# 6.3000 Quiz 2 Spring 2025

**Name:** Solutions

**Kerberos (Athena) username:**

## Please WAIT until we tell you to begin.

This quiz is closed book, but you may use two $8.5 \times 11$ sheets of notes (four sides).

**You may NOT use any electronic devices (such as calculators and phones).**

If you have questions, please **come to us** at the front of the room to ask.

### Please enter all solutions in the boxes provided.

Work on other pages with QR codes will be considered for partial credit.
Please provide a note if you continue work on worksheets at the end of the exam.

### Please do not write on the QR codes at the bottom of each page.

We use those codes to identify which pages belong to each student.

### Trigonometric Identities Reference

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$
$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b)$$
$$\cos(a+b)-\cos(a-b) = -2\sin(a)\sin(b)$$
$$\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$$
$$\sin(a+b)-\sin(a-b) = 2\cos(a)\sin(b)$$
$$2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b)$$
$$2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$$
$$2\sin(a)\cos(b) = \sin(a+b)+\sin(a-b)$$
$$2\cos(a)\sin(b) = \sin(a+b)-\sin(a-b)$$

# 1 Short-Answer Questions (24 points)

**Part a.** The left column of the following table shows the input/output relations for four different systems, where $x$ represents the input and $y$ represents the output of the system. For each system, determine if that system is additive and/or homogeneous and/or time-invariant, and enter **Yes** or **No** in the boxes to the right.

| | additive? (**Yes** or **No**) | homogeneous? (**Yes** or **No**) | time-invariant? (**Yes** or **No**) |
|---|---|---|---|
| $y[n] = \left(-\frac{1}{2}\right)^n \left(x[n]+1\right)$ | **No** | **No** | **No** |
| $y[n] = \sin\left(x[n]\right)$ | **No** | **No** | **Yes** |
| $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ | **Yes** | **Yes** | **Yes** |
| $y(t) = tx(t)$ | **Yes** | **Yes** | **No** |

**Part a1.** Determine if the output of the sum of two inputs ($x_1[n]+x_2[n]$) is the sum of outputs ($y_1[n]$ and $y_2[n]$).

$$y_1[n] + y_2[n] = \left(-\frac{1}{2}\right)^n \left(\left(x_1[n]+\mathbf{1}\right) + \left(x_2[n]+\mathbf{1}\right)\right) \neq \left(-\frac{1}{2}\right)^n \left(x_1[n]+x_2[n]+\mathbf{1}\right) \neq y_1[n]+y_2[n]$$

Determine if scaling the input by $\alpha$ scales the output by $\alpha$.

$$y_{scaled}[n] = \left(-\frac{1}{2}\right)^n \left(\alpha x[n]+\mathbf{1}\right) = \left(-\frac{1}{2}\right)^n \left(\alpha x[n]+1\right) \neq \left(-\frac{1}{2}\right)^n \left(\alpha x[n]+\alpha\right)$$

Determine if shifting the input shifts the output.

$$\left(-\frac{1}{2}\right)^n \left(x[n-m]+1\right) \neq \left(-\frac{1}{2}\right)^{n-\mathbf{m}} \left(x[n-m]+1\right)$$

**Part a2.**

$$\sin(x_1[n] + x_2[n]) \neq \sin(x_1) + \sin(x_2)$$
$$\alpha \sin(x_1[n]) \neq \sin(\alpha x_1[n])$$
$$\sin(x[n-m]) = \sin(x[n-m])$$

**Part a3.**

The integral of a sum is the sum of the integrals.
Scaling the integrand scales the integral.

$$\int_{-\infty}^{t} x(\tau - t_0)d\tau = \int_{-\infty}^{t-t_0} x(\lambda)d\lambda = y(t-t_0)$$

**Part a4.**

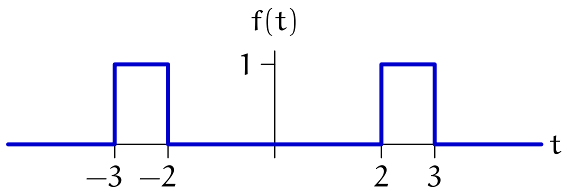
$tx_1(t) + tx_2(t) = t(x_1(t) + x_2(t))$

$t\alpha x(t) = \alpha t x(t)$

$t(x(t{-}t_0)) \neq (t{-}t_0)x(t{-}t_0)$

**Part b.** Determine the frequency response $F(\omega)$ of a linear, time-invariant system with the following impulse response:

$$f(t) = \begin{cases} 1 & \text{if } 2 < |t| < 3 \\ 0 & \text{otherwise} \end{cases}$$




Enter a closed form expression[1] for $F(\omega)$ in the box below.

$$F(\omega) = \boxed{2\frac{\sin 3\omega}{\omega} - 2\frac{\sin 2\omega}{\omega}}$$

The signal $f(t)$ can be written as the difference between a rectangular pulse that extends from $-3$ to 3 and a rectangular pulse that extends from $-2$ to 2. The Fourier transform of the former is

$$\int_{-3}^{3} e^{-j\omega t} dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-3}^{3} = 2\frac{\sin 3\omega}{\omega}$$

Similarly, the Fourier transform of the latter is $2\frac{\sin 2\omega}{\omega}$. Thus the total answer is

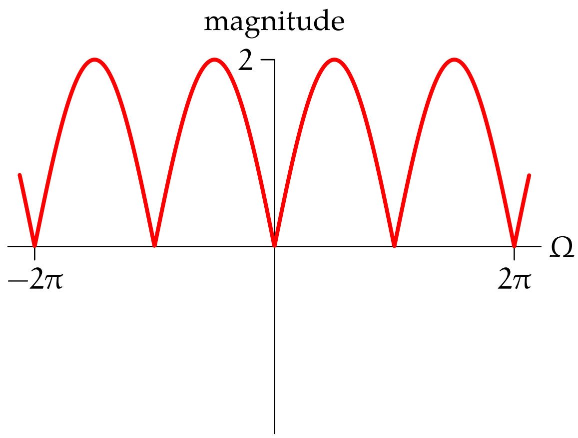
$$F(\omega) = 2\frac{\sin 3\omega}{\omega} - 2\frac{\sin 2\omega}{\omega}$$

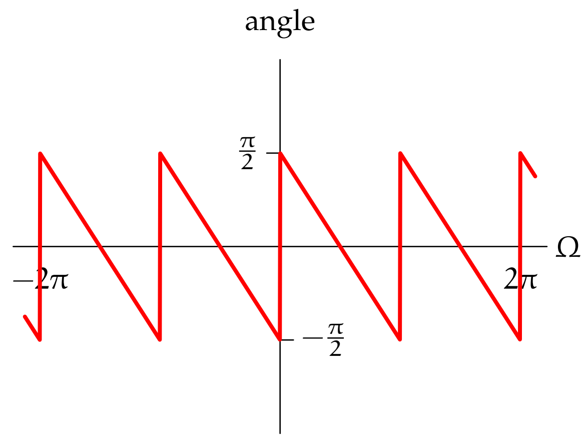
[1] Your expression should not include integrals or derivatives.

**Part c.** Determine the frequency response of a linear, time-invariant system with the following unit-sample response:

$$g[n] = \delta[n] - \delta[n-2]$$

Sketch the magnitude and angle of the frequency response on the axes below. Label the key points.




The frequency response is the Fourier transform of the unit-sample response:

$$G(\Omega) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\Omega n} = 1 - e^{-j2\Omega}$$

This frequency response can be simplified by realizing that $g[n]$ is a time-delayed version of $\delta[n+1]-\delta[n-1]$, which would correspond to a sinusoid in frequency. We can take advantage of this fact by factoring the time delay term out of the expression for $G(\Omega)$, as follows.

$$G(\Omega) = 1 - e^{-j2\Omega} = e^{-j\Omega}\left(e^{j\Omega} - e^{-j\Omega}\right) = j2\sin(\Omega)e^{-j\Omega}$$

The magnitude of the frequency response is the magnitude of $2\sin(\Omega)$.

The angle of the frequency response is determined by three factors. First, the $j$ in $j2\sin(\Omega)e^{-j\Omega}$ contributes $+\pi/2$. Second, $\sin(\theta)$ is negative when $\pi < \theta < 2\pi$. Third, the phase term $e^{-j\Omega}$ contributes a linear term that decreases in proportion to $\Omega$. When $\Omega$ is a small positive number, only the first factor contributes, so the angle of the frequency response is $\pi/2$. As $\Omega$ increases, the linear term decreases the angle of the frequency response until $\Omega = \pi/2$. At that point, the sign of $\sin(\Omega)$ is negative, so the phase jumps by $\pi$. Then the cycle repeats. Thus these three factors combine to give rise to the sawtooth function above.

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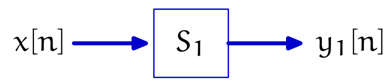
# 2 Cascaded Systems (22 points)

### Part a.

Let $S_1$ represent a linear, time-invariant (LTI) system whose unit-sample response $h_1[n]$ is a unit step:




$$h_1[n] = \begin{cases} 1 & \text{if } n \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

Determine the response $y_1[n]$ of this system when the input $x[n]$ is the following geometric sequence:

$$x[n] = \begin{cases} (0.9)^n & \text{if } n \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

Enter the first 5 values of $y_1[n]$ in the boxes below.

$y_1[0] =$ $1$

$y_1[1] =$ $1 + 0.9$

$y_1[2] =$ $1 + 0.9 + 0.9^2$

$y_1[3] =$ $1 + 0.9 + 0.9^2 + 0.9^3$

$y_1[4] =$ $1 + 0.9 + 0.9^2 + 0.9^3 + 0.9^4$

Enter a closed-form expression[2] for $y_1[n]$ in the box below.

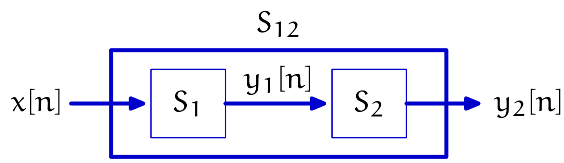
$$y_1[n] = \frac{1-(0.9)^{n+1}}{1-0.9} = 10 - 9(0.9)^n$$

$$y_1[n] = \sum_{m=-\infty}^{\infty} h_1[m]\,u[n-m] = \sum_{m=-\infty}^{\infty} 0.9^m u[m]\,u[n-m] = \sum_{m=0}^{n} 0.9^m = \frac{1-0.9^{n+1}}{1-0.9} = 10 - 9\,(0.9)^n$$

[2] Your expression can contain additions, subtractions, multiplications, divisions, and exponentiations, but no other operators. Also, the number of operations required for each $y_1[n]$ should be bounded by a constant as $n \to \infty$.

**Part b.**

Let $S_{12}$ represent the linear, time-invariant (LTI) system that results when two LTI systems ($S_1$ and $S_2$) are connected in cascade (so that the output of $S_1$ is the input to $S_2$) as shown in the figure to the right.




Determine the unit-sample response $h_{12}[n]$ of $S_{12}$ when the unit-sample response $h_1[n]$ of system $S_1$ is a unit-step function $u[n]$ and the unit-sample response $h_2[n]$ of system $S_2$ is also a unit-step function $u[n]$:

$$h_1[n] = u[n]$$
$$h_2[n] = u[n]$$

Enter a closed-form expression[3] for $h_{12}[n]$ in the box below.

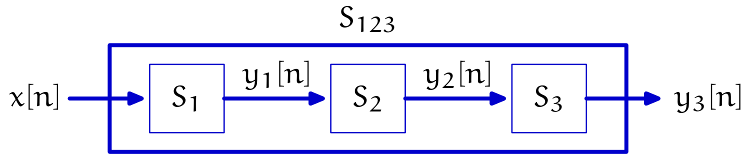
$h_{12}[n] =$ $(n+1)\,u[n]$

$$\begin{aligned} h_{12}[n] &= (u * u)[n] \\ &= \sum_{m=-\infty}^{\infty} u[m]u[n-m] \\ &= \sum_{m=0}^{n} u[m]u[n-m] \\ &= (n+1)\,u[n] \end{aligned}$$

[3] Your expression can contain additions, subtractions, multiplications, divisions, and exponentiations, but no other operators. Also, the number of operations required for each $h_{12}[n]$ should be bounded by a constant as $n \to \infty$.

**Part c.**

Let $S_{123}$ represent the LTI system that results when three LTI systems ($S_1$, $S_2$, and $S_3$) are connected in cascade, as shown in the figure to the right.




Determine the unit-sample response $h_{123}[n]$ of $S_{123}$ when the unit-sample response $h_1[n]$ of system $S_1$ is a unit-step function $u[n]$, the unit-sample response $h_2[n]$ of system $S_2$ is a unit-step function $u[n]$, and the unit-sample response $h_3[n]$ of system $S_3$ is also a unit-step function $u[n]$:

$$h_1[n] = u[n]$$
$$h_2[n] = u[n]$$
$$h_3[n] = u[n]$$

Enter the first 5 values of $h_{123}[n]$ in the boxes below.

$h_{123}[0] =$ 1

$h_{123}[1] =$ 3

$h_{123}[2] =$ 6

$h_{123}[3] =$ 10

$h_{123}[4] =$ 15

$$\begin{aligned}(u*u*u)[n] &= ((u*u)*u)[n] \\ &= \sum_{m=-\infty}^{\infty} (m+1)\, u[m]\, u[n-m] \\ &= \sum_{m=0}^{n} (m+1) \\ &= \frac{(n+1)(n+2)}{2}\end{aligned}$$

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# 3 Pulsed Relations (27 points)

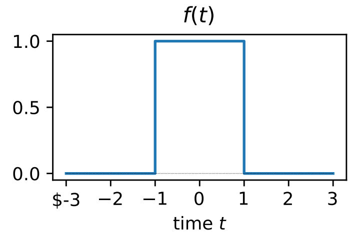
Let $f(t)$ represent the following signal

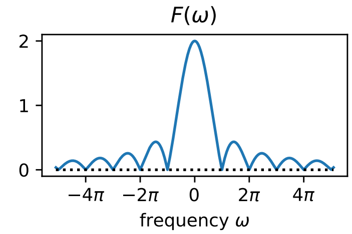
$$f(t) = \begin{cases} 1 & \text{if } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

and let $F(\omega)$ represent its Fourier transform

$$F(\omega) = 2\,\frac{\sin(\omega)}{\omega}$$

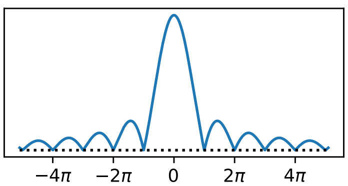
as shown on the right.



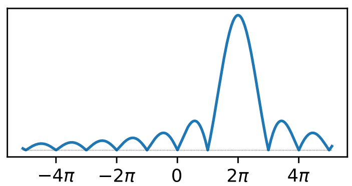




Each of the following plots shows the magnitude of the Fourier transform of a signal derived from $f(t)$. Note that the magnitude scales for these plots are not specified and may differ from one another.

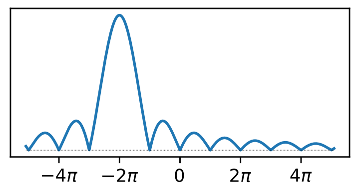
A



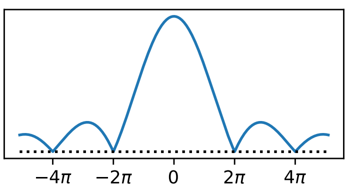

B



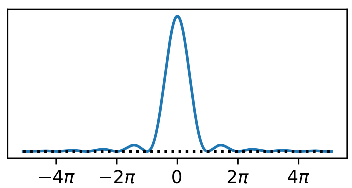

C



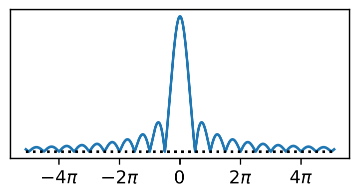

D



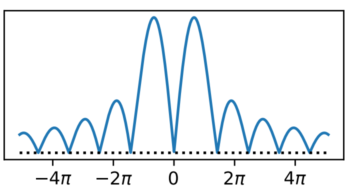

E



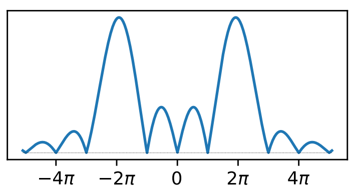

F



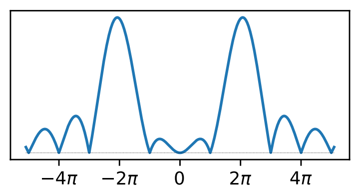

G




H




I




Identify which plot (A-I) shows the magnitude of each of the following derived signals.

| | | | |
|---|---|---|---|
| $f(t/2)$ : | **F** | $f(t)\sin(2\pi t)$ : | **H** |
| $(f * f)(t)$ : | **E** | $f(t)\cos(2\pi t)$ : | **I** |
| $f(2t)$ : | **D** | $f(t)e^{j2\pi t}$ : | **B** |
| $f(t-1)$ : | **A** | $f(t)e^{-j2\pi t}$ : | **C** |
| $tf(t)$ : | **G** | | |

**Part 3a.**

$$F_1(\omega) = \int_{-\infty}^{\infty} f(t/2)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(\tau)e^{-j\omega 2\tau}2d\tau = 2F(2\omega)$$

Answer: **F**

**Part 3b.**

$$F_2(\omega) = \int_{-\infty}^{\infty} (f*f)(t)e^{-j\omega t}dt = F^2(\omega)$$

Answer: **E**

**Part 3c.**

$$F_3(\omega) = \int_{-\infty}^{\infty} f(2t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau/2}d\tau/2 = F(\omega/2)/2$$

Answer: **D**

**Part 3d.**

$$F_4(\omega) = \int_{-\infty}^{\infty} f(t-1)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(\tau)e^{-j\omega(\tau+1)}dt = e^{-j\omega}\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}dt = F(\omega)$$

Answer: **A**

**Part 3e.**

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$\frac{dF(\omega)}{d\omega} = \int_{-\infty}^{\infty} -jtf(t)e^{-j\omega t}dt$$

$$F_5(\omega) = j\frac{dF(\omega)}{d\omega}$$

Answer: **G**

**Part 3h.**

$$F_8(\omega) = \int_{-\infty}^{\infty} f(t)e^{j2\pi t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(t)e^{j2\pi t}e^{-j(\omega-2\pi)t}dt = F(\omega-2\pi)$$

Answer: **B**

**Part 3i.**

$$F_9(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(t)e^{j2\pi t}e^{-j(\omega+2\pi)t}dt = F(\omega+2\pi)$$

Answer: **C**

**Part 3g.**

$$F_7(\omega) = \int_{-\infty}^{\infty} f(t)\cos(2\pi t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(t)\frac{1}{2}\left(e^{j2\pi t}+e^{-j2\pi t}\right)e^{-j\omega t}dt = \frac{1}{2}\Big(F(\omega{-}2\pi)+F(\omega{+}2\pi)\Big)$$

Answer: **I**

**Part 3f.**

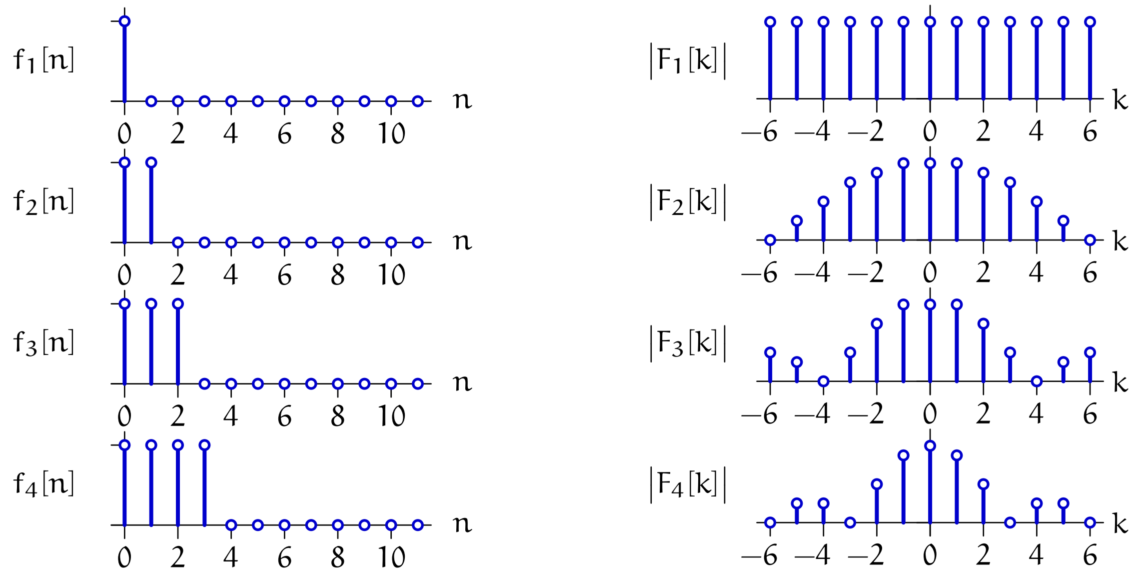
$$F_6(\omega) = \int_{-\infty}^{\infty} f(t)\sin(2\pi t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(t)\frac{1}{j2}\left(e^{j2\pi t}-e^{-j2\pi t}\right)e^{-j\omega t}dt = \frac{1}{j2}\Big(F(\omega{-}2\pi)-F(\omega{+}2\pi)\Big)$$

Answer: **H**

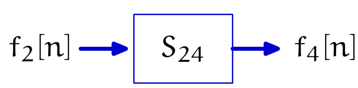
The sidelobes near $\omega$=0 in panels I and H have different peak magnitudes. For the cosine case, the two sidelobes near $\omega$=0 have opposite signs. For the sine case, the have the same signs. Therefore, the case with the larger sidebands corresponds to the sine case.

# 4 Pulses In Pulses Out (27 points)

The first 12 samples of four **periodic** signals that are each periodic in N=12 are shown in the left column below, and the magnitudes of their Fourier series coefficients are shown in the right column.




**Part a.** Consider a system $S_{24}$ that produces $f_4[n]$ as output when $f_2[n]$ is its input.

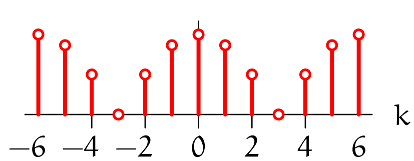



Determine the unit-sample response $h_{24}[n]$ and frequency response $H_{24}[k]$ of system $S_{24}$ and enter expressions for $h_{24}[n]$ and $H_{24}[k]$ in the boxes below.

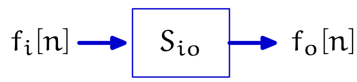
$h_{24}[n] =$ $\delta[n] + \delta[n-2]$

$H_{24}[k] =$ $1 + e^{-j\pi k/3}$

Sketch the magnitude of the frequency response $H_{24}[k]$ on the axes below.



**Part b.** Consider 16 possible systems that are each defined by its input signal $f_i[n]$ and output signal $f_o[n]$:




Some of these systems could be linear and time invariant (LTI); others cannot.

Determine which of the 16 systems cannot possibly be LTI, and enter **X** in the corresponding box below.

| i \ o | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | | | | |
| 2 | X | | X | |
| 3 | X | X | | X |
| 4 | X | X | X | |

If a system is LTI, then its frequency response $H_{ij}[k]$ is the ratio of the Fourier series coefficients of its output $F_j[k]$ divided by its input $F_i[k]$:

$$H_{ij}[k] = \frac{F_j[k]}{F_i[k]}$$

Notice that this ratio is infinite if $F_i[k] = 0$ when $F_j[k] \neq 0$. This relation follows from the filtering property of LTI systems: if $F_i[k] = 0$ then there is no possible value of $H_{ij}[k]$ for which $F_j[k]$ could be nonzero, and the system cannot be LTI.

Row 1: The Fourier series coefficients $F_1[k]$ are all nonzero, so we can define $H_{1j}[k] = F_1[k]/F_j[k]$ for all k. All of the entries in the first row correspond to possible LTI systems.

Row 3: The fourth harmonic of $f_3[n]$ is missing (i.e., its amplitude is zero). Therefore if $f_3[n]$ is the input signal, then none of the other signals other than $f_3[n]$ could result if the corresponding system is LTI. Therefore $S_{31}$, $S_{32}$, and $S_{34}$ must be nonlinear. Of course $S_{33}$ could be LTI (with $h33[k] = \delta[k]$).

Row 2: The sixth harmonic is missing from $f_2[n]$ but not from $f_1[n]$ or $f_3[n]$. therefore $S_{21}$ and $S_{23}$ must be nonlinear. The sixth harmonic is missing from both $f_2[n]$ and $f_4[n]$. Therefore the ratio of $F_4[n]$ to $F_2[n]$ is indeterminant. It follows that $H_{24}$ could be any number and $S_{24}$ could be LTI.

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*Worksheet (intentionally blank)*

*Worksheet (intentionally blank)*