

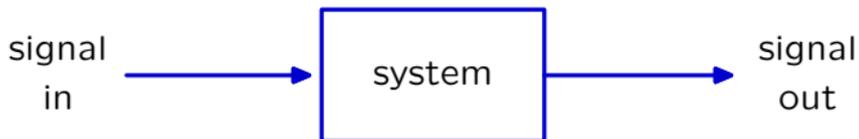
6.3000: Signal Processing

Unit-Sample Response and Convolution

- Unit Sample Signal and Unit Sample Response
- Discrete-Time Convolution
- Impulse Function and Impulse Response
- Continuous-Time Convolution

Last Time: The System Abstraction

Represent a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



Properties of Systems

We will focus primarily on systems that have two important properties:

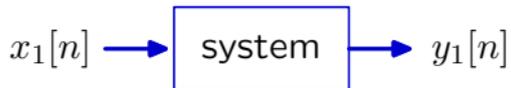
- **linearity**
- **time invariance**

Such systems are both useful and mathematically tractable.

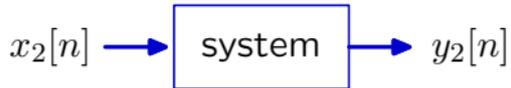
Additivity

A system is additive if its **response to a sum** of signals is equal to the **sum of the responses** to each signal taken one at a time.

Given



and



the **system is additive** if



for all possible inputs and all times n .

Homogeneity

A system is homogeneous if multiplying its input signal by a constant multiplies the output signal by the same constant.

Given



the **system is homogeneous** if

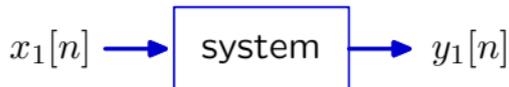


for all α and all possible inputs and all times n .

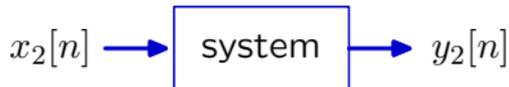
Linearity

A system is linear if its **response to a weighted sum** of input signals is equal to the **weighted sum of its responses** to each of the input signals.

Given



and



the **system is linear** if



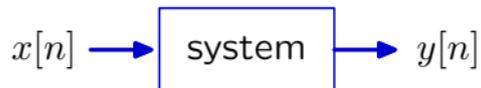
for all α and β and all possible inputs and all times n .

A system is linear if it is both additive and homogeneous.

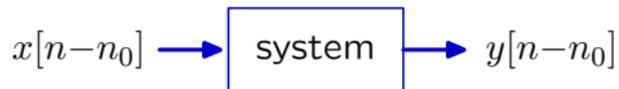
Time-Invariance

A system is time-invariant if delaying the input signal simply delays the output signal by the same amount of time.

Given



the **system is time invariant** if



for all n_0 and for all possible inputs and all times n .

Linear Difference Equations with Constant Coefficients

If it is initially at rest,¹ a system whose input/output relation obeys a linear difference equation with constant coefficients is linear and time-invariant.

General form:

$$\sum_l c_l y[n-l] = \sum_m d_m x[n-m]$$

Additivity: output of sum of inputs is the sum of corresponding outputs

$$\sum_l c_l (y_1[n-l] + y_2[n-l]) = \sum_m d_m (x_1[n-m] + x_2[n-m]) \quad \checkmark$$

Homogeneity: scaling the input scales the resulting output

$$\sum_l \alpha c_l y[n-l] = \sum_m \alpha d_m x[n-m] \quad \checkmark$$

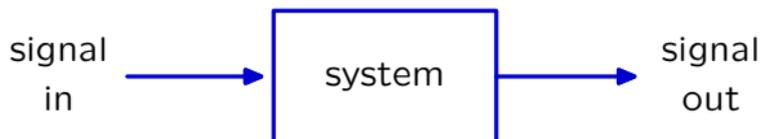
Time invariance: delaying an input delays the resulting output

$$\sum_l c_l y[(n-n_0)-l] = \sum_m d_m x[(n-n_0)-m] \quad \checkmark$$

¹ so that the homogeneous part of the output signal is zero

Today: Representing a System by its Unit-Sample Response

Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



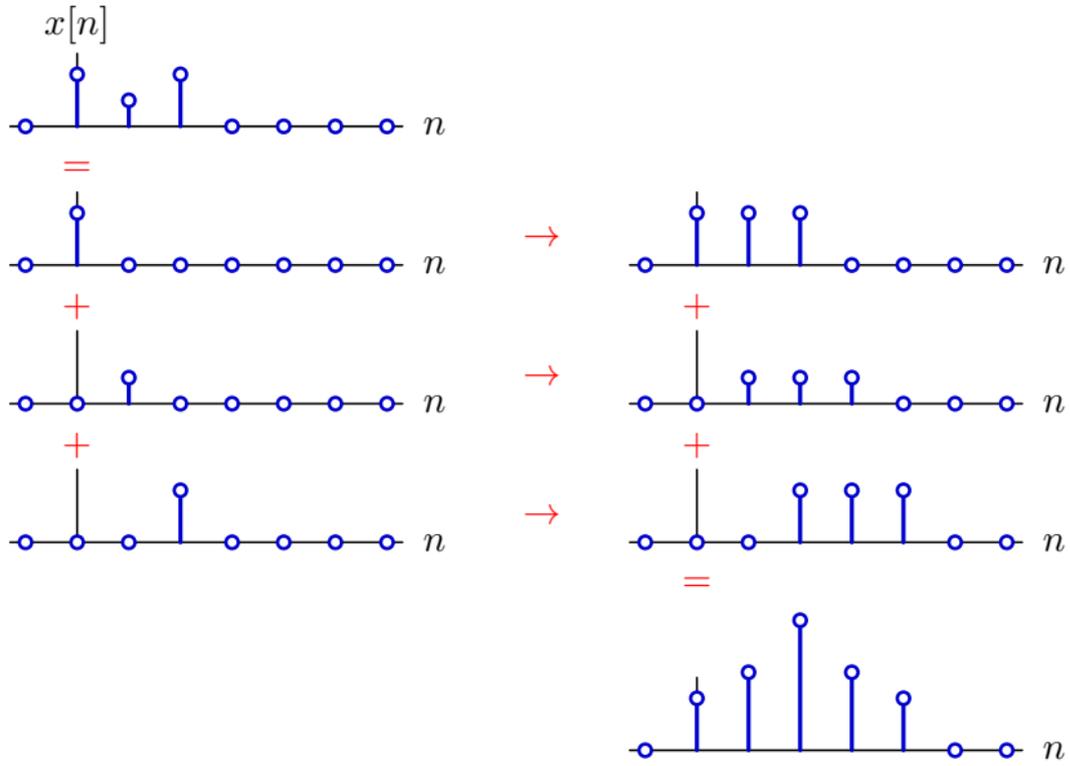
This abstraction is particularly powerful for **linear and time-invariant** systems, which are both **prevalent** and **mathematically tractable**.

Three important representations for LTI systems:

- **Difference Equation:** algebraic **constraint** on samples ✓
- **Convolution:** represent a system by its **unit-sample response**
- **Filter:** represent a system by its **frequency response**

Superposition

Response of an LTI system is determined by the system's response to $\delta[n]$.



Unit-Sample Response and Convolution

If a system is linear and time-invariant (LTI), its input-output relation is **completely specified** by the system's unit-sample response $h[n]$.

1. Find the unit-sample response of a system.



2. Time invariance implies that shifting the input simply shifts the output.



3. Homogeneity implies that scaling the input simply scales the output.



4. Additivity implies that the response to a sum is the sum of responses.

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

This rule for combining the input $x[n]$ with the unit-sample response $h[n]$ is called **convolution**.

Convolution

The response of an LTI system to an arbitrary input $x[n]$ can be found by **convolving** that input with the **unit-sample response** $h[n]$ of the system.



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \equiv (x * h)[n]$$

This is an amazing result.

We can represent the operation of an LTI system by a **single signal!**

Notation

Convolution is conventionally represented with an asterisk.

$$\sum_{m=-\infty}^{\infty} x[m]h[n-m] \equiv (x * h)[n]$$

It is customary (but confusing) to abbreviate this notation:

$$(x * h)[n] = x[n] * h[n]$$

$x[n] * h[n]$ looks like an operation of samples; but it is not!

$$x[1] * h[1] \neq (x * h)[1]$$

Convolution operates on signals not samples.

Unambiguous notation:

$$y = x * h$$

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \equiv (x * h)[n]$$

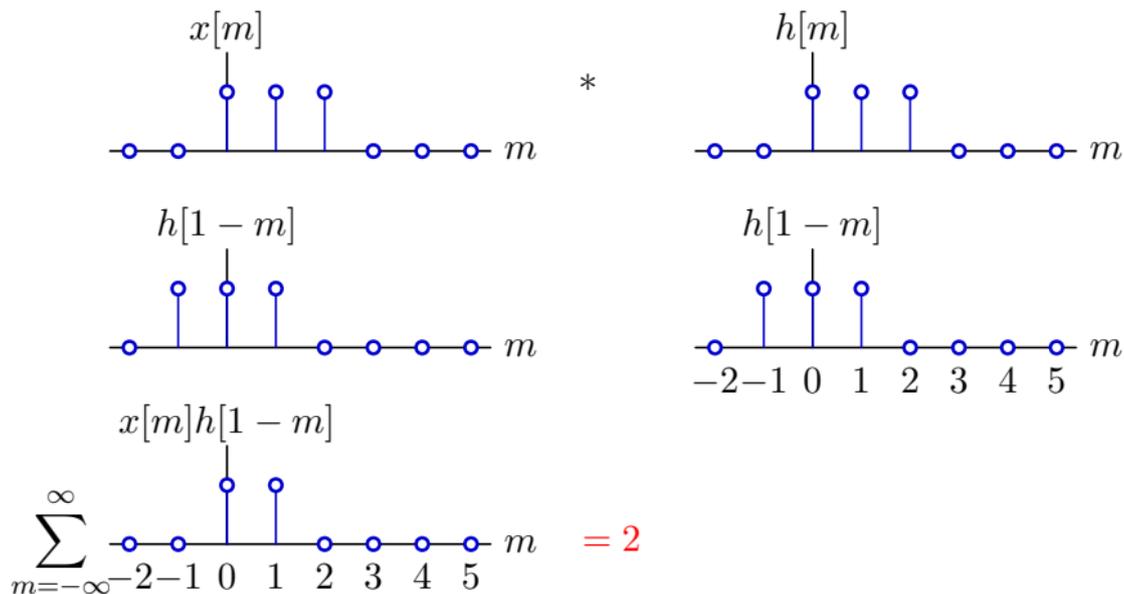
The symbols x and h represent DT signals.

Convolving x with h generates a new DT signal $y = x * h$.

Structure of Convolution

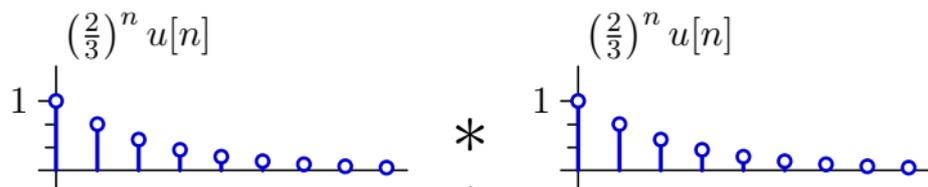
Focus on computing the n^{th} output sample.

$$y[1] = \sum_{m=-\infty}^{\infty} x[m]h[1-m]$$

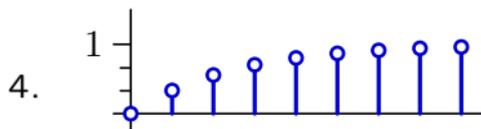
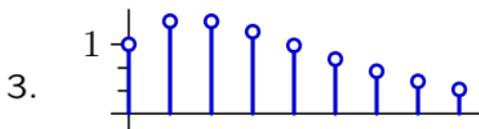
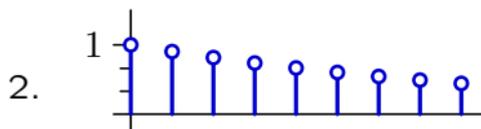
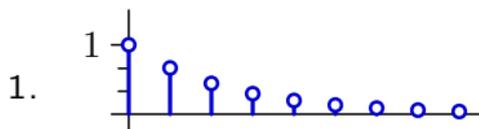


Check Yourself

Consider the convolution of two geometric sequences:



Which plot below shows the result of the convolution above?

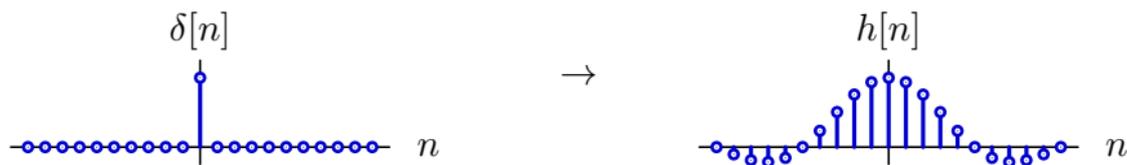


Unit-Sample Response

The unit-sample response is a **complete** description of an LTI system.



The response of a linear system to a unit sample signal



can be used to compute the response to any arbitrary input signal.

$$y[n] = (x * h)[n] \equiv \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Continuous-Time Systems

Superposition and convolution are also important for CT systems.

Linear Differential Equations with Constant Coefficients

If it is initially at rest,² a system whose input/output relation obeys a linear differential equation with constant coefficients is linear and time-invariant.

General form:

$$\sum_l c_l \frac{d^l y(t)}{dt^l} = \sum_m d_m \frac{d^m x(t)}{dt^m}$$

Additivity: output of sum is sum of outputs

$$\sum_l c_l \left(\frac{d^l y_1(t)}{dt^l} + \frac{d^l y_2(t)}{dt^l} \right) = \sum_m d_m \left(\frac{d^m x_1(t)}{dt^m} + \frac{d^m x_2(t)}{dt^m} \right)$$

Homogeneity: scaling an input scales its output

$$\sum_l c_l \left(\alpha \frac{d^l y(t)}{dt^l} \right) = \sum_m d_m \left(\alpha \frac{d^m x(t)}{dt^m} \right)$$

Time invariance: delaying an input delays its output

$$\sum_l c_l \frac{d^l y(t - \tau)}{dt^l} = \sum_m d_m \frac{d^m x(t - \tau)}{dt^m}$$

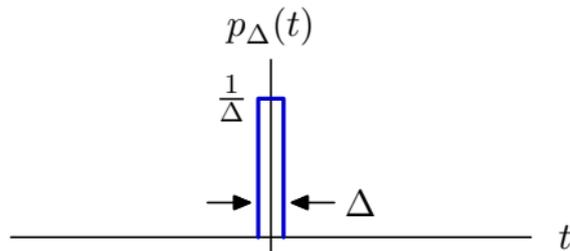
² so that the homogeneous part of the output signal is zero

Impulse Response

A CT system is completely characterized by its **impulse response**, much as a DT system is completely characterized by its unit-sample response.

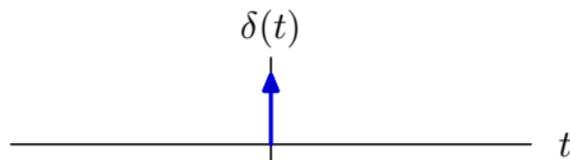
We have worked with the impulse (Dirac delta) function $\delta(t)$ previously. It's defined in a limit as follows.

Let $p_{\Delta}(t)$ represent a pulse of width Δ and height $\frac{1}{\Delta}$ so that its area is 1.



Then

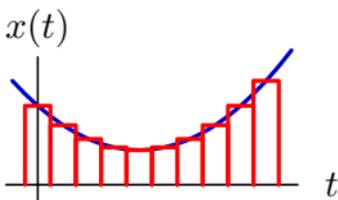
$$\delta(t) = \lim_{\Delta \rightarrow 0} p_{\Delta}(t)$$



Impulse Response

An arbitrary CT signal can be represented by an infinite sum of infinitesimal impulses (which define an integral).

Approximate an arbitrary signal $x(t)$ (blue) as a sum of pulses $p_{\Delta}(t)$ (red).



$$x_{\Delta}(t) = \sum_{m=-\infty}^{\infty} x(m\Delta)p_{\Delta}(t - m\Delta)\Delta$$

and the limit of $x_{\Delta}(t)$ as $\Delta \rightarrow 0$ will approximate $x(t)$.

$$\lim_{\Delta \rightarrow 0} x_{\Delta}(t) = \lim_{\Delta \rightarrow 0} \sum_{m=-\infty}^{\infty} x(m\Delta)p_{\Delta}(t - m\Delta)\Delta \rightarrow \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau) d\tau$$

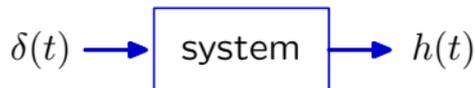
The result in CT is much like the result for DT:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau) d\tau \qquad x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta(n - m)$$

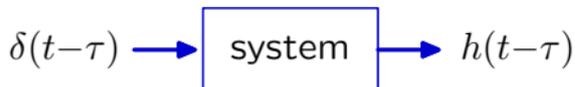
Impulse Response

If a system is linear and time-invariant (LTI), its input-output relation is **completely specified** by the system's impulse response $h(t)$.

1. One can always find the impulse response of a system.



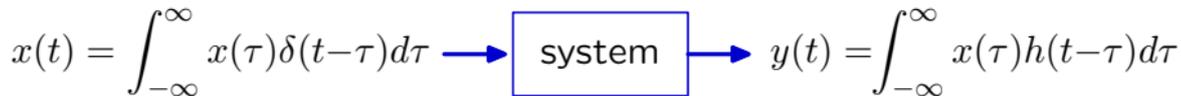
2. Time invariance implies that shifting the input simply shifts the output.



3. Homogeneity implies that scaling the input simply scales the output.



4. Additivity implies that the response to a sum is the sum of responses.



$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \rightarrow \boxed{\text{system}} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

This rule for combining the input $x(t)$ with the impulse response $h(t)$ is called **convolution**.

Impulse Response

The impulse response is a **complete** description of an LTI system.



The response of a linear system to an impulse function



can be used to compute the response to any arbitrary input signal.

$$y(t) = (x * h)(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

Comparison of CT and DT Convolution

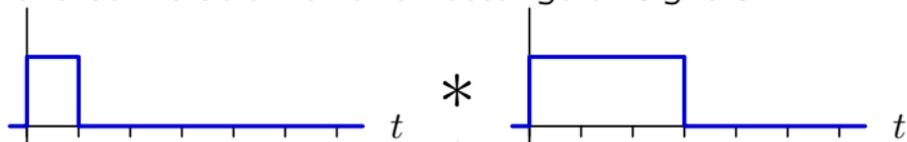
Convolution of CT signals is analogous to convolution of DT signals.

$$\text{DT: } y[n] = (x * h)[n] = \sum_{m=-\infty}^{\infty} x[m]h[n - m]$$

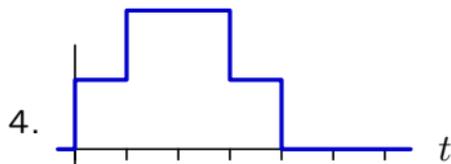
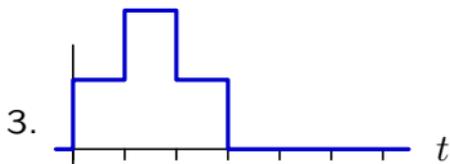
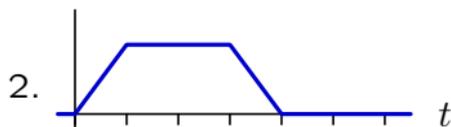
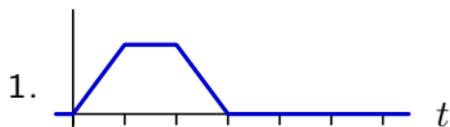
$$\text{CT: } y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

Check Yourself

Consider the convolution of two rectangular signals:



Which plot shows the result of the convolution above?



5. none of the above

Check Yourself

Match expressions on the left with functions on the right where

$$f(t) = e^{-t} u(t)$$

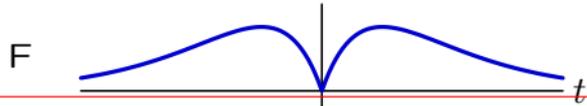
$$g(t) = e^t u(-t)$$

$$(f * f)(t) \quad \square$$

$$(g * g)(t) \quad \square$$

$$(f * g)(t) \quad \square$$

$$(g * f)(t) \quad \square$$



Summary

The response of a discrete-time, LTI system to an input $x[n]$ can be computed by convolving the input with the system's **unit sample response**.



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \equiv (x * h)[n]$$

The response of a continuous-time, LTI system to an input $x(t)$ can be computed by convolving the input with the system's **impulse response**.



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \equiv (x * h)(t)$$

Convolution allows us to represent a system by a **single signal!**