6.003 Quiz 2 Spring 2022

Name: Solutions

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This quiz is closed book, but you may use two 8.5×11 sheets of notes (four sides).

You may NOT use any electronic devices (such as calculators and phones).

If you have questions, please **come to us** at the front of the room to ask.

Please enter all solutions in the boxes provided.

Work on other pages with QR codes will be considered for partial credit.
Please provide a note if you continue work on worksheets at the end of the exam.

Please do not write on the QR codes at the bottom of each page.

We use those codes to identify which pages belong to each student.

Trigonometric Identities Reference

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b)$$
$$\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$$
$$2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b)$$
$$2\sin(a)\cos(b) = \sin(a+b)+\sin(a-b)$$

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$
$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\cos(a+b)-\cos(a-b) = -2\sin(a)\sin(b)$$
$$\sin(a+b)-\sin(a-b) = 2\cos(a)\sin(b)$$
$$2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$$
$$2\cos(a)\sin(b) = \sin(a+b)-\sin(a-b)$$

1 Closing Sums (12 points)

In each of the following parts, k and n represent integers, and N is an integer multiple of 2 that is greater than 0. Simplify each expression by closing the sum and removing indeterminate forms (such as divisions by 0). Enter your answers in the boxes provided. Your answers should be valid for $0 \leqslant k < N$, and need not apply outside that range.

$$f_0[k] = \sum_{n=0}^{N-1} e^{-j\frac{2\pi kn}{N}} = \boxed{N\delta[k]}$$

$$\sum_{n=0}^{N-1} e^{-j\frac{2\pi kn}{N}} = \frac{1-e^{-j\frac{2\pi kN}{N}}}{1-e^{-j\frac{2\pi k}{N}}} = \frac{1-e^{-j2\pi k}}{1-e^{-j\frac{2\pi k}{N}}}$$

When $k = 0$, both the numerator and denominator are zero, and the answer is indeterminant. We can resolve the indeterminacy using L'Hôpital's rule or by substituting $k = 0$ into the original sum. Both methods show that the answer is N when $k = 0$. For $0 < k < N$, the numerator is zero and the denominator is not zero. Therefore the answer is 0 for $0 < k < N$.

$$f_1[k] = \sum_{n=0}^{N-1} e^{j\frac{2\pi k_0 n}{N}} e^{-j\frac{2\pi kn}{N}} = \boxed{N\delta[k-k_0]}$$

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi k_o n}{N}} e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{N-1} e^{-j\frac{2\pi(k-k_o)n}{N}}$$

This result is the same as the previous result with $k-k_o$ substituted for k.

$$f_2[k] = \sum_{n=0}^{N-1} e^{-j\frac{4\pi kn}{N}} = \boxed{N\delta[k] + N\delta[k-\tfrac{N}{2}]}$$

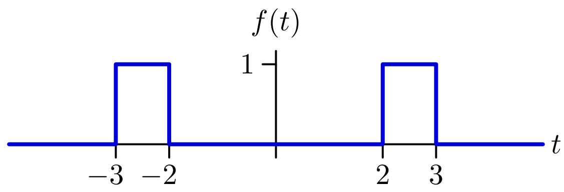
$$\sum_{n=0}^{N-1} e^{-j\frac{4\pi kn}{N}} = \frac{1-e^{-j\frac{4\pi kN}{N}}}{1-e^{-j\frac{4\pi k}{N}}} = \frac{1-e^{-j4\pi k}}{1-e^{-j\frac{4\pi k}{N}}}$$

As in the first case, the numerator is zero for all k. However, in this case, the denominator is zero at $k = 0$ and $k = N/2$ (provided N is even). Resolving the indeterminacy by l'Hôpital's rule or by substituting $k = N/2$ into the original sum gives the same result that the sum is N.

2 Frequency Responses (20 points)

Part a. Determine the frequency response $F(\omega)$ of a linear, time-invariant system with the following impulse response:

$$f(t) = \begin{cases} 1 & \text{if } 2 < |t| < 3 \\ 0 & \text{otherwise} \end{cases}$$



Enter a closed form expression for $F(\omega)$ in the box below.

$$F(\omega) = \boxed{2\frac{\sin 3\omega}{\omega} - 2\frac{\sin 2\omega}{\omega}}$$

The signal $f(t)$ can be written as the difference between a rectangular pulse that extends from -3 to 3 and a rectangular pulse that extends from -2 to 2. The Fourier transform of the former is

$$\int_{-3}^{3} e^{-j\omega t}\,dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-3}^{3} = 2\frac{\sin 3\omega}{\omega}$$

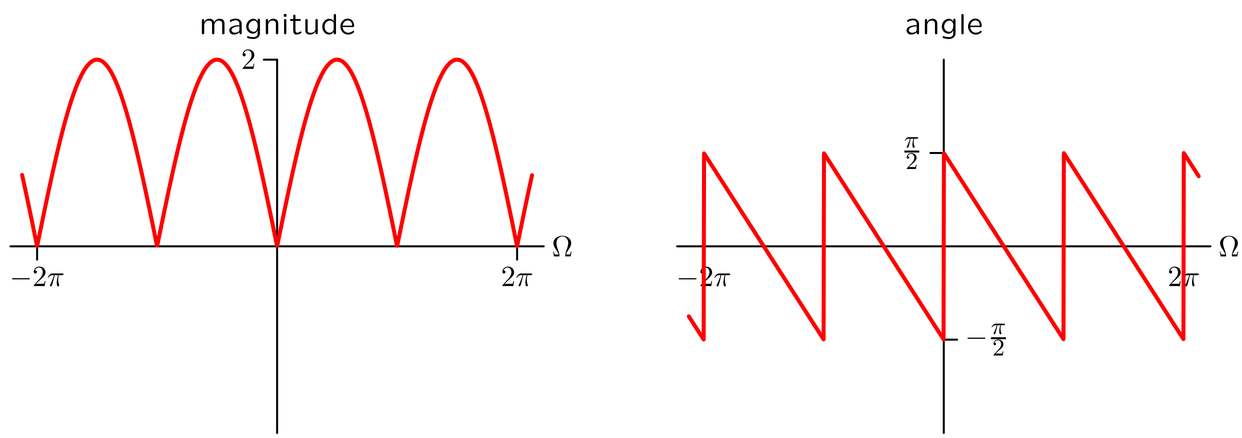
Similarly, the Fourier transform of the latter is $2\frac{\sin 2\omega}{\omega}$. Thus the total answer is

$$F(\omega) = 2\frac{\sin 3\omega}{\omega} - 2\frac{\sin 2\omega}{\omega}$$

Part b. Determine the frequency response of a linear, time-invariant system with the following unit-sample response.

$$g[n] = \delta[n] - \delta[n-2]$$

Sketch the magnitude and angle of the frequency response on the axes below. Label the key points.



The frequency response is the Fourier transform of the unit-sample response:

$$G(\Omega) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\Omega n} = 1 - e^{-j2\Omega}$$

This frequency response can be simplified by realizing that $g[n]$ is a time-delayed version of $\delta[n+1]-\delta[n-1]$, which would correspond to a sinusoid in frequency. We can take advantage of this fact by factoring the time delay term out of the expression for $G(\Omega)$, as follows.

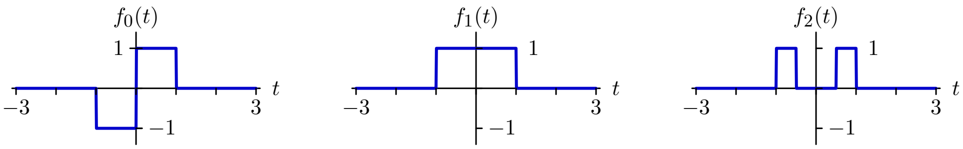
$$G(\Omega) = 1 - e^{-j2\Omega} = e^{-j\Omega}\left(e^{j\Omega} - e^{-j\Omega}\right) = j2\sin(\Omega)e^{-j\Omega}$$

The magnitude of the frequency response is the magnitude of $2\sin(\Omega)$.

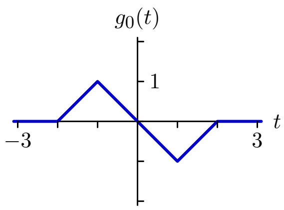
The angle of the frequency response is determined by three factors. First, the j in $j2\sin(\Omega)e^{-j\Omega}$ contributes $+\pi/2$. Second, $\sin(\theta)$ is negative when $\pi < \theta < 2\pi$. Third, the phase term $e^{-j\Omega}$ contributes a linear term that decreases in proportion to Ω. When Ω is a small positive number, only the first factor contributes, so the angle of the frequency response is $\pi/2$. As Ω increases, the linear term decreases the angle of the frequency response until $\Omega = \pi/2$. At that point, the sign of $\sin(\Omega)$ is negative, so the phase jumps by π. Then the cycle repeats. Thus these three factors combine to give rise to the sawtooth function above.

3 Convolving Steps (24 points)

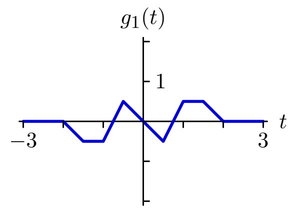
Let $f_0(t)$, $f_1(t)$, and $f_2(t)$ represent the following signals.



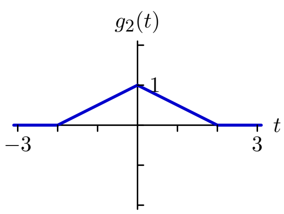
Determine if each of the signals below can be constructed by convolving (f_0 or f_1 or f_2) with (f_0 or f_1 or f_2). If it can, then enter the signals to be convolved in the corresponding boxes. If it cannot, then put an X in both boxes.



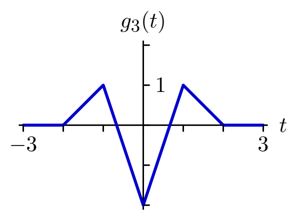
= [X] * [X]



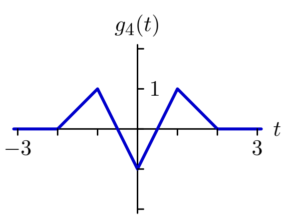
= [f_0 f_2] * [f_2 f_0]



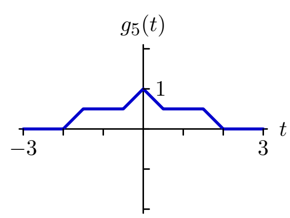
= [X] * [X]



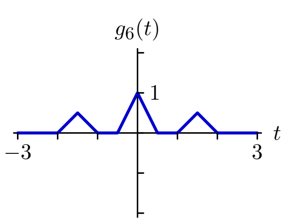
= [f_0] * [f_0]



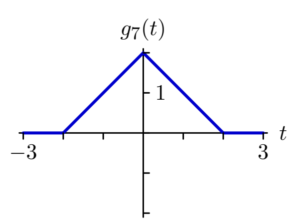
= [X] * [X]



= [f_1 f_2] * [f_2 f_1]



= [f_2] * [f_2]



= [f_1] * [f_1]

One of three signals (f_0, f_1, or f_2) is convolved with one of three signals (f_0, f_1, or f_2), so there are nine possible outcomes. However, convolution is commutative, so only six of the convolution can be unique.

Start by evaluating the convolution integral

$$(f_0 * f_0)(t) = \int_{-\infty}^{\infty} f_0(\tau)f_0(t-\tau)d\tau$$

at integer values of t. At $t = -2$, the non-zero parts of $f_0(\tau)$ and $f_0(t-\tau)$ are non-overlapping, so the convolution is 0. At $t = -1$, the negative parts of both functions overlap, so the convolution yields 1. At $t = 0$, the positive parts overlap negative parts and vice verse, so the convolution yields -2. At $t = 1$, the positive parts of both functions overlap, so the convolution yields 1. As the signals slide by each other, the overlapping areas change linearly, so convolution produces a piecewise linear function. The answer is therefore g_3.

By similar reasoning, $f_1 * f_1$ is 1 when $t = -1$, 2 when $t = 0$, and 1 when $t = 1$. The answer is therefore g_7.

For $f_0 * f_1$, the convolution is -1 when $t = -1$, 0 when $t = 0$, and 1 when $t = 1$. The answer is $-g_0$, which is not an option.

For convolutions with f_2, we need to evaluate the convolution integral for half integer steps in time: $t = -2.5$, then -2, then -2.5, ... 2.5.

For $f_2 * f_0$ we get $-1/2$, $-1/2$, $1/2$, 0, $-1/2$, $1/2$, and $1/2$. The answer is g_1.

For $f_2 * f_1$ we get 1/2, 1/2, 1/2, 1, 1/2, 1/2, and 1/2. The answer is g_5.

For $f_2 * f_2$ we get 1/2, 0, 0, 1, 0, 0, and 1/2. The answer is g_6.

Answers g_0, g_2, and g_3 cannot be generated by convolving any pair of the input signals.

4 Systems (20 points)

Part a. Let $\mathcal{S}$ represent a linear, time-invariant system with unit-sample response

$$h[n] = \begin{cases} \left(\frac{1}{2}\right)^n & \text{if } n \text{ is both even and greater than or equal to } 0 \\ 0 & \text{otherwise} \end{cases}$$

If the input to $\mathcal{S}$ is

$$x[n] = \cos(\pi n/4)$$

then the output can be written in the following form:

$$y[n] = A\cos(\pi n/4) + B\sin(\pi n/4)$$

Determine A and B and enter these numbers in the following boxes.

$A = \frac{16}{17}$

$B = \frac{4}{17}$

The frequency response of $\mathcal{S}$ is

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\Omega n} = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{2m} e^{-j\Omega 2m} = \frac{1}{1 - \frac{1}{4}e^{-j2\Omega}}$$

The input signal can be written as

$$x[n] = \cos(\pi n/4) = \text{Re}\left(e^{j\pi n/4}\right)$$

and the corresponding output is then

$$y[n] = \text{Re}\left(H\left(\frac{\pi}{4}\right)e^{j\pi n/4}\right) = \text{Re}\left(\frac{1}{1 - \frac{1}{4}e^{-j\pi/2}}e^{j\pi n/4}\right)$$

$$= \text{Re}\left(\frac{1}{1 + \frac{1}{4}j}e^{j\pi n/4}\right) = \text{Re}\left(\frac{4}{4 + j}e^{j\pi n/4}\right)$$

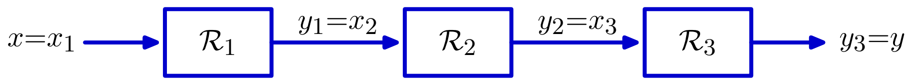
$$= \text{Re}\left(\left(\frac{16 - 4j}{17}\right)\left(\cos\left(\frac{\pi n}{4}\right) + j\sin\left(\frac{\pi n}{4}\right)\right)\right)$$

$$= \frac{16}{17}\cos\left(\frac{\pi n}{4}\right) + \frac{4}{17}\sin\left(\frac{\pi n}{4}\right)$$

Part b. A system is constructed from three identical subsystems: $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$. The input signal x_i and output signal y_i of each subsystem are related by the following differential equation:

$$y_i(t) + \frac{d}{dt}y_i(t) = x_i(t)$$

where i is 1, 2, and 3 for $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$, respectively. The three subsystems are connected so that the output of $\mathcal{R}_1$ becomes the input to $\mathcal{R}_2$, and the output of $\mathcal{R}_2$ becomes the input to $\mathcal{R}_3$ as shown in the following diagram.



If the input to the combined system is

$$x(t) = x_1(t) = \cos(t)$$

then the output can be written in the following form:

$$y(t) = y_3(t) = C\cos(t - \phi)$$

Determine C and ϕ, and enter these numbers in the following boxes.

$C =$ $\frac{1}{2\sqrt{2}}$

$\phi =$ $\frac{3\pi}{4}$

Determine the frequency response of each subsystem by taking the Fourier transform of its differential equation:

$$Y_i(\omega) + j\omega Y_i(\omega) = X_i(\omega)$$

Then the frequency response of the i^{th} subsystem is

$$H_i(\omega) = \frac{Y_i(\Omega)}{X_i(\Omega)} = \frac{1}{1+j\omega}$$

The frequency response of the composite system is the product of those for each subsystem:

$$H(\omega) = \frac{Y(\Omega)}{X(\Omega)} = \left(\frac{1}{1+j\omega}\right)^3$$

The input signal can be written as a sum of complex exponentials:

$$x(t) = \cos(t) = \frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt}$$

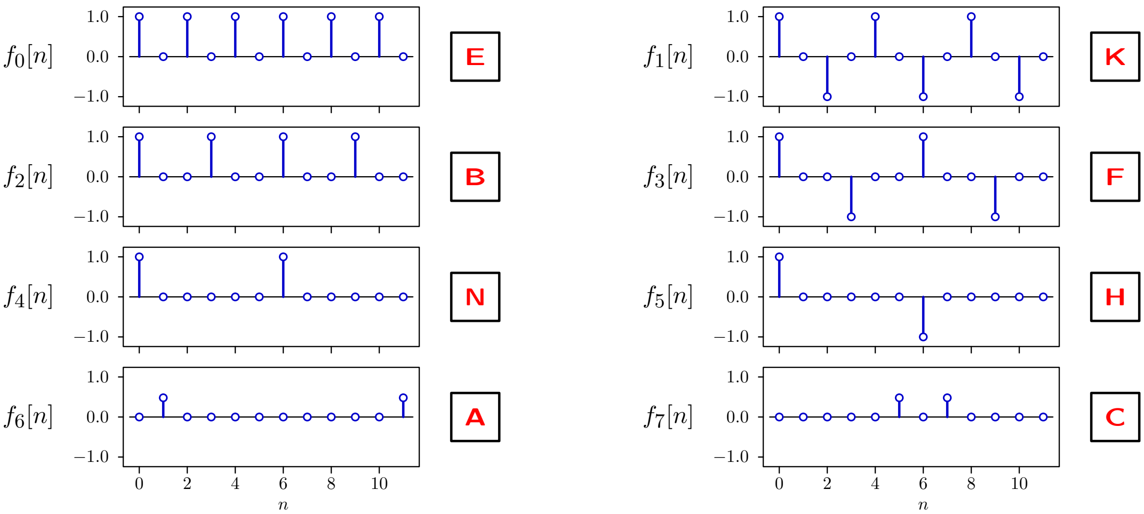
and the resulting output signal is then

$$y(t) = \frac{1}{2}H(1)e^{jt} + \frac{1}{2}H(-1)e^{-jt} = \frac{1}{2}\left(\frac{1}{1+j}\right)^3 e^{jt} + \frac{1}{2}\left(\frac{1}{1-j}\right)^3 e^{-jt}$$

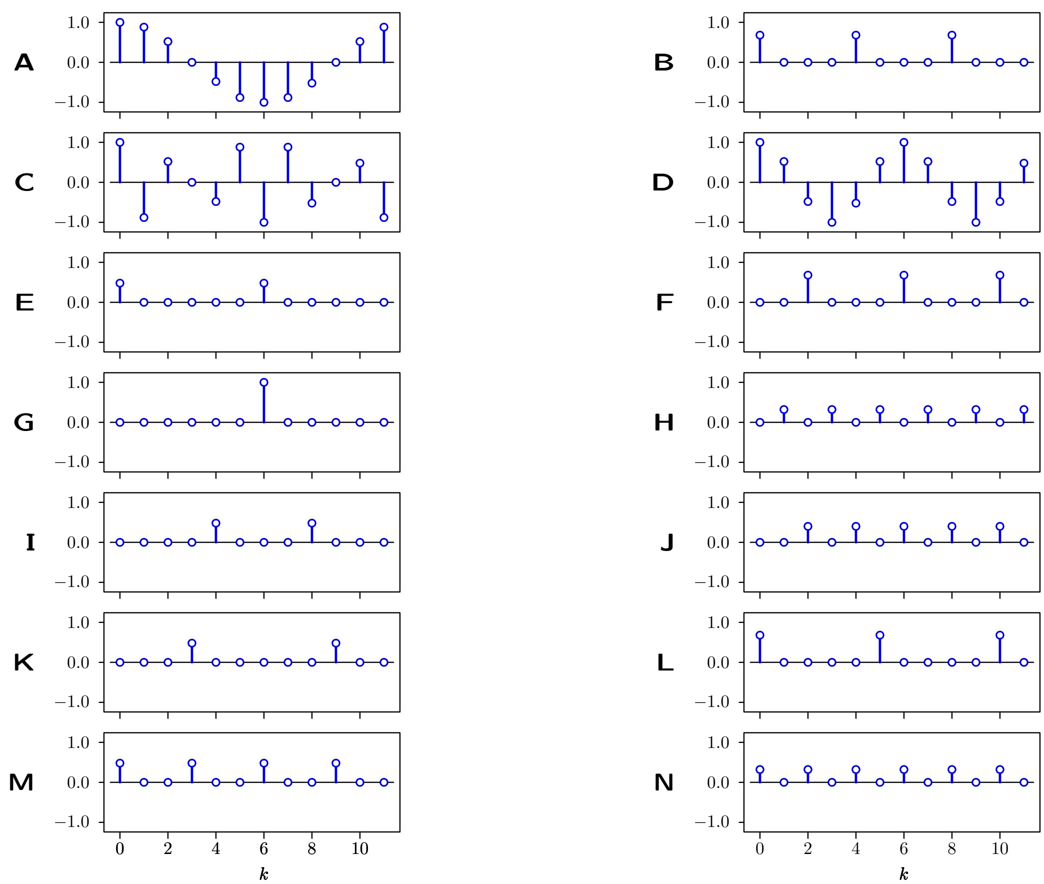
$$= \text{Re}\left(\left(\frac{1}{1+j}\right)^3 e^{jt}\right) = \text{Re}\left(\left(\frac{1}{\sqrt{2}}e^{-j\pi/4}\right)^3 e^{jt}\right) = \text{Re}\left(\frac{1}{2\sqrt{2}}e^{j(t-3\pi/4)}\right) = \frac{1}{2\sqrt{2}}\cos\left(t - \frac{3\pi}{4}\right)$$

5 DFT Matching (24 points)

Each of the following plots shows samples $n=0$ through $n=11$ of a discrete-time signal $f_i[n]$.



Determine which of the following plots (**A-N**) shows the real part of the 12-point DFT of each of the preceding signals ($f_0[n]$–$f_7[n]$), and enter that letter in the corresponding box above.



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