin(\pi n/N)}$$



Let $H[k]$ represent the complementary high-pass filter given by the following plot:

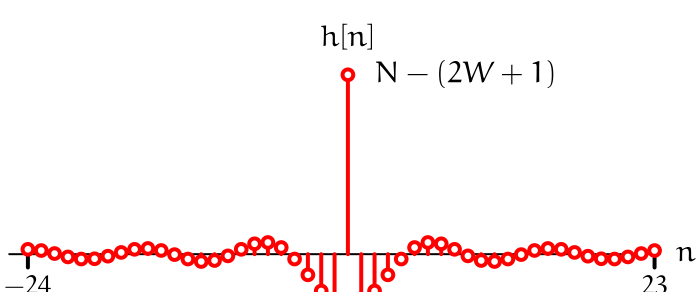
$$H[k] = \begin{cases} 1 & \text{if } W < |k| \leqslant N/2 \\ 0 & \text{otherwise} \end{cases}$$



Determine a closed-form expression for $h[n]$. Enter your expression in the box below.

$$h[n] = N\delta[n] - g[n] = N\delta[n] - \frac{\sin(\pi(2W+1)n/N)}{\sin(\pi n/N)}$$

Plot the discrete-time function $h[n]$ as a function of n on the axes below. Label the important features.

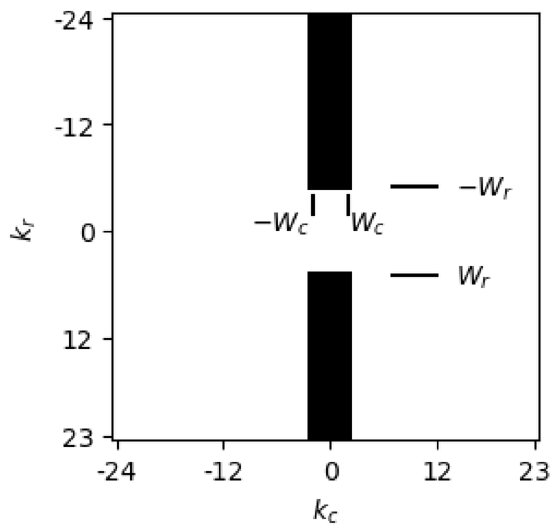


Part b. Next, we wish to specify a two-dimensional image-processing filter with the following properties:

- The filter should pass low frequencies in the horizontal direction ($|k_c| \leqslant W_c$ where W_c is a positive integer constant) and remove high frequencies in the horizontal direction ($W_c < |k_c| \leqslant \frac{C}{2}$).
- The filter should also pass high frequencies in the vertical direction ($W_r < |k_r| \leqslant \frac{R}{2}$ where W_r is a positive integer constant) and remove low frequencies in the vertical direction ($|k_r| \leqslant W_r$).

The images have R=48 rows of pixels (indexed by r, where $-\frac{R}{2} \leqslant r < \frac{R}{2}$) and C=48 columns of pixels (indexed by c, where $-\frac{C}{2} \leqslant c < \frac{C}{2}$), and the unit-sample response of the image-processing filter is represented by $h[r, c]$.

Let $H[k_r, k_c]$ represent the DFT of $h[r, c]$. The values of $H[k_r, k_c]$ should be 1 for frequency indices k_r, k_c that are passed by the filter and 0 for frequency indices that are not passed. On the diagram below, indicate the region (or regions) of $H[k_r, k_c]$ that are equal to 1. Fully label the region(s) in terms of the constants W_r, and W_c (where $W_r = 5$ and $W_c = 2$).



Black pixels represents regions in the passband (i.e., their values are 1) and white pixels represents regions in the stopband (i.e., their values are 0).

Enter a closed-form expression for the unit-sample response $h[r, c]$ in the box below.

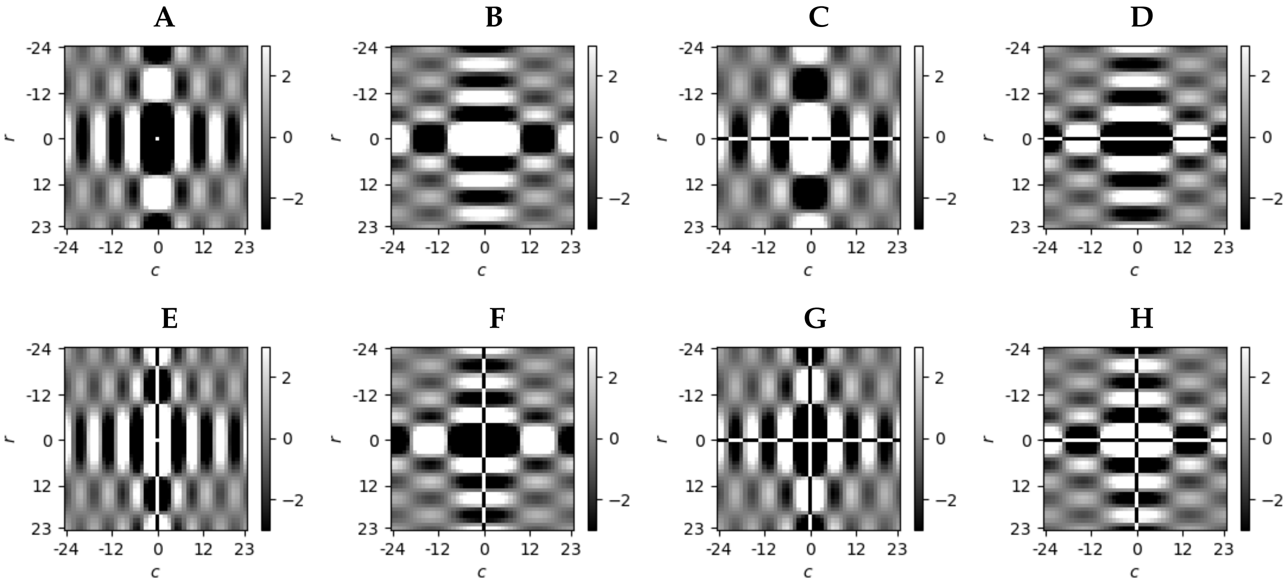
$$h[r,c] = \left(\frac{\sin(\pi(2W_c+1)c/C)}{\sin(\pi c/C)}\right)\left(R\delta[r] - \frac{\sin(\pi(2W_r+1)r/R)}{\sin(\pi r/R)}\right)$$

$$h[r,c] = \frac{1}{RC}\sum_{|k_c|\leqslant W_c}\ \sum_{W_r\leqslant|k_r|\leqslant\frac{R}{2}} e^{-j2\pi k_c c/C} e^{-j2\pi k_r r/R} = \frac{1}{RC}\sum_{|k_c|\leqslant W_c} e^{-j2\pi k_c c/C} \sum_{W_r<|k_r|\leqslant\frac{R}{2}} e^{-j2\pi k_r r/R}$$

Notice that $h[r, c]$ is the product of a lowpass filter in the c direction times a highpass filter in the r direction:

$$h[r,c] = \left(\frac{\sin(\pi(2W_c+1)c/C)}{\sin(\pi c/C)}\right)\left(R\delta[r] - \frac{\sin(\pi(2W_r+1)r/R)}{\sin(\pi r/R)}\right)$$

Part c. Each of the following panels (A-H) shows the unit-sample response of a two-dimensional image-processing filter. The brightness scale is the same for each figure, with white representing pixel values that are greater than zero and black representing pixel values that are less than zero.



Determine which panel corresponds to the unit-sample response $h[r, c]$ of the filter that passes low frequencies in the horizontal direction and high frequencies in the vertical direction (as described in part b). Enter your answer in the box below.

answer (A-H) = D

Worksheet (*intentionally blank*)

Worksheet (intentionally blank)