6.3000 Quiz 2 Fall 2022

Name: Solutions

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This quiz is closed book, but you may use two 8.5×11 sheets of notes (four sides).

You may NOT use any electronic devices (such as calculators and phones).

If you have questions, please **come to us** at the front of the room to ask.

Please enter all solutions in the boxes provided.

Work on other pages with QR codes will be considered for partial credit.
Please provide a note if you continue work on worksheets at the end of the exam.

Please do not write on the QR codes at the bottom of each page.

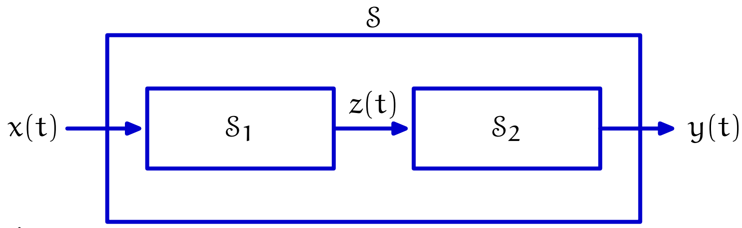
We use those codes to identify which pages belong to each student.

Trigonometric Identities Reference

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$
$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b)$$
$$\cos(a+b)-\cos(a-b) = -2\sin(a)\sin(b)$$
$$\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$$
$$\sin(a+b)-\sin(a-b) = 2\cos(a)\sin(b)$$
$$2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b)$$
$$2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$$
$$2\sin(a)\cos(b) = \sin(a+b)+\sin(a-b)$$
$$2\cos(a)\sin(b) = \sin(a+b)-\sin(a-b)$$

1 System Identification (20 points)

Part a. Two linear, time-invariant, continuous-time systems, $\mathcal{S}_1$ and $\mathcal{S}_2$ are connected so that the output of $\mathcal{S}_1$ is the input to $\mathcal{S}_2$ as shown in the following figure.



The impulse response of $\mathcal{S}_1$ is

$$h_1(t) = e^{-t}u(t)$$

and the impulse response of $\mathcal{S}_2$ is

$$h_2(t) = e^{-2t}u(t)$$

where $u(t)$ represents the unit-step function:

$$u(t) = \begin{cases} 1 & \text{if } t \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

Find a differential equation of the form

$$a_0 y(t) + a_1 y'(t) + a_2 y''(t) + \cdots = b_0 x(t) + b_1 x'(t) + b_2 x''(t) + \cdots$$

to relate $x(t)$ and $y(t)$, where $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ represent constants, primes represent derivatives, double primes represent double derivatives, and the ellipses represent higher order terms.

Note that your answer should not contain any $z(t)$ terms.

Enter your differential equation in the box below.

$$2y(t) + 3y'(t) + y''(t) = x(t)$$

Find the frequency responses of $\mathcal{S}_1$ and $\mathcal{S}_2$:

$$H_1(\omega) = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-j\omega t}dt = \int_0^{\infty} e^{-(1+j\omega)t}dt = \frac{1}{1+j\omega}$$

$$H_2(\omega) = \int_{-\infty}^{\infty} e^{-2t}u(t)e^{-j\omega t}dt = \int_0^{\infty} e^{-(2+j\omega)t}dt = \frac{1}{2+j\omega}$$

The frequency response of $\mathcal{S}$ is the product of H_1 and H_2, which is equal to the ratio of the Fourier transform of $y(t)$ over that of $x(t)$:

$$H(\omega) = H_1(\omega)H_2(\omega) = \frac{1}{(1+j\omega)(2+j\omega)} = \frac{Y(\omega)}{X(\omega)}$$

Therefore

$$X(\omega) = (2 + 3j\omega + (j\omega)^2)Y(\omega)$$

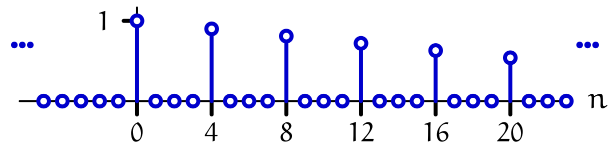
Taking the inverse Fourier transform yields the desired differential equation:

$$2y(t) + 3y'(t) + y''(t) = x(t)$$

Multiplying each term by the same constant results in an equally acceptable solution.

Part b. Consider a linear, time-invariant, discrete-time system with the unit-sample response $h[n]$ given below:

$$h[n] = \sum_{i=0}^{\infty} \alpha^i \delta[n-4i]$$



where $0 < \alpha < 1$.

Determine a difference equation to relate the input $x[n]$ and output $y[n]$ of this system.

Enter your difference equation in the box below.

$$y[n] - \alpha y[n-4] = x[n]$$

Note: Your solution does not have to be closed-form to receive full credit.

Method 1. The convolution sum can be used to find a relation between input and output samples, as follows.

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = \sum_{m=-\infty}^{\infty} \sum_{i=0}^{\infty} \alpha^i \delta[m-4i]x[n-m] = \sum_{i=0}^{\infty} \sum_{m=-\infty}^{\infty} \alpha^i \delta[m-4i]x[n-m] = \sum_{i=0}^{\infty} \alpha^i x[n-4i]$$

$$y[n] = x[n] + \alpha x[n-4] + \alpha^2 x[n-8] + \alpha^3 x[n-12] + \cdots$$

This expression has an infinite number of terms, but it can be simplified by using the linear time-invariant properties of the system.

$$y[n] = x[n] + \alpha x[n-4] + \alpha^2 x[n-8] + \alpha^3 x[n-12] + \cdots$$

$$\alpha y[n-4] = \alpha x[n-4] + \alpha^2 x[n-8] + \alpha^3 x[n-12] + \alpha^4 x[n-16] + \cdots$$

$$y[n] - \alpha y[n-4] = x[n]$$

Method 2. An alternative approach is to work in the frequency domain. The frequency response of the system is the Fourier transform of the unit-sample response.

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} = \sum_{m=0}^{\infty} \alpha^i \delta[n-4i]e^{-j\Omega n} = \sum_{m=0}^{\infty} \alpha^i e^{-j\Omega 4i} = \frac{1}{1-\alpha e^{-j4\Omega}}$$

The frequency response is the ratio of the Fourier transforms of y and x:

$$H(\Omega) = \frac{1}{1-\alpha e^{-j4\Omega}} = \frac{Y(\Omega)}{X(\Omega)}$$

$$X(\Omega) = (1-\alpha e^{j4\Omega})Y(\Omega) = Y(\Omega) - \alpha e^{j4\Omega}Y(\Omega)$$

We can find the difference equation by taking the inverse Fourier transforms of each term:

$$y[n] - \alpha y[n-4] = x[n]$$

2 Convolutions (27 points)

Part a. Let f_1 represent the following signal:

$$f_1[n] = \begin{cases} (-1)^n & \text{if } 0 \leqslant n < 6 \\ 0 & \text{otherwise} \end{cases}$$

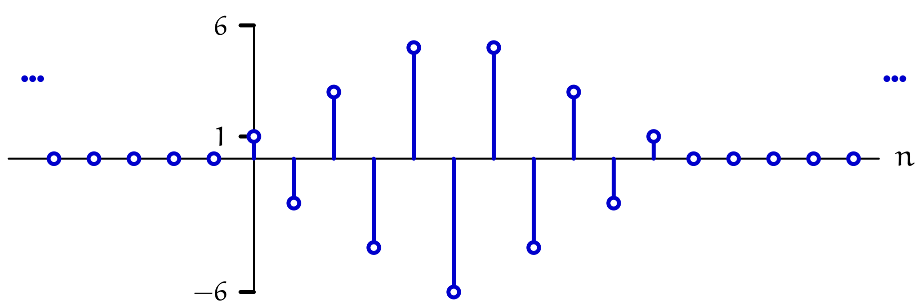
and let f_2 represent the signal that results when f_1 is convolved with itself:

$$f_2[n] = (f_1 * f_1)[n]$$

Determine the first 15 samples of f_2 and enter their values in the boxes below.

$f_2[0]$:	1	$f_2[5]$:	−6	$f_2[10]$:	1
$f_2[1]$:	−2	$f_2[6]$:	5	$f_2[11]$:	0
$f_2[2]$:	3	$f_2[7]$:	−4	$f_2[12]$:	0
$f_2[3]$:	−4	$f_2[8]$:	3	$f_2[13]$:	0
$f_2[4]$:	5	$f_2[9]$:	−2	$f_2[14]$:	0

$$g[n] = (f * f)[n] = \sum_{n=-\infty}^{\infty} f[m]f[n-m]$$



Part b. Let g_1 represent a signal whose even-numbered samples are 1 and whose odd-numbered samples are zero. Let g_2 represent the signal that would result from the following procedure:

- use an analysis width $N = 6$ (i.e., $0 \leqslant n < 6$) to calculate the DFT G_1 of g_1,
- let $G_2[k] = G_1^2[k]$, and
- take the inverse DFT of G_2 and multiply by $N = 6$ to find g_2.

Determine $g_2[0]$ through $g_2[5]$ and enter their values in the boxes below.

$g_2[0]$: 3

$g_2[1]$: 0

$g_2[2]$: 3

$g_2[3]$: 0

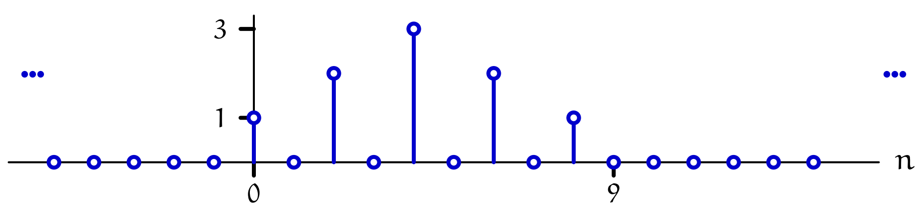
$g_2[4]$: 3

$g_2[5]$: 0

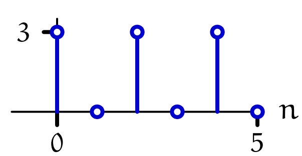
The procedure described for computing g_2 is equivalent to circularly convolving the first N samples of g_1 with itself using an analysis window $N = 10$. Circular convolution is equivalent to conventional convolution followed by aliasing as follows:

$$(g_1 \circledast g_1)[n] = \sum_{m=-\infty}^{\infty} (g_1 * g_1)[n + iN]$$

We can compute the conventional convolution of g_1 with itself using superposition or flip-and-shift. Either way, we get the following result.



After aliasing, the following signal results:

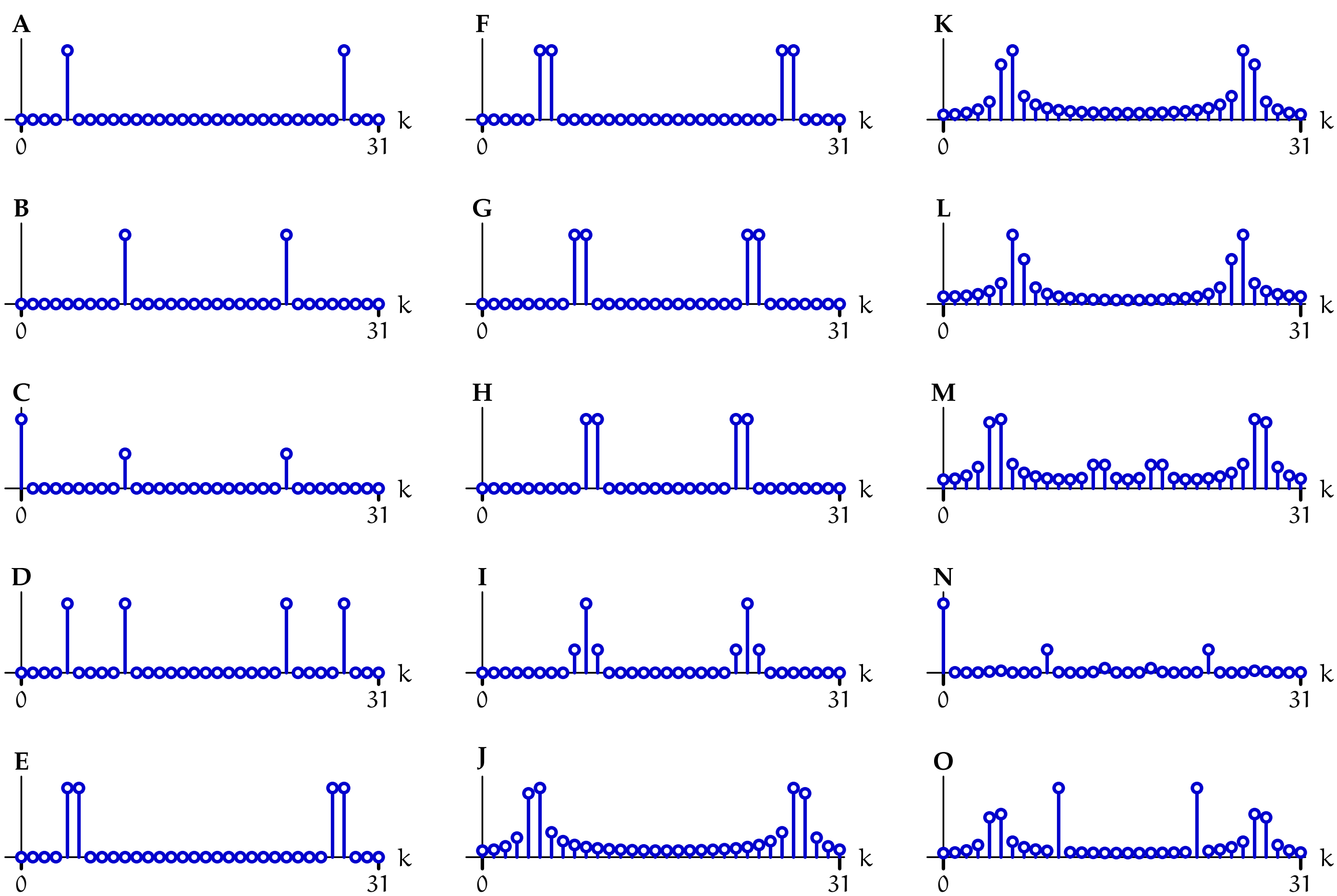


3 DFTs of Sinusoids (32 points)

Each of the following eight signals ($x_0[n]$ through $x_7[n]$) are defined by functions of discrete time n.

$x_0[n] = \cos(9\pi n/32)$:	J	$x_4[n] = \cos(2\pi n/5)$:	L
$x_1[n] = \cos(9\pi n/16)$:	B	$x_5[n] = \left\lvert\cos(9\pi n/32)\right\rvert$:	N
$x_2[n] = \cos(9\pi n/26)$:	K	$x_6[n] = \cos^2(9\pi n/32)$:	C
$x_3[n] = \cos(9\pi n/36)$:	A	$x_7[n] = \cos^3(9\pi n/32)$:	M

Discrete Fourier Transforms (DFTs) of length $N = 32$ are computed for each of these signals. Identify which of the plots below shows the magnitude of each of these DFTs as a function of frequency index k. Then write the letter of that plot in the corresponding box above.

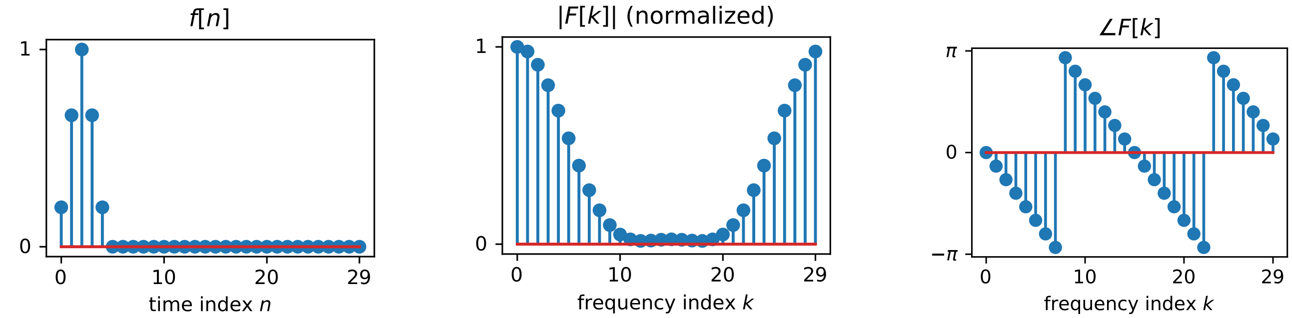


The vertical scales for these plots differ. Each has been normalized so that its peak magnitude is 1.

Worksheet (*intentionally blank*)

4 Related DFTs (40 points)

Let $f[n]$ represent a periodic, discrete-time signal with period N=30 so that $f[n] = f[n+30]$ for all n. The following plots show one period of $f[n]$ along with the magnitude and angle of its Discrete Fourier Transform (DFT) computed with an analysis window of length N=30. The magnitude plot has been scaled so that the maximum magnitude is 1.



Five signals ($g_1[n]$ through $g_5[n]$) are derived from $f[n]$ as shown below.

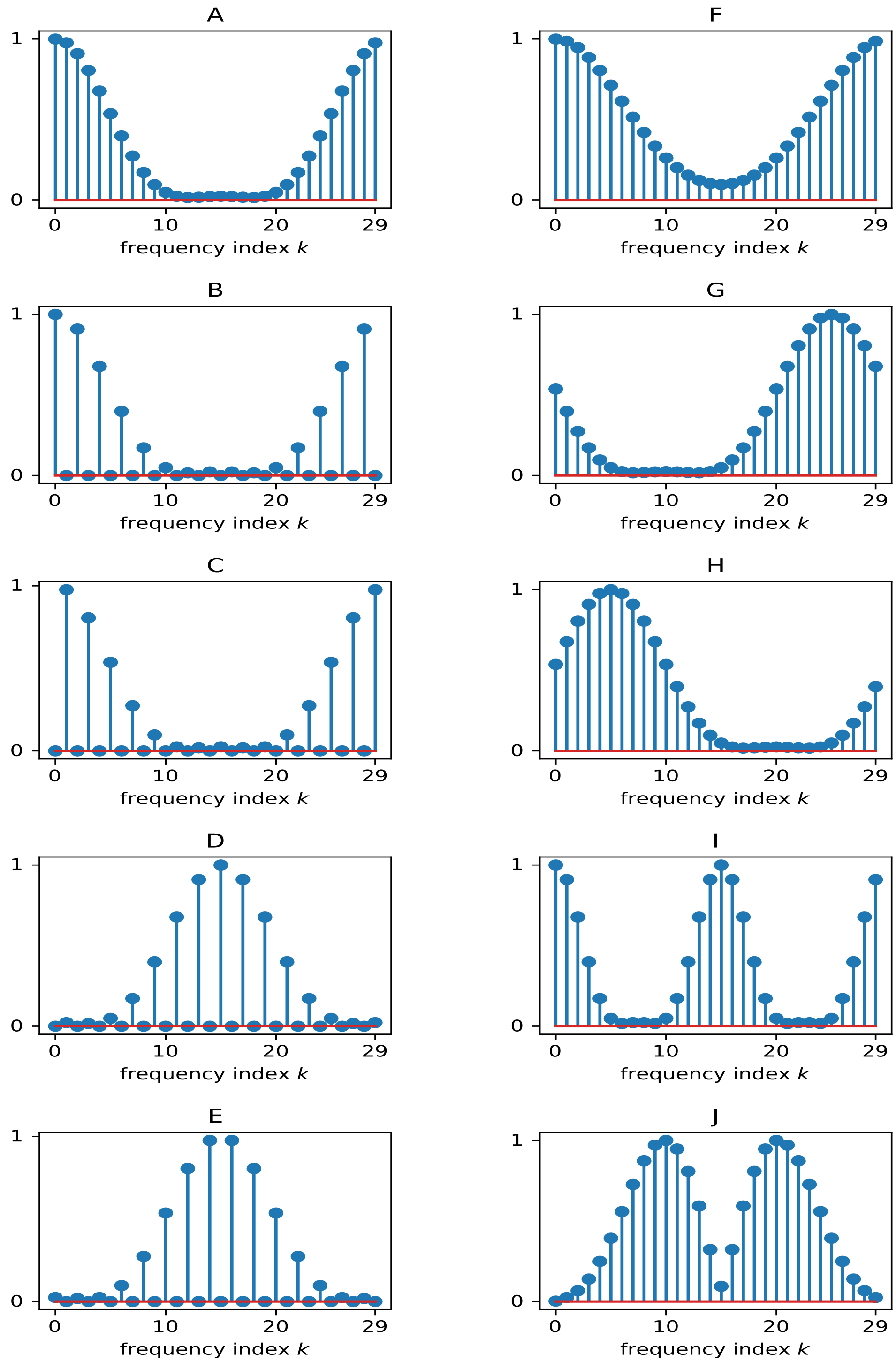
	normalized magnitude	angle
$g_1[n] = f[-n]$	A	j
$g_2[n] = f[n-1]$	A	h
$g_3[n] = f[n+1]$	A	b
$g_4[n] = f^2[n]$	F	c
$g_5[n] = \begin{cases} f[n/2] & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$	I	i

Determine which of the panels (A-J) on the following page shows the magnitude of the DFT of each of the derived signals and write the letter for that panel in the corresponding box in the magnitude column above.

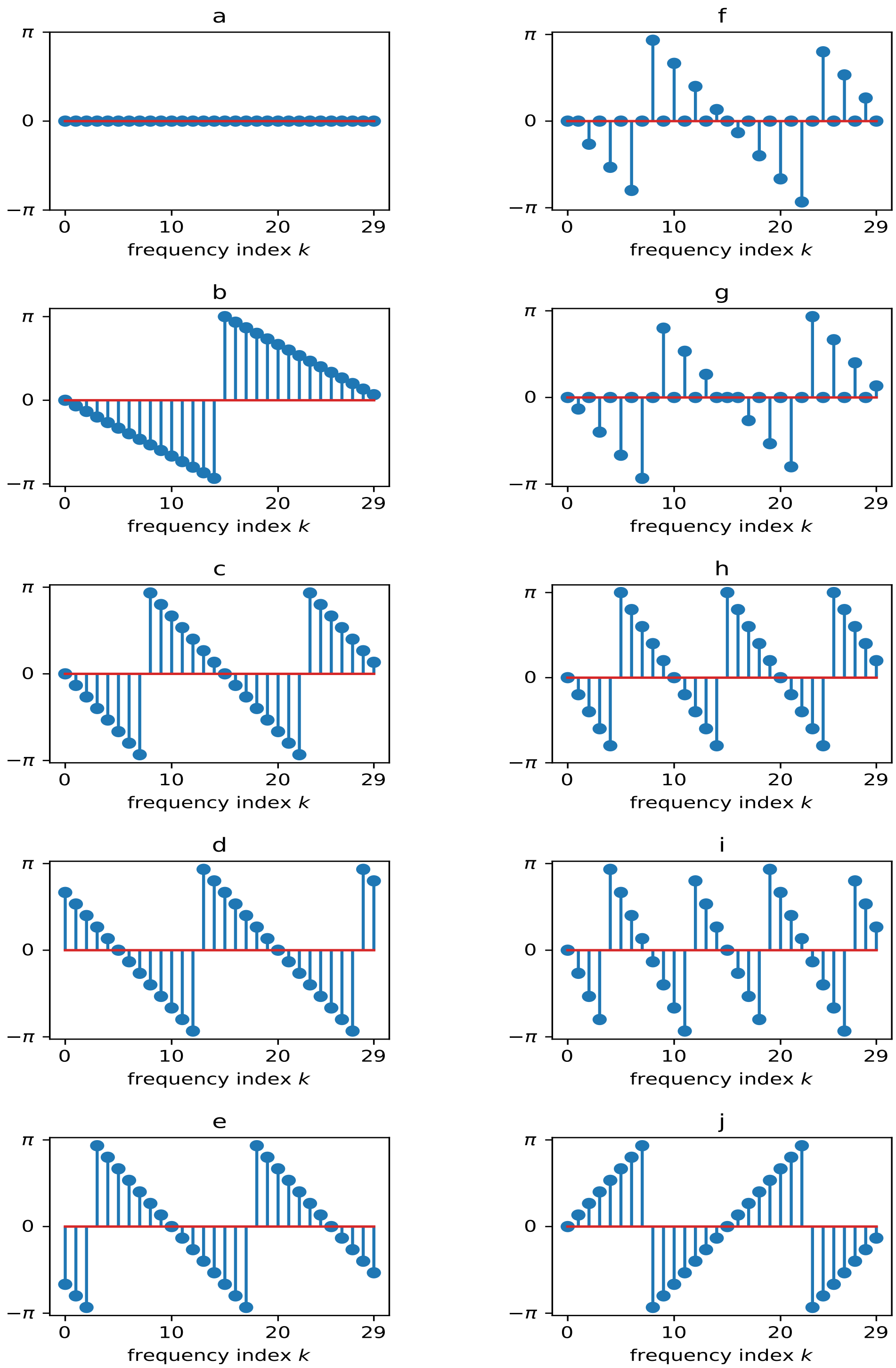
Similarly, determine which of the panels (a-j) on the subsequent page shows the angle of the DFT of each of the derived signals and write the letter for that panel in the corresponding box in the angle column above.

The same panel may be used more than once.

Magnitude Plots for Problem 4: the magnitudes in each plot have been scaled so that the largest magnitude in each plot is 1.



Angle Plots for Problem 4.



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