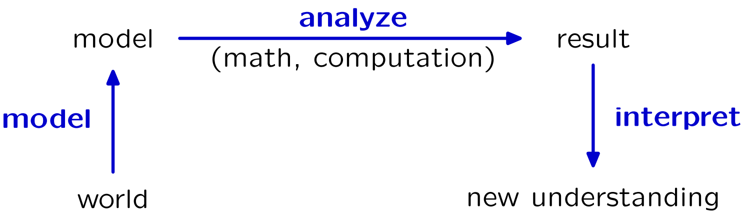
6.3000: Signal Processing

Synthetic Aperture Optics

Context: Fourier methods are **widely used** in math, science, engineering.

- MRI – imaging in the frequency domain
- Vibrating String – Fourier series in the real world
- Fourier Optics – two-dimensional Fourier transforms in the real world
- Synthetic Aperture Microscopy



May 07, 2026

Fourier methods in math, science, engineering

Fourier methods build on useful and interesting **mathematical properties**

→ harmonically related sinusoids are **orthogonal** over the period T.

→ conceptually **simple** and computationally **efficient** (e.g., FFT).

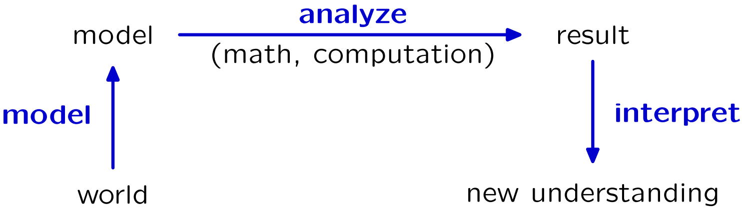
Fourier methods also play special roles in many branches of **physics**

→ one example is MRI (last lecture)

→ physics of **waves** underlie many applications:

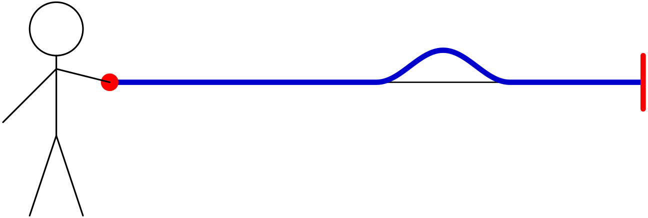
→ waves of motion (strings): 1D wave equation

→ waves of light (optics): 2D/3D wave equation



Physical Example: Vibrating String

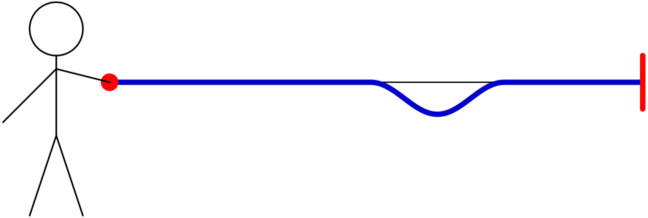
A taut string supports wave motion.



The speed V of the wave depends on the tension and mass of the string.

Physical Example: Vibrating String

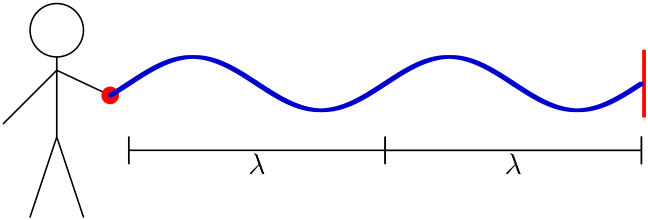
The wave reflects off a rigid boundary.



The amplitude of the reflected wave is opposite that of the incident wave.

Physical Example: Vibrating String

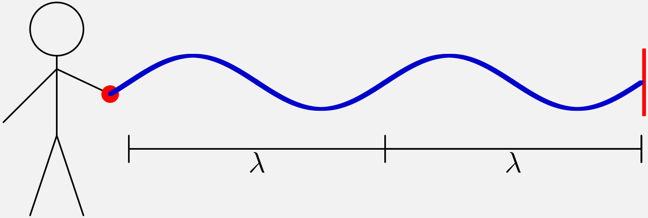
Sinusoidal excitation (frequency f=0.0208 cycles per distance step)



launches a sinusoidal “traveling wave with wavelength λ (distance steps).

Check Yourself

Sinusoidal excitation launches a sinusoidal "traveling wave" with wavelength λ.



What is the relation between the wavelength λ and the frequency of the excitation (ω)? Assume the velocity of the traveling wave is V.

1. $\lambda = \frac{V}{\omega}$
2. $\lambda = \frac{2\pi V}{\omega}$
3. $\lambda = \frac{1}{\omega V}$
4. $\lambda = \frac{2\pi}{V}$
5. none of the above

Check Yourself

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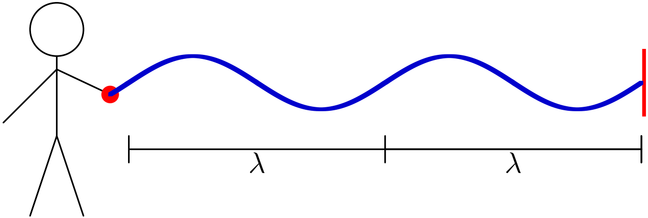

The wavelength λ = distance traveled during one period T of excitation.

$$\text{velocity} = \frac{\text{distance}}{\text{time}}$$

$$V = \frac{\lambda}{T} = \lambda f = \frac{\lambda\omega}{2\pi}$$

$$\lambda = \frac{2\pi V}{\omega}$$

Answer = 2

Check Yourself

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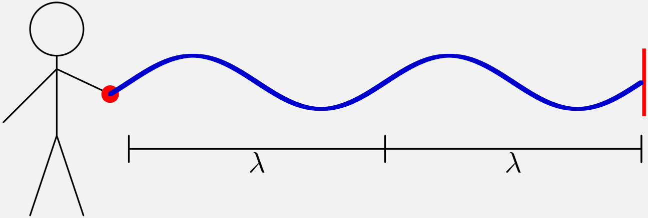

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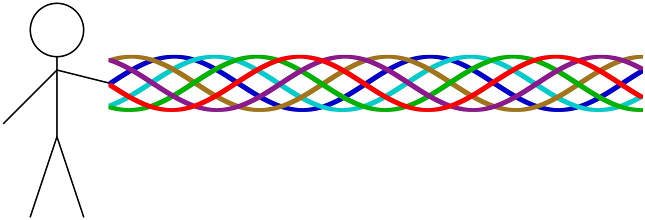


Pass 1: wave launches from left and travels to right.
Pass 2: wave reflects off right boundary and travels to left.
Pass 3: wave reflects off left and travels to right.
Pass 4: wave reflects off right boundary and travels to left.
Pass 5: wave reflects off left and travels to right.
Pass 6: wave reflects off right boundary and travels to left.

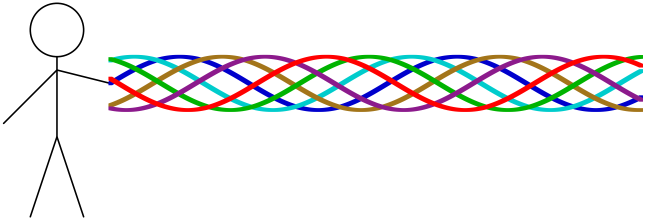
Initial wave (pass 1) and reflections (passes >1) sum.

The resulting interference can be constructive or destructive.

Sum of passes 1 to 6 (shown here) is zero: destructive interference.

Physical Example: Vibrating String

Changing frequency 0.0208 to 0.0192: different result but similar pattern.



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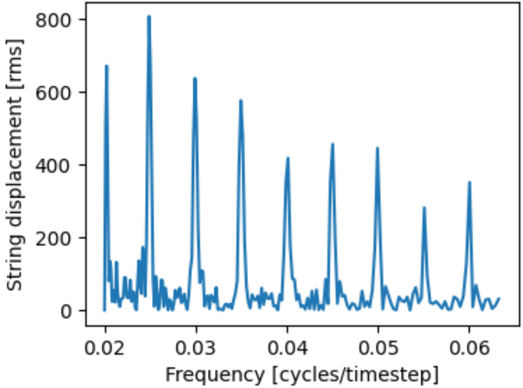
Frequency Response of Vibrating String

The RMS string displacement depends on the frequency of excitation. Many frequencies produce small responses due to destructive interference. But a few excitation frequencies generate large responses, as shown here:



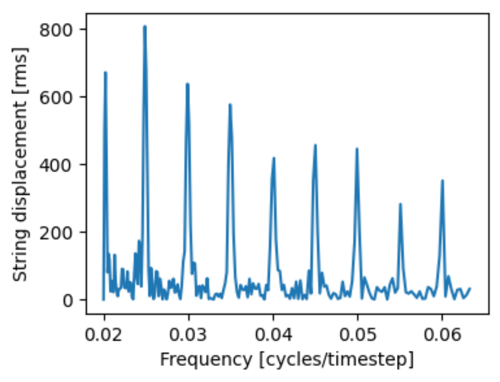
Statistics of Peaks

frequency	period	length/N
f	$\lambda = T$	L/λ
0.020	50.0	2.00
0.025	40.0	2.50
0.030	33.3	3.00
0.035	28.6	3.50
0.040	25.0	4.00
0.045	22.2	4.50
0.050	20.0	5.00
0.055	18.2	5.50
0.060	16.6	6.00

Frequency f is in cycles per timestep: the corresponding period $T = 1/f$ is in timesteps, which is numerically equal to the wavelength λ since the velocity $V = 1$ distance step per timestep. The length of the string is $L = 100$ distance steps.

Check Yourself

Frequency response of vibrating string.



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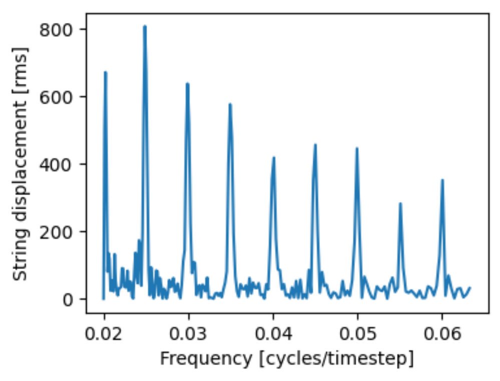
Why are there large displacements at some freq's but not most?

Check Yourself

Frequency response of vibrating string.



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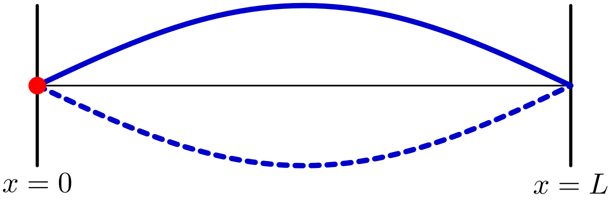
Why are there large displacements at some freq's but not most?

If the round-trip travel time is an integer multiple of the period of excitation, then there is purely constructive interference beween the passes. There are only countably many such frequencies.

Physical Example: Vibrating String

We get constructive interference if round-trip travel time equals the period.

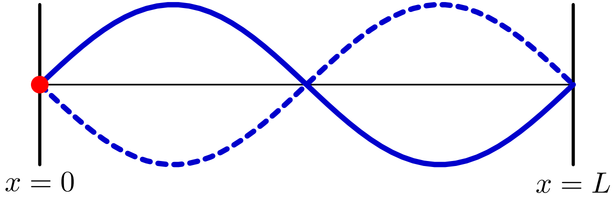


Round-trip travel time $= \dfrac{2L}{V} = T$

$$\omega_o = \frac{2\pi}{T} = \frac{2\pi}{2L/V} = \frac{\pi V}{L}$$

Physical Example: Vibrating String

We also get constructive interference if round-trip travel time is $2T$.

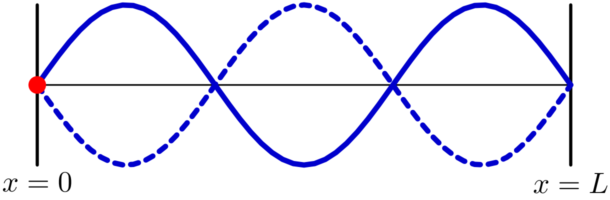


Round-trip travel time $= \dfrac{2L}{V} = 2T$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{L/V} = \frac{2\pi V}{L} = 2\omega_o$$

Physical Example: Vibrating String

In fact, we also get constructive interference if round-trip travel time is kT.



Round-trip travel time $= \dfrac{2L}{V} = kT$

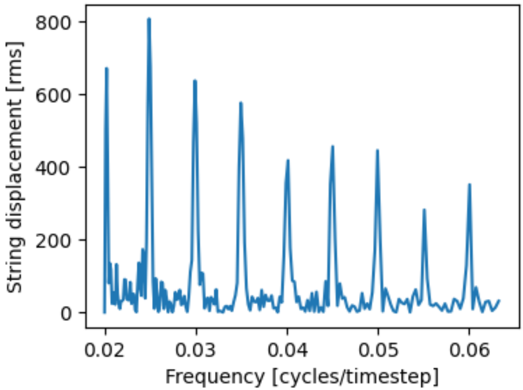
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2L/kV} = \frac{k\pi V}{L} = k\omega_o$$

Only certain frequencies persist: harmonics of $\omega_o = \pi V/L$.

This is the basis of stringed instruments.

Vibrating strings: relations to Fourier series

Strings vibrate most at certain harmonically related frequencies.



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These frequencies are determined by the physics of strings, and are well matched to analysis with Fourier series: the usefulness of Fourier series comes from physics.

Fourier series capture an important feature of the nature of vibrating strings.

Vibrating strings: relations to Fourier series

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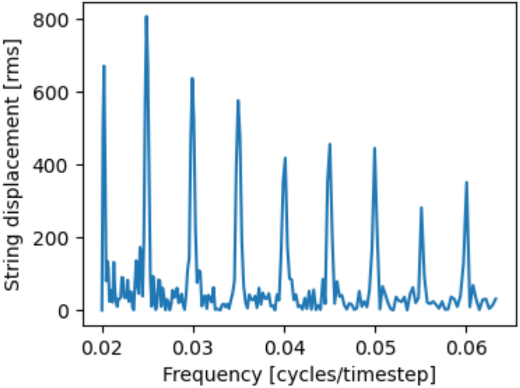

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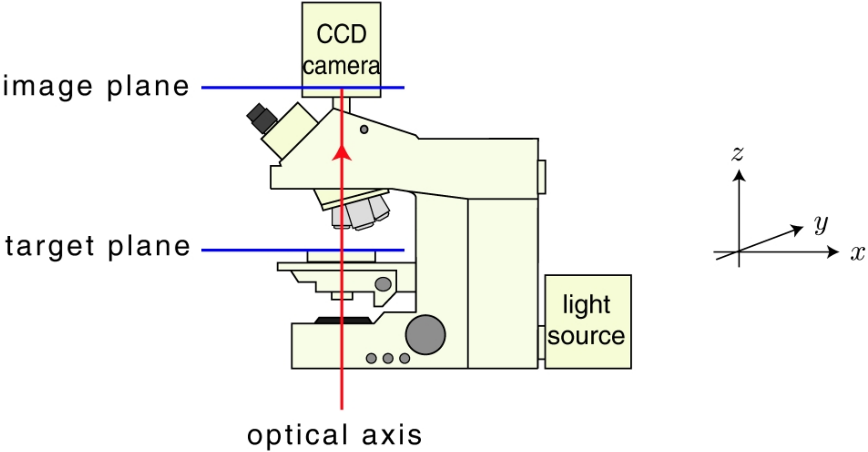
Fourier series similarly capture important features of other physical systems, e.g., optics!

Optics: 2D/3D waves

Optical Imaging

Images from even the best microscopes are blurred.



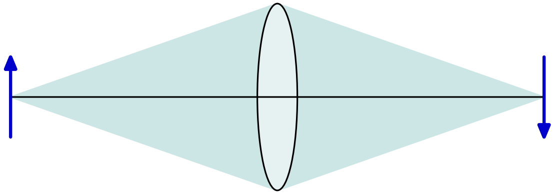
Blurring is a fundamental property of lenses.

We'd like to understand what limits the resolution of a light microscope.

Optical Imaging

A perfect lens transforms a spherical wave of light from a target into a spherical wave that converges to an image of the target.

However, to be perfect, the lens would have to be infinitely large.



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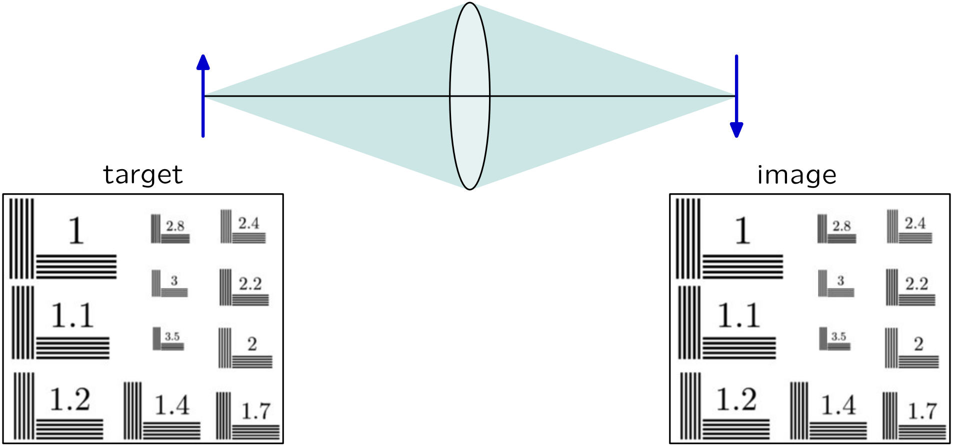

Blurring is inversely related to the diameter of the lens.

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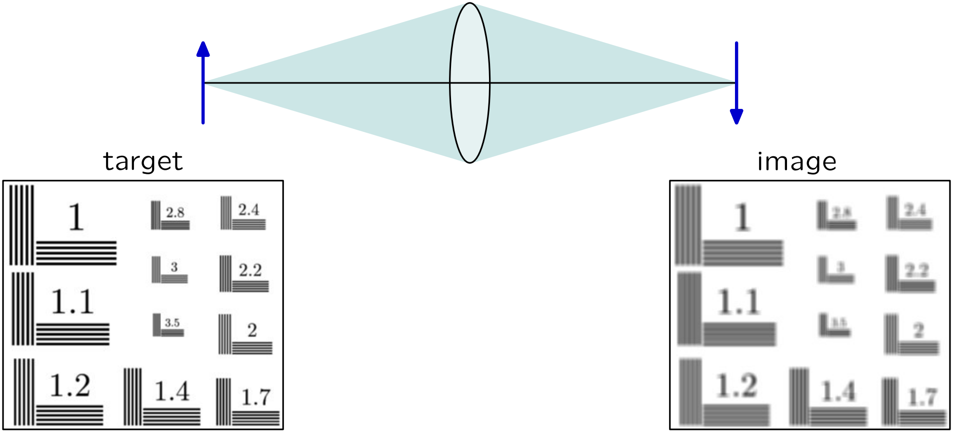

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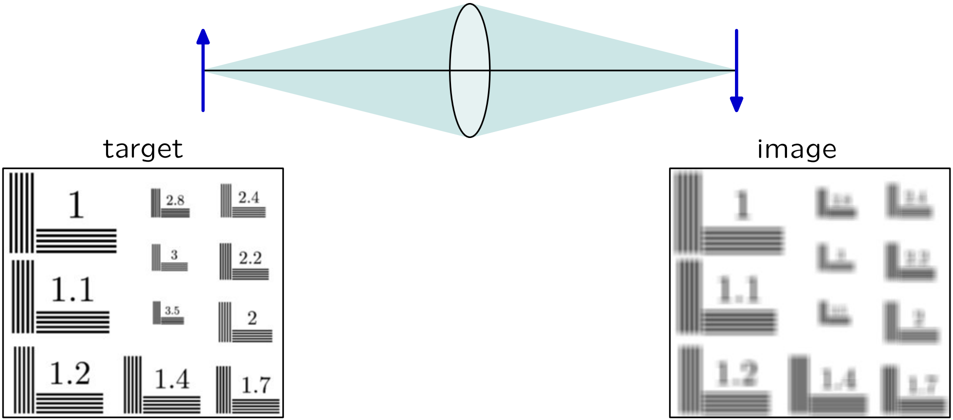
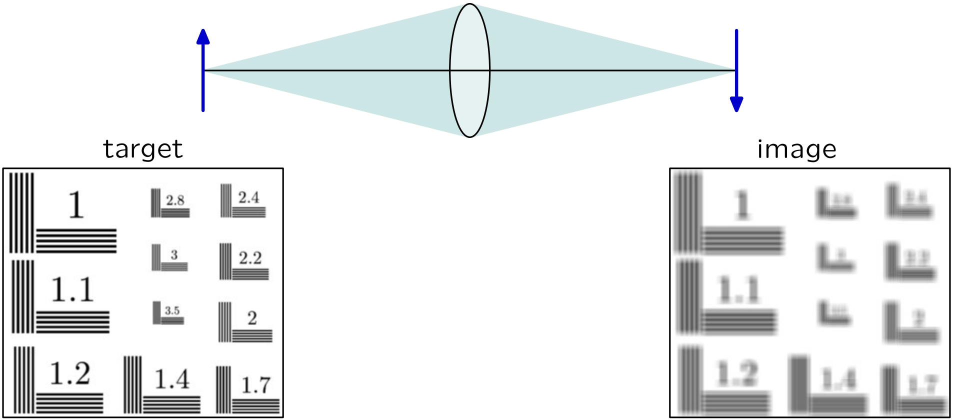


Blurring is inversely related to the diameter of the lens.

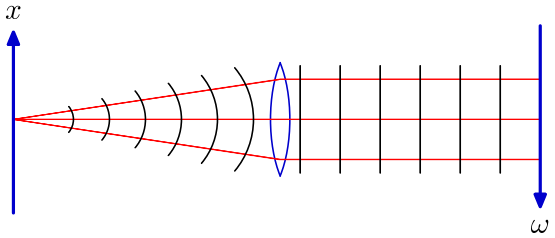
Optical Imaging

Our goal is to understand how the size of a lense affects image resolution. We will see how Fourier representations can be used to understand (and even overcome some of) these limitations.



Fourier Optics

If a target is located in the focal plane of a lens, light from a point on the target forms a plane wave as it passes through the lens.



If the target point lies on the axis of the lens, then the plane wave is perpendicular to the imaging plane.

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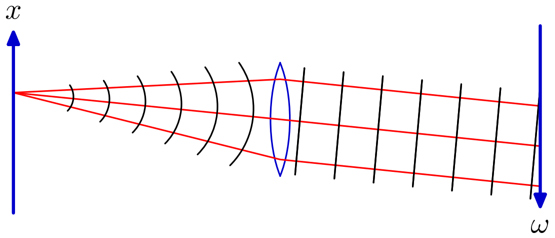
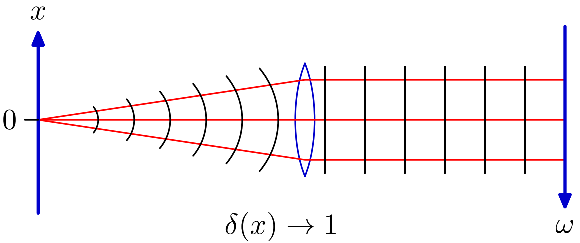


If the target point lies on the axis of the lens, then the plane wave is perpendicular to the imaging plane.

If the target point lies off the axis of the lens, then the plane wave is no longer perpendicular to the image plane. The light striking the image plane has linearly increasing phase delay with distance.

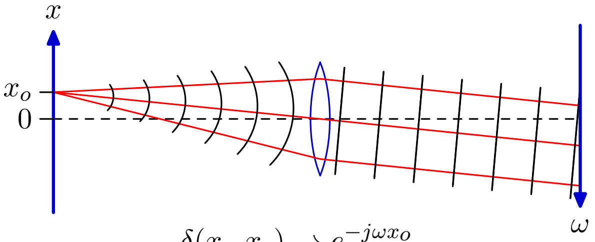
Fourier Optics

Light from the point $x=0$ generates a plane wave, that is everywhere in phase at the imaging plane.



$$\delta(x) \rightarrow 1$$

Light from $x=x_o$ generates a plane wave with linearly increasing phase lag.

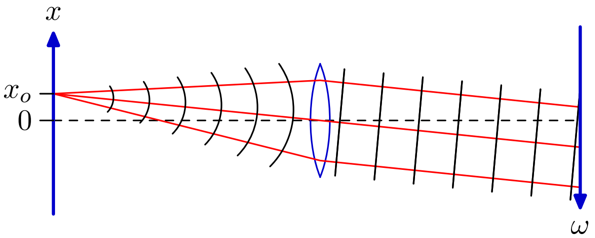


$$\delta(x-x_o) \rightarrow e^{-j\omega x_o}$$

Fourier Optics

The target can be described as a collection of point sources of light

$$f(x) = \int f(x_o)\delta(x - x_o)\, dx_o$$



and the result in the image plane is a superposition of plane waves, one for each point in the target.

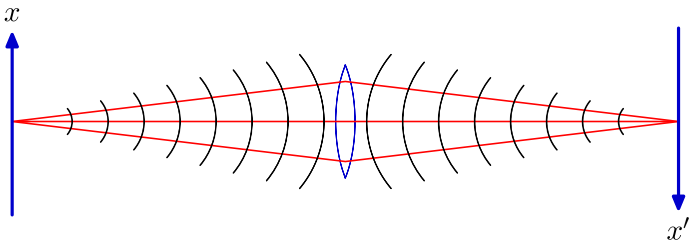
$$g(\omega) = \int f(x)\, e^{-j\omega x}\, dx = F(\omega)$$

Notice that $g(\omega) = F(\omega)$ is the Fourier transform of $f(x)$.

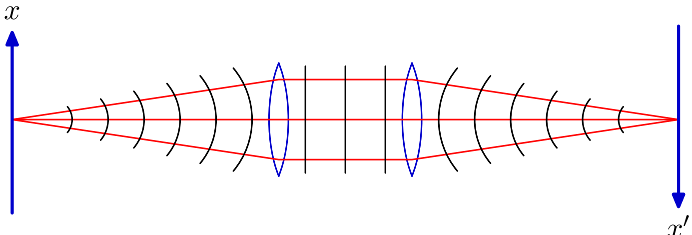
Fourier Optics: $f(x) \stackrel{\text{CTFT}}{\Longrightarrow} F(\omega)$

Fourier Optics

If an object is more than one focal distance from the lens, then the light converges to create an image of the object in the image plane.

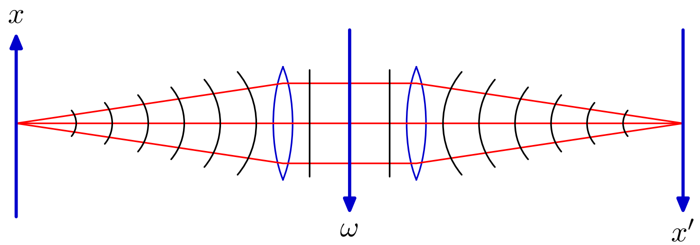


This one-lens system is equivalent to a system with two lenses: one located a focal distance from the object and one located a focal distance from the image.



Fourier Optics

Now the Fourier transform relation holds for both halves of the system.



$$F(\omega) = \int f(x) e^{-j\omega x} dx$$

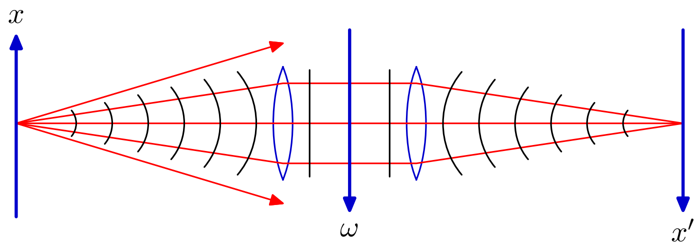
$$f'(x') = \frac{1}{2\pi} \int F(\omega) e^{j\omega x'} d\omega$$

Ideally, both limits of integration would be infinite.

However the finite diameter of the lens limits the highest frequencies $|\omega|$.

Fourier Optics

Light emanating from the target at large angles is not captured by the lens.



$$F(\omega) = \int f(x) e^{-j\omega x} dx$$

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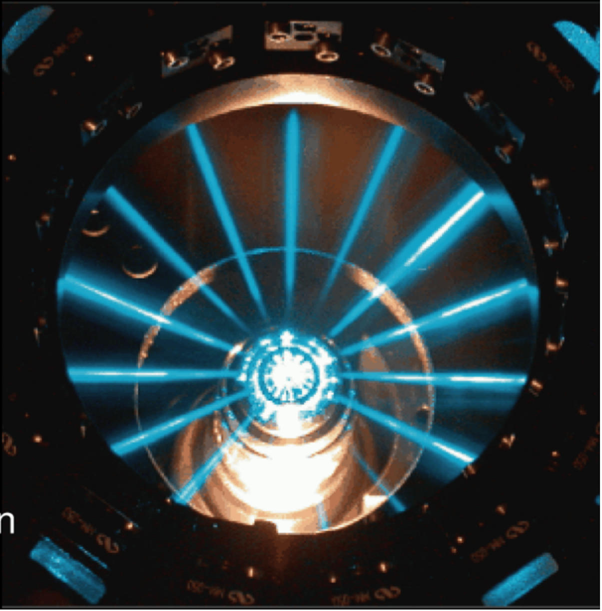
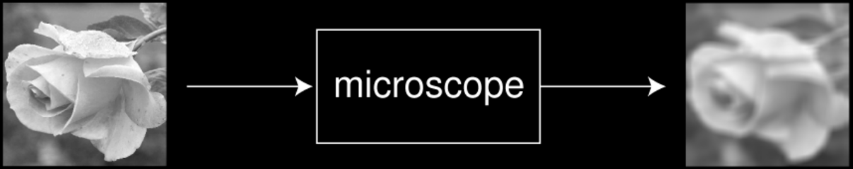
As a result, the image at x' is a lowpass version of the target at x.

Microscopy with 6.003

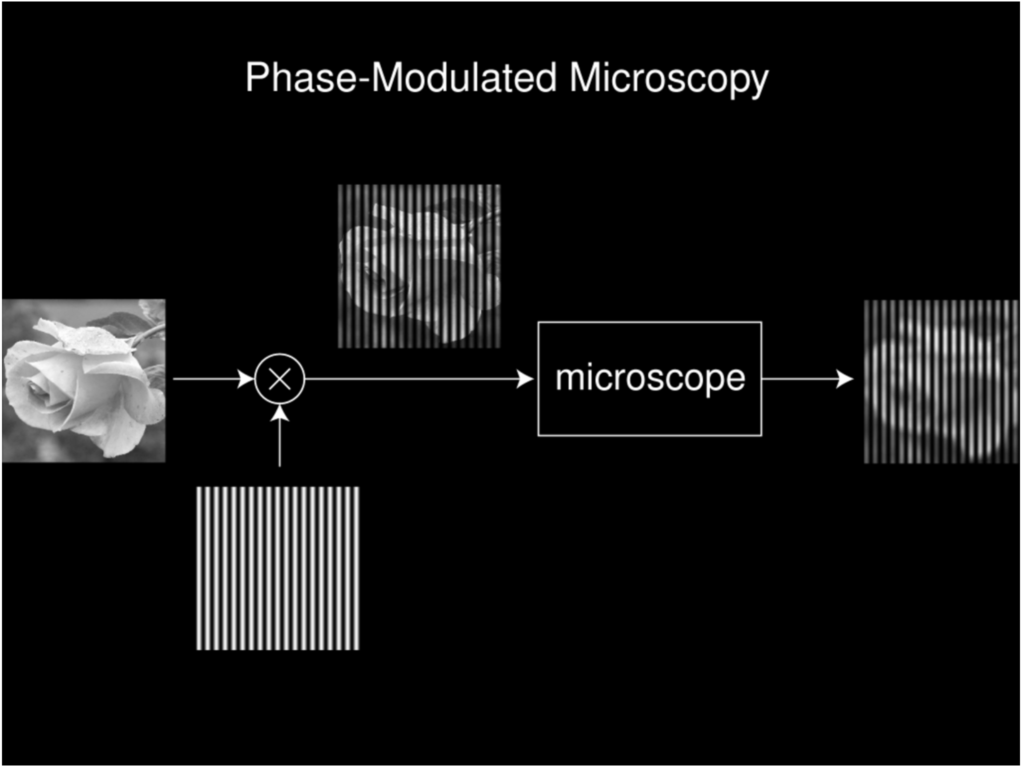


Dennis M. Freeman
Stanley S. Hong
Jekwan Ryu
Michael S. Mermelstein
Berthold K. P. Horn

6.003 Model of a Microscope



Microscope = low-pass filter

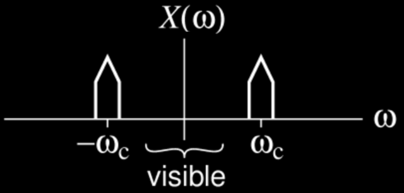

Phase-Modulated Microscopy
microscope

Demonstration

Demonstration

Phase-Modulated Microscopy

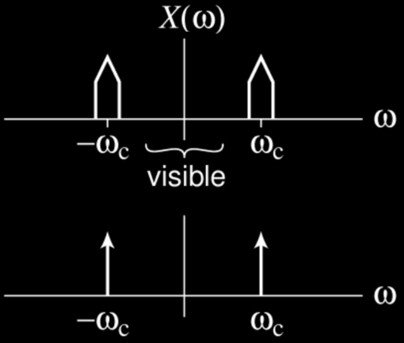
Poster: $\cos(\omega_c y + f(x,y))$



Phase-Modulated Microscopy

Poster: $\cos(\omega_c y + f(x,y))$

Projector: $\cos(\omega_c y)$

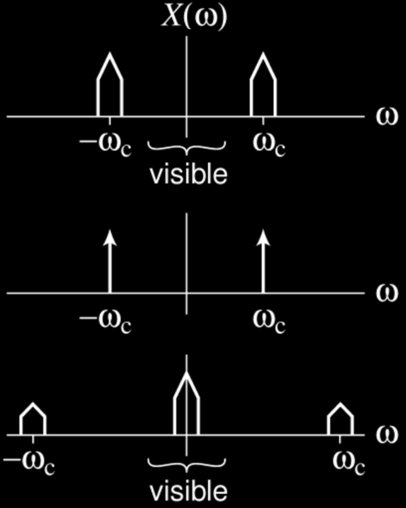


Phase-Modulated Microscopy

Poster: $\cos(\omega_c y + f(x,y))$

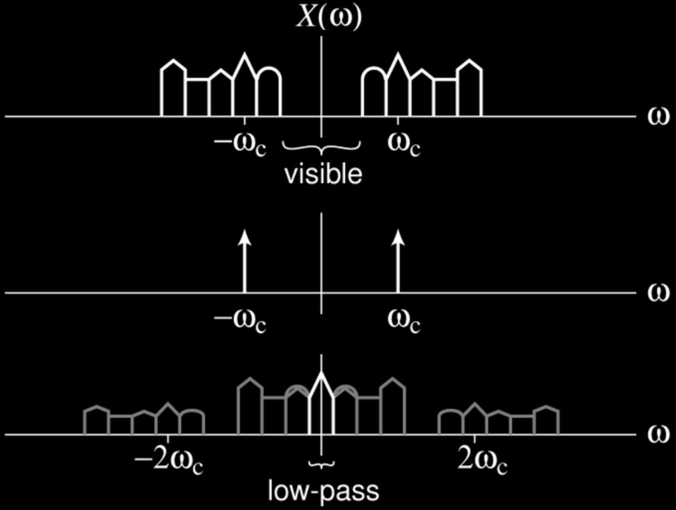
Projector: $\cos(\omega_c y)$

Poster with Projector: $\cos(\omega_c y)\,\cos(\omega_c y + f(x,y))$



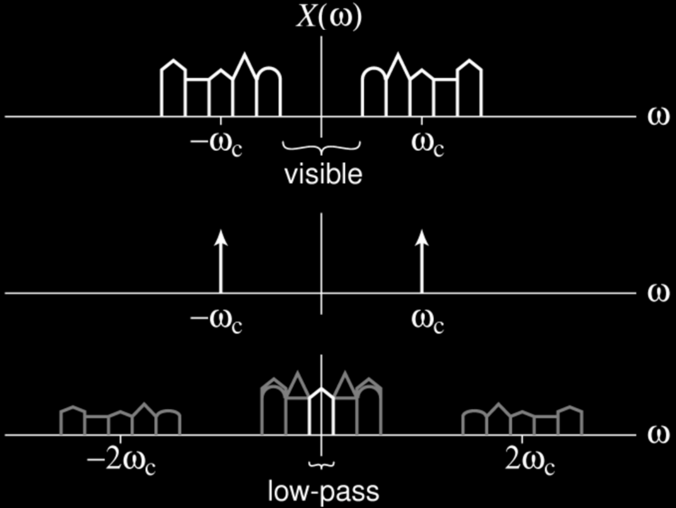
Modulated illumination enables low-pass system (eyes) to detect high spatial frequencies

Phase-Modulated Microscopy



Modulated illumination enables low-pass system (eyes) to detect high spatial frequencies

Phase-Modulated Microscopy



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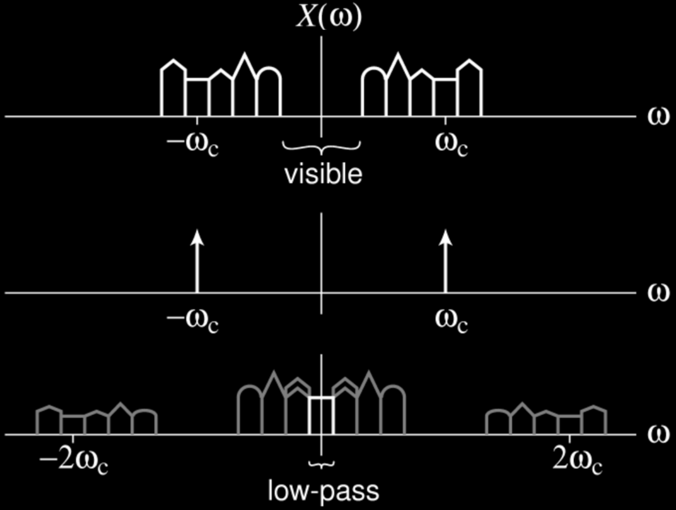
Phase-Modulated Microscopy



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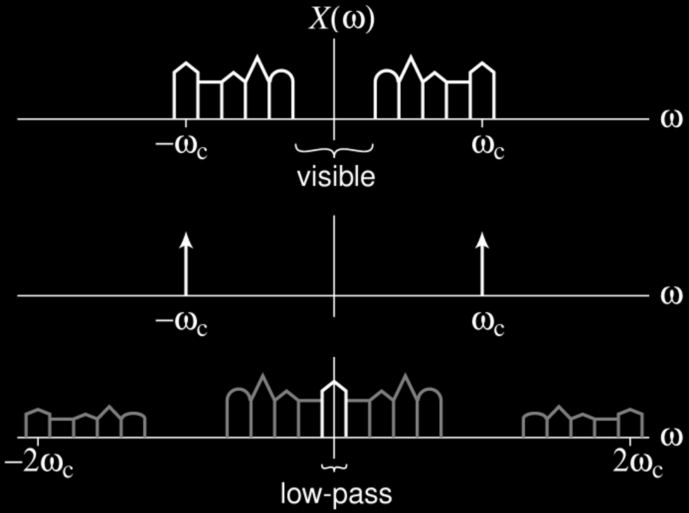
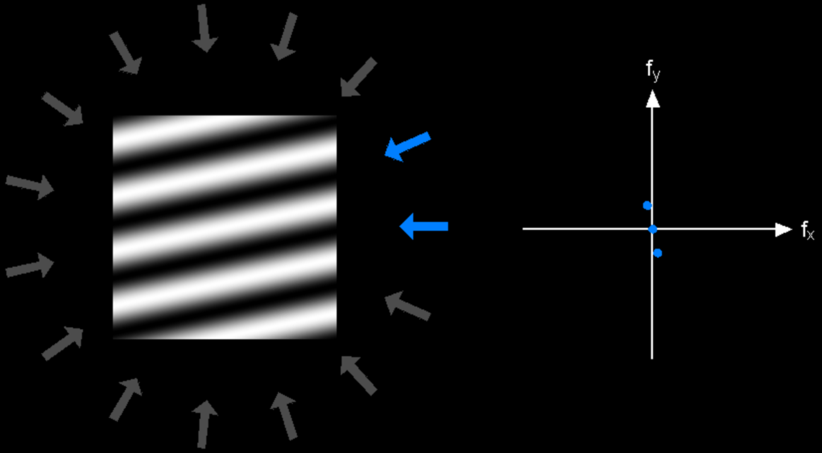
Phase-Modulated Microscopy



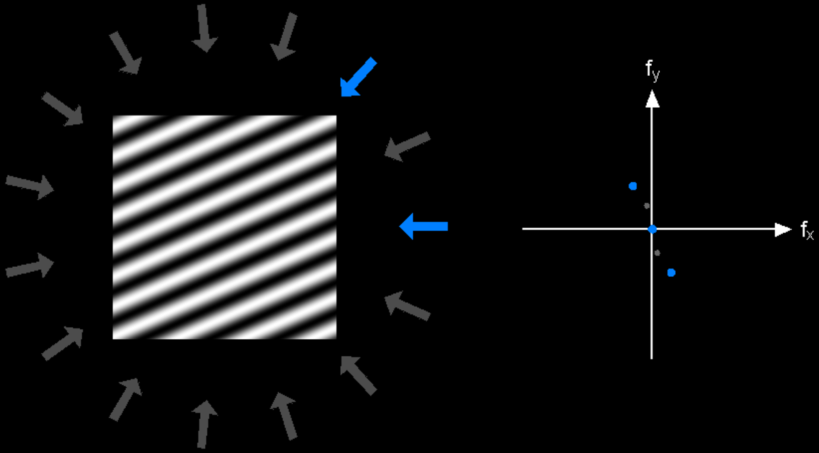
Modulated illumination enables low-pass system (eyes) to detect high spatial frequencies

Standing-wave illumination spectrum



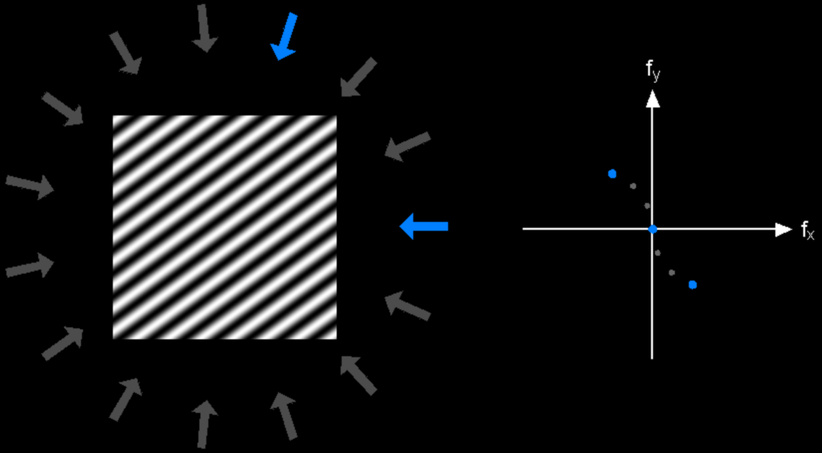
Thanks to M. Mermelstein

Standing-wave illumination spectrum



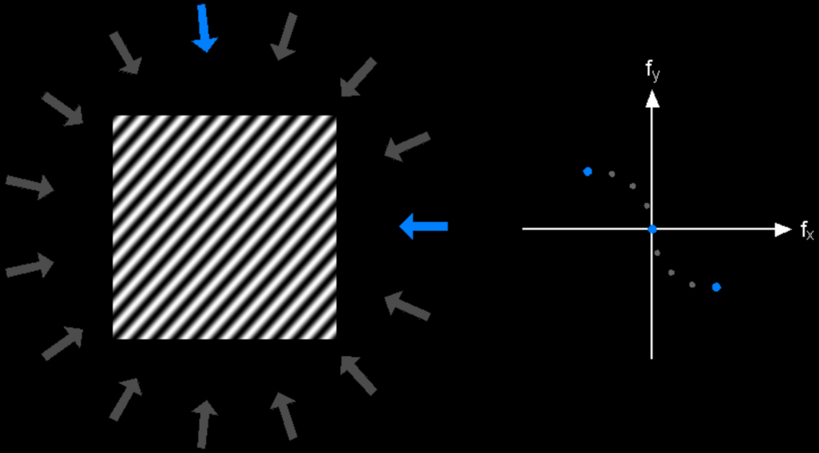
Thanks to M. Mermelstein

Standing-wave illumination spectrum



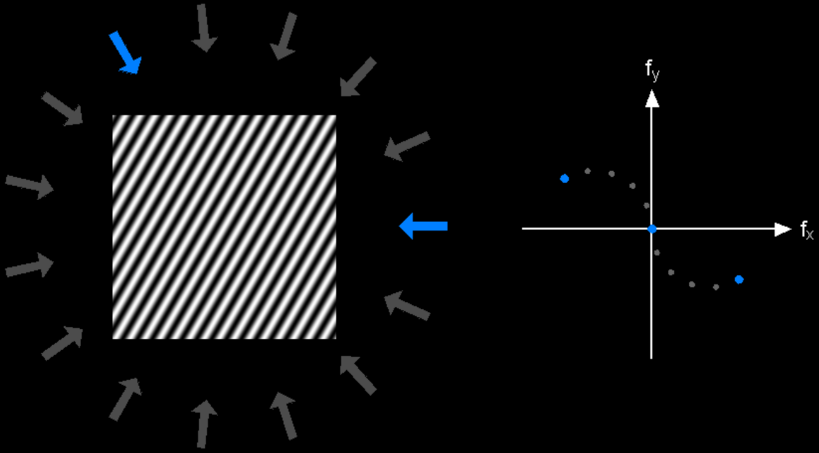
Thanks to M. Mermelstein

Standing-wave illumination spectrum



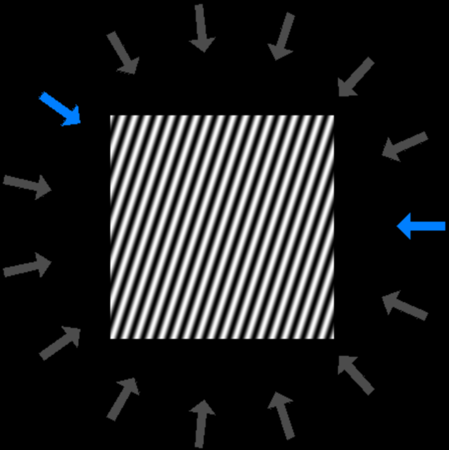
Thanks to M. Mermelstein

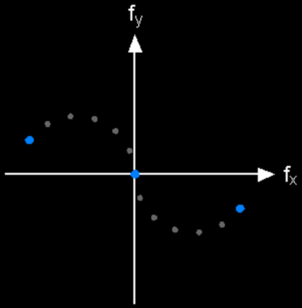
Standing-wave illumination spectrum



Thanks to M. Mermelstein

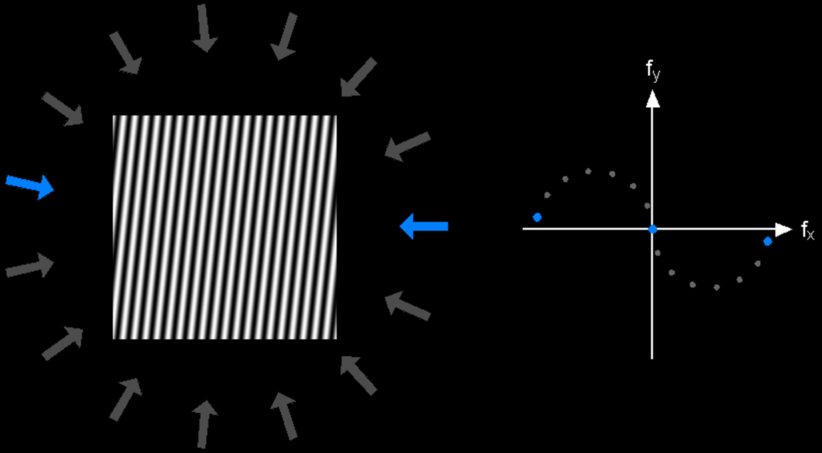
Standing-wave illumination spectrum





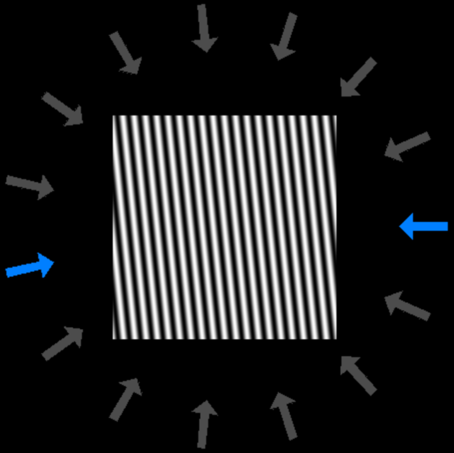
Thanks to M. Mermelstein

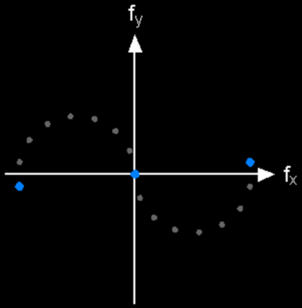
Standing-wave illumination spectrum



Thanks to M. Mermelstein

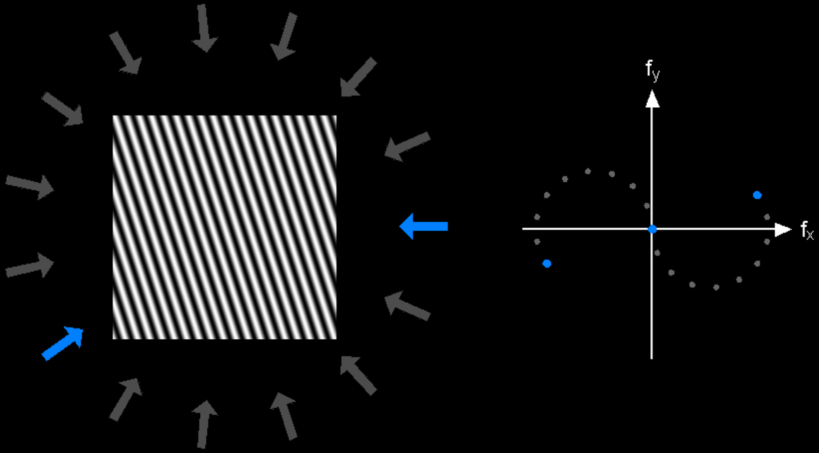
Standing-wave illumination spectrum





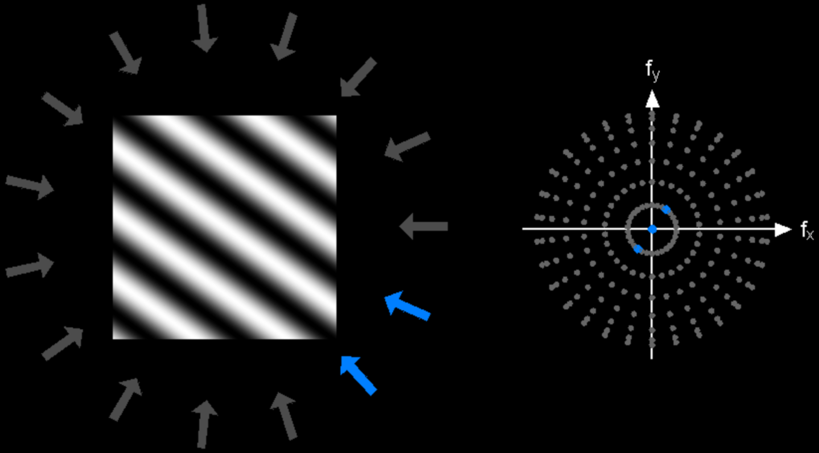
Thanks to M. Mermelstein

Standing-wave illumination spectrum



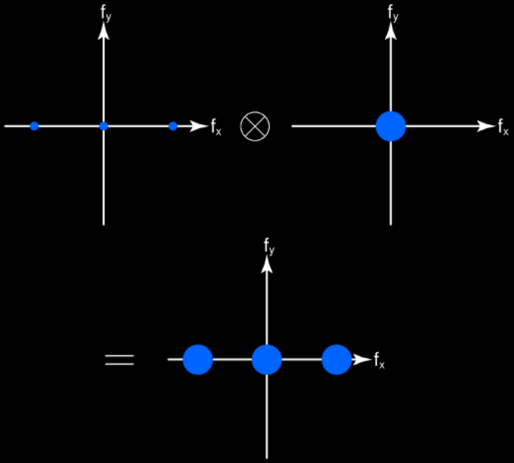
Thanks to M. Mermelstein

Standing-wave illumination spectrum



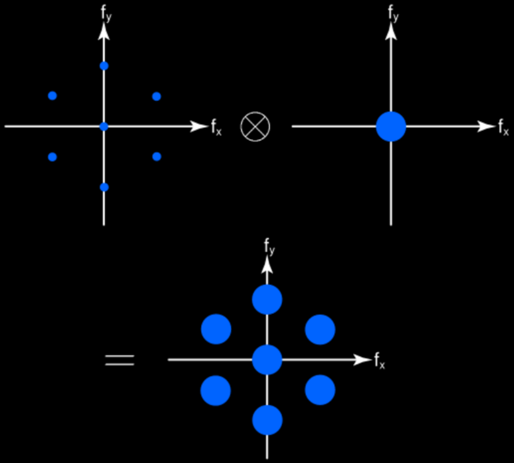
Thanks to M. Mermelstein

Optical transfer function



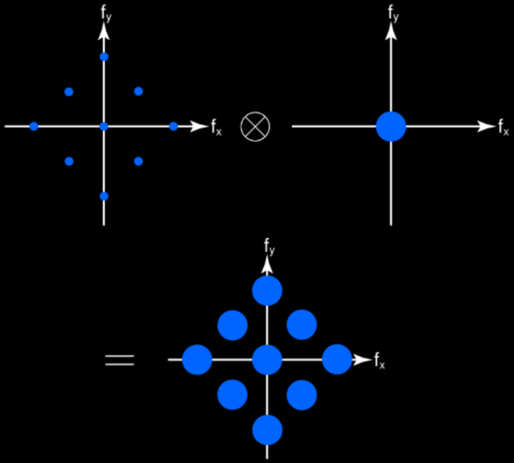
2 beams

Optical transfer function



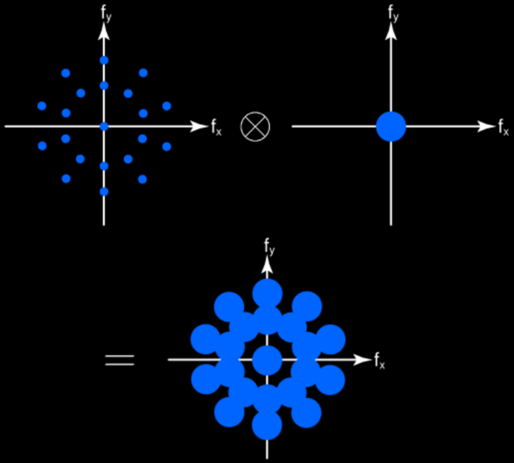
3 beams

Optical transfer function



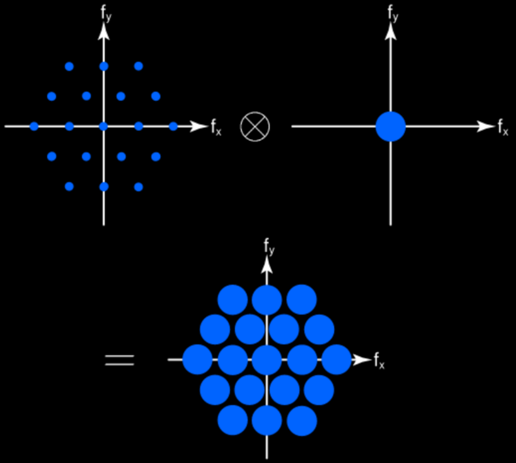
4 beams

Optical transfer function



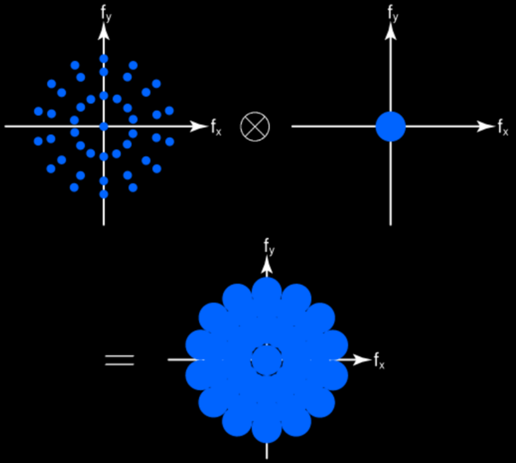
5 beams

Optical transfer function



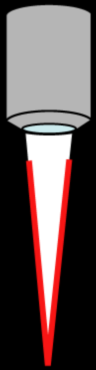
6 beams

Optical transfer function

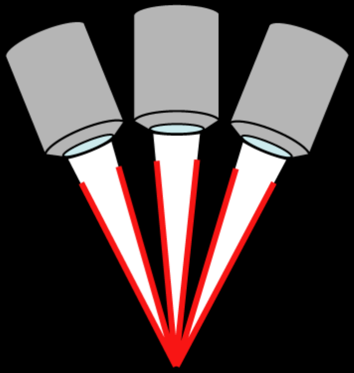


7 beams

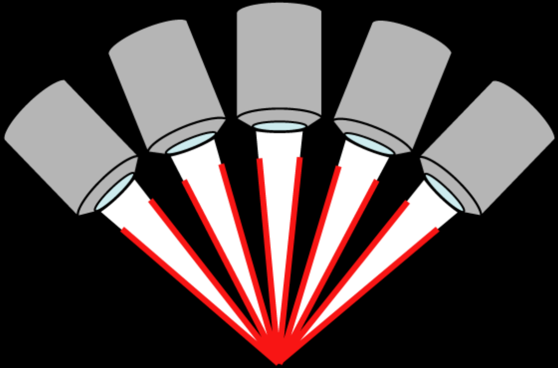
Aperture synthesis



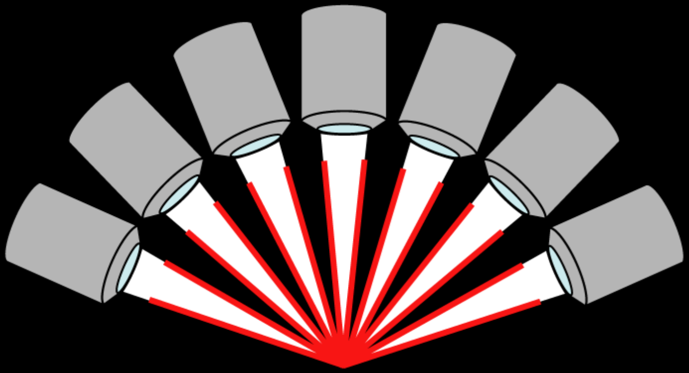
Aperture synthesis



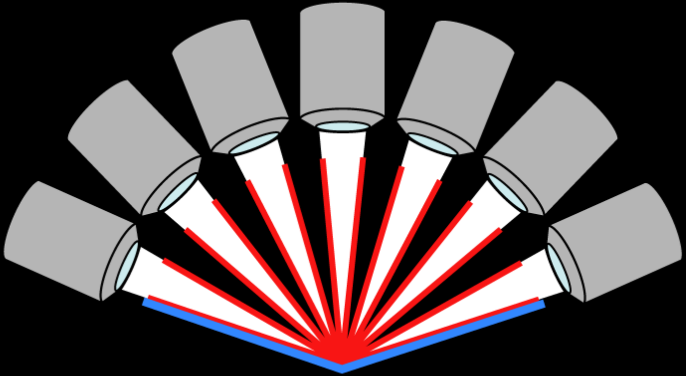
Aperture synthesis



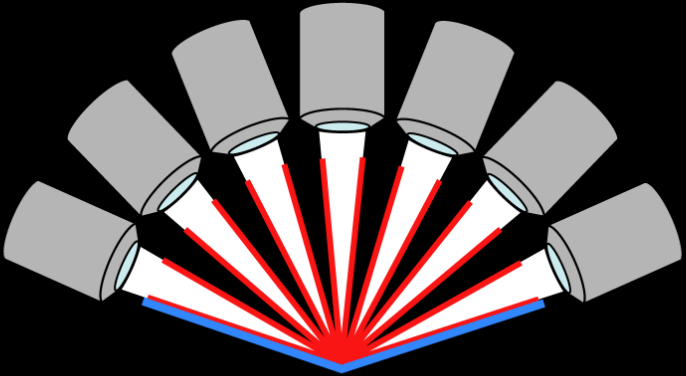
Aperture synthesis



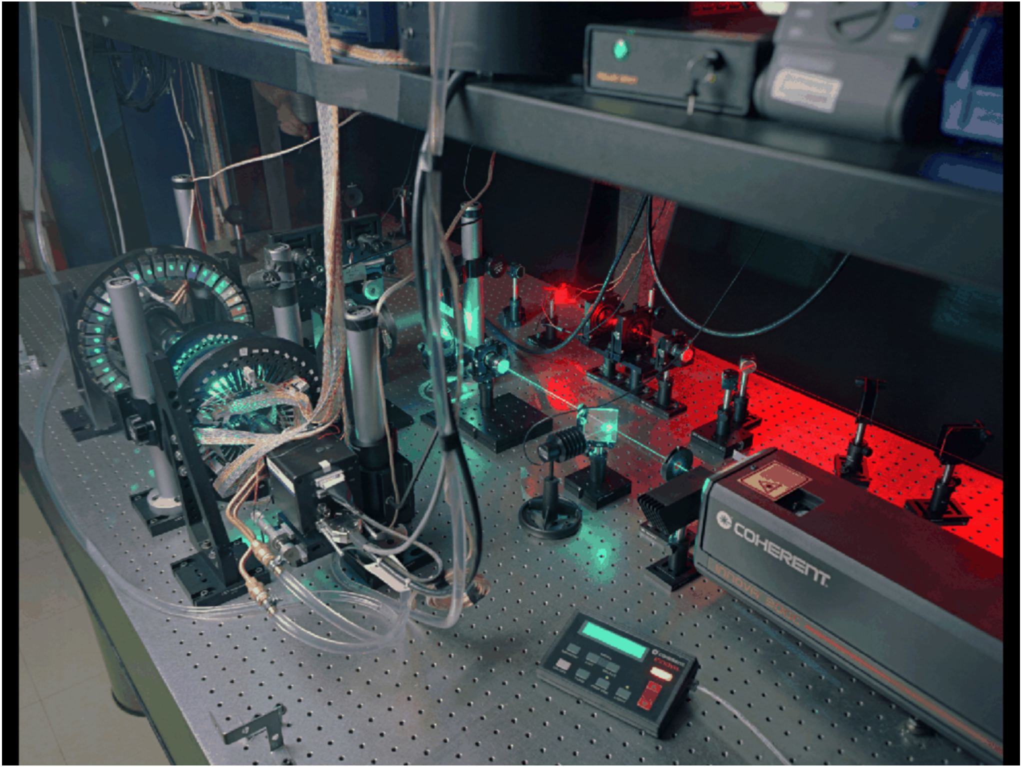
Aperture synthesis

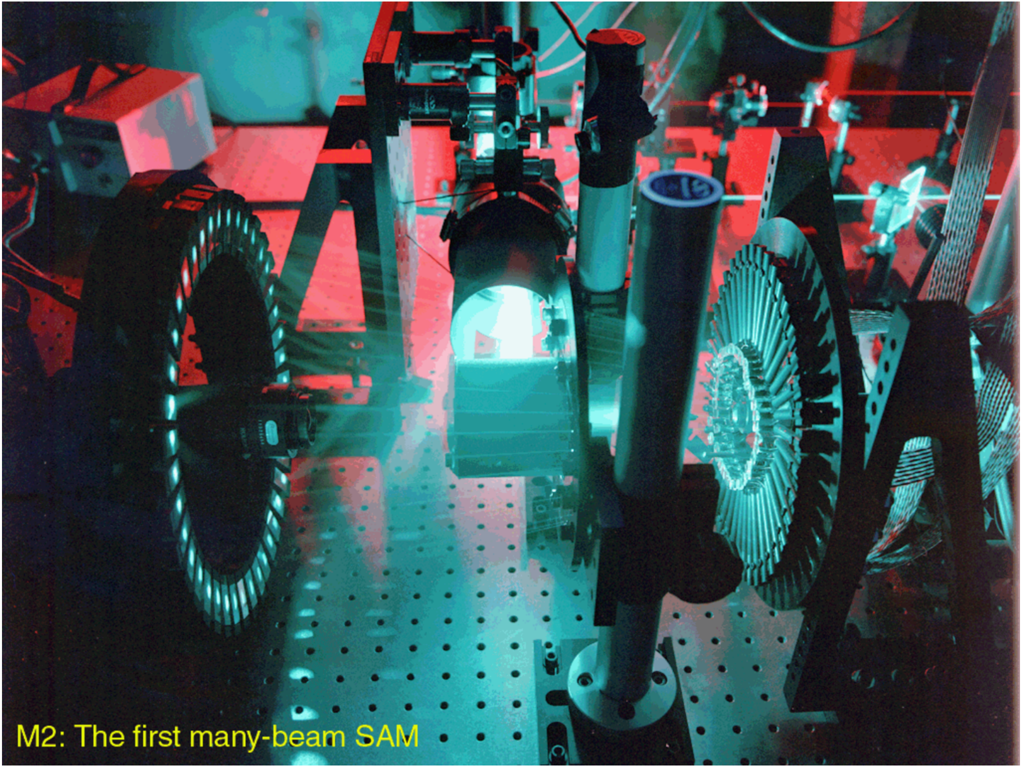


Aperture synthesis



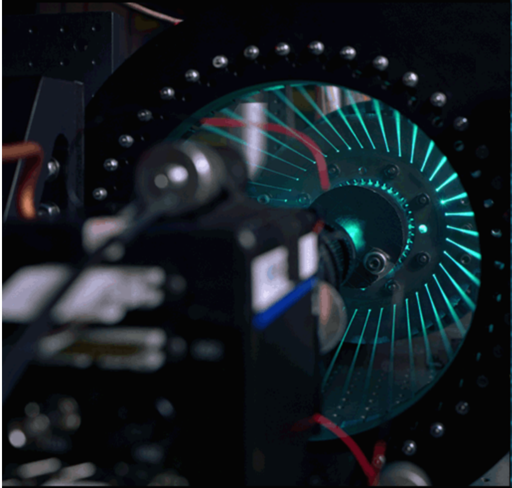
Combine multiple low-NA optics to *synthesize* high NA


COHERENT

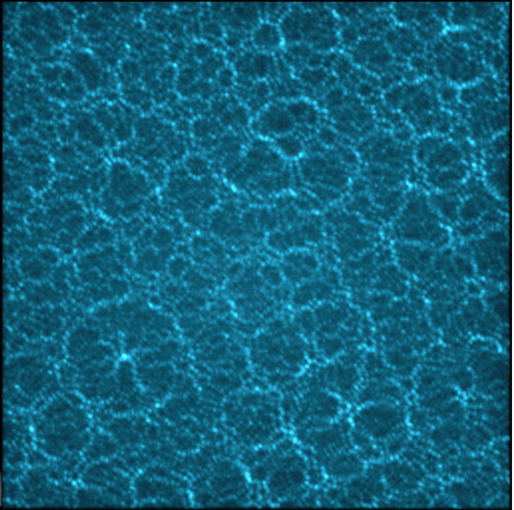


M2: The first many-beam SAM

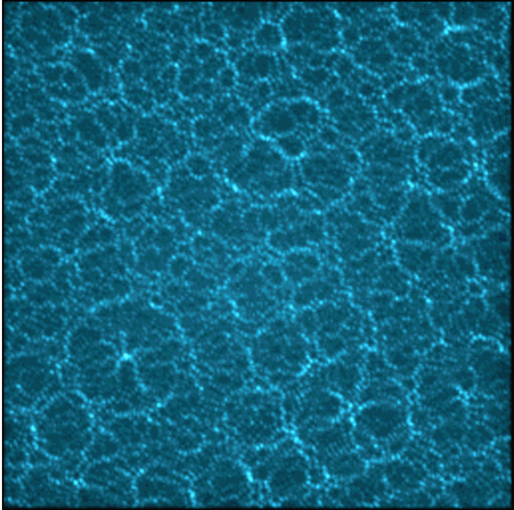
41 BEAMS IN A RING



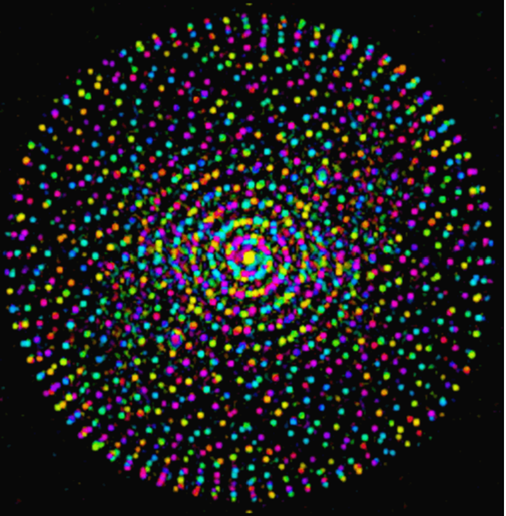
MAKE PATTERNS LIKE THIS



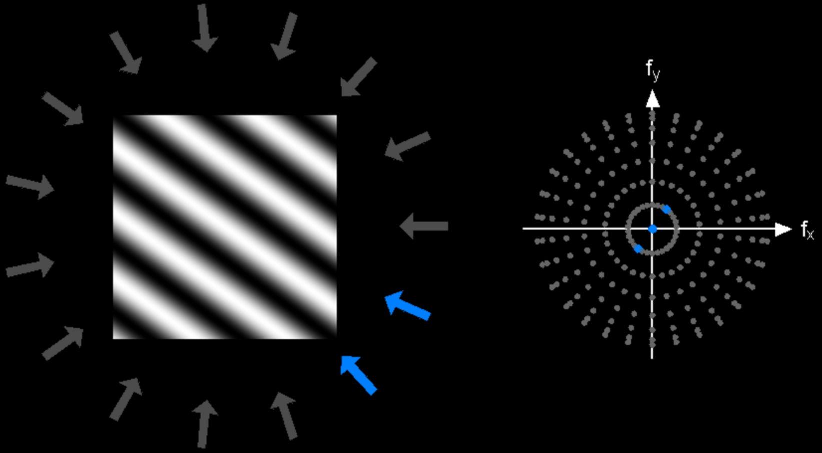
PATTERNS LIKE THIS



HAVE TRANSFORMS LIKE THIS

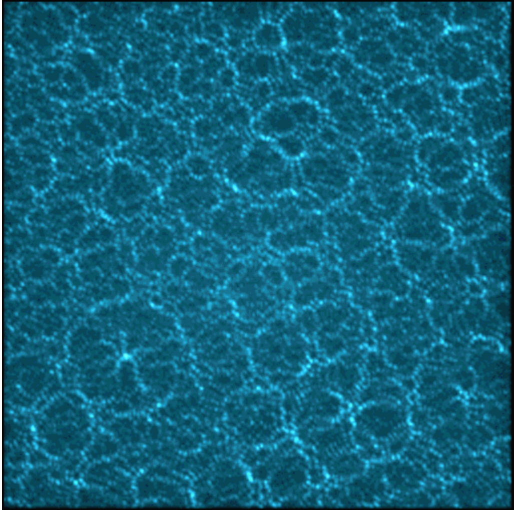


Standing-wave illumination spectrum

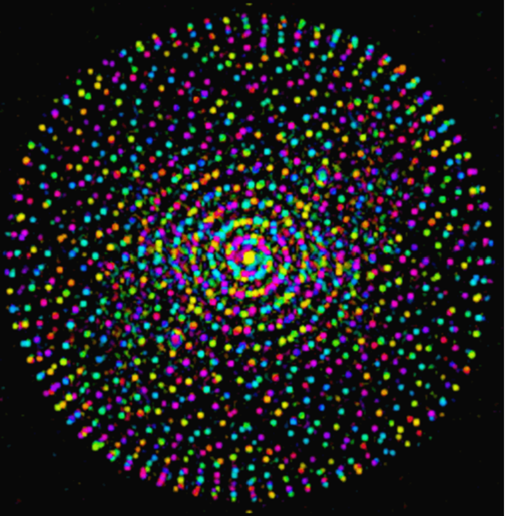


Thanks to M. Mermelstein

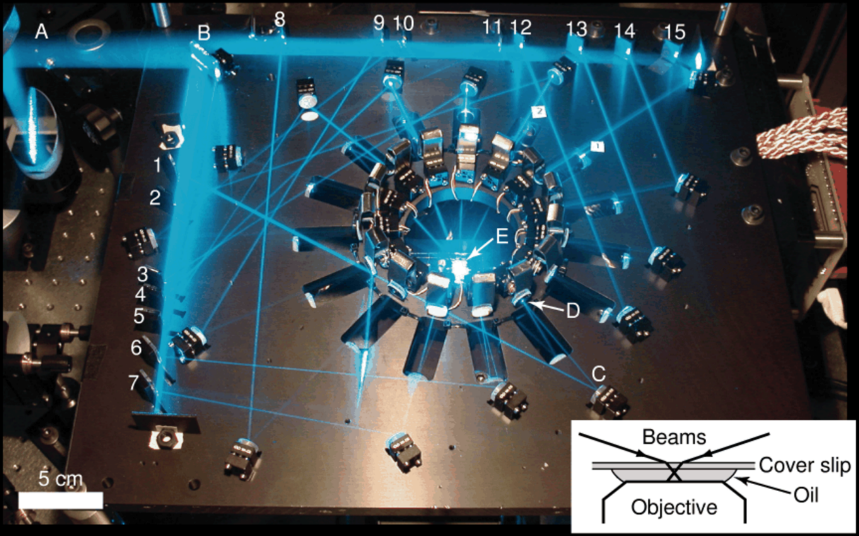
PATTERNS LIKE THIS



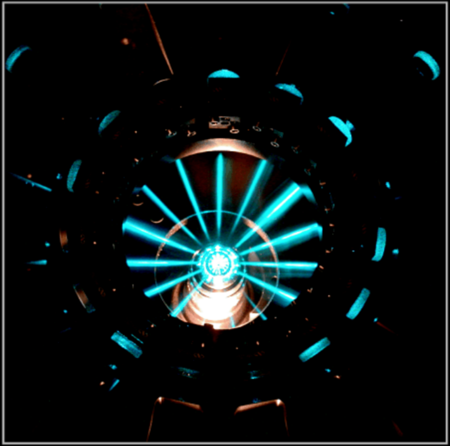
HAVE TRANSFORMS LIKE THIS

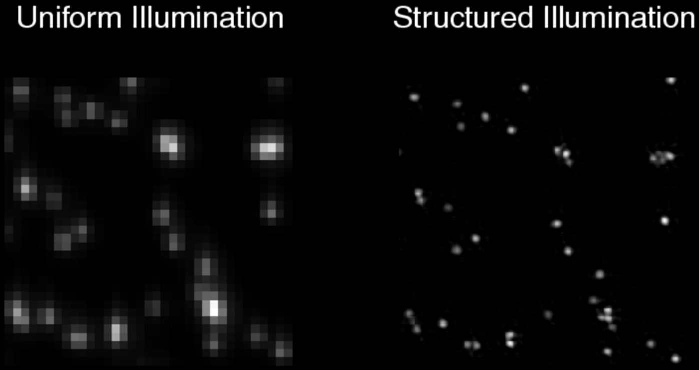


Experimental apparatus

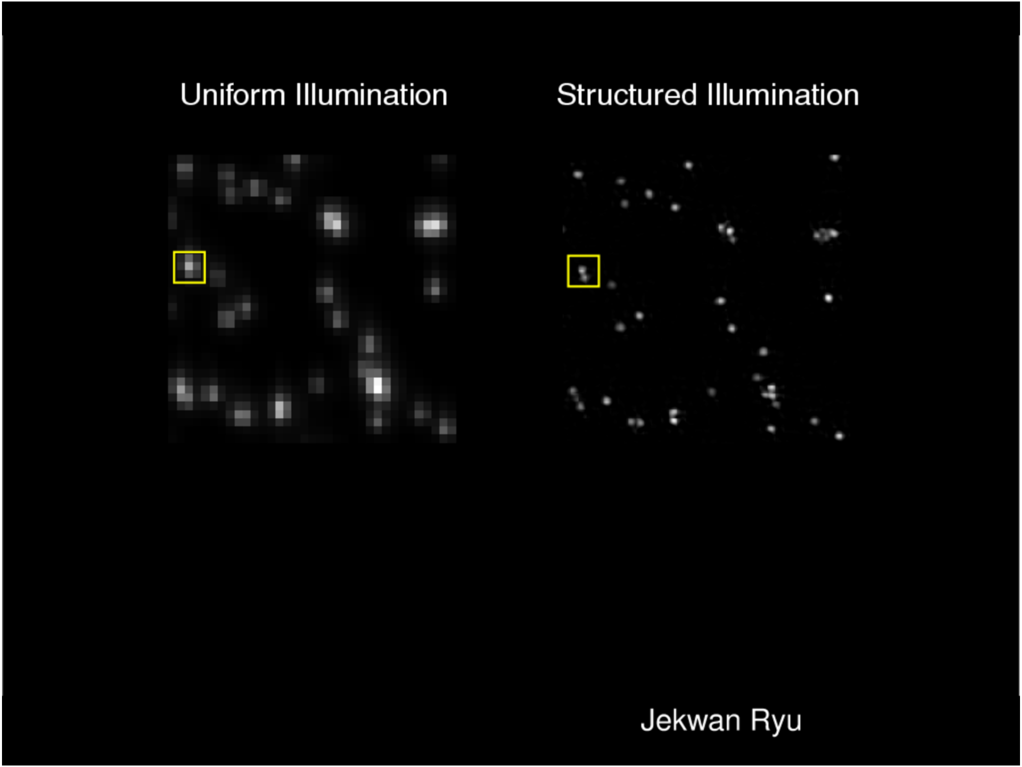


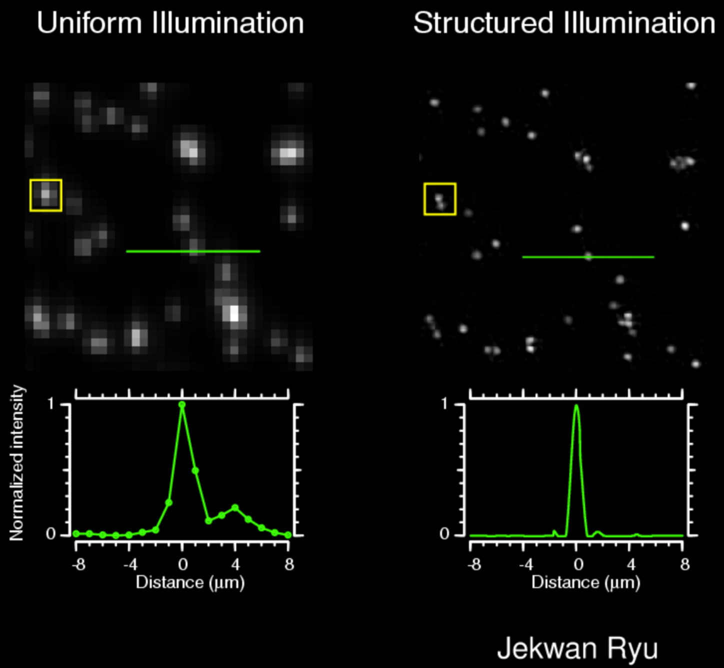
Stanley S. Hong





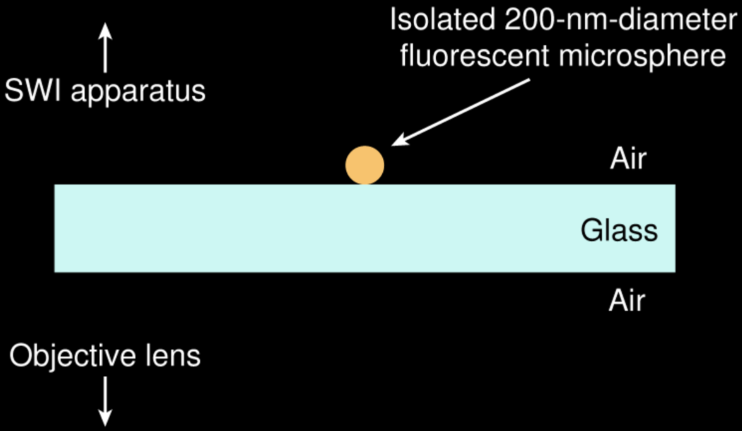
Jekwan Ryu


Uniform Illumination
Structured Illumination
Jekwan Ryu



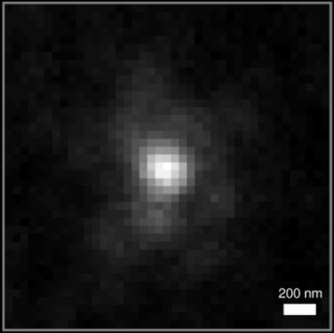
Uniform Illumination
Structured Illumination
Normalized intensity
1
0
-8
-4
0
4
8
Distance (µm)
Jekwan Ryu

Measurement of PSF

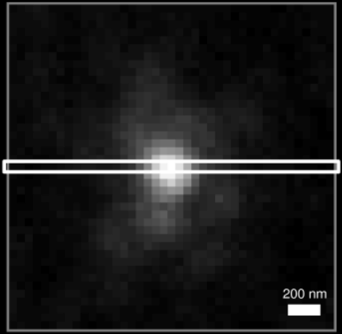


(Cross section, not to scale)

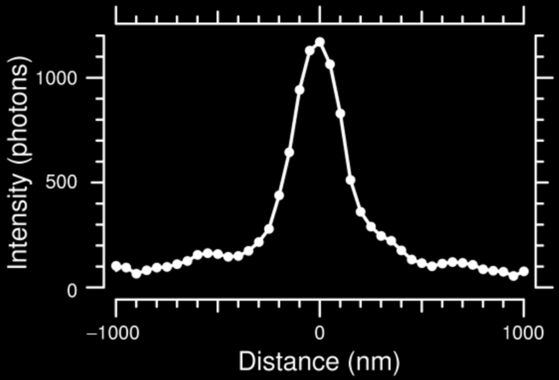
Measurement of PSF



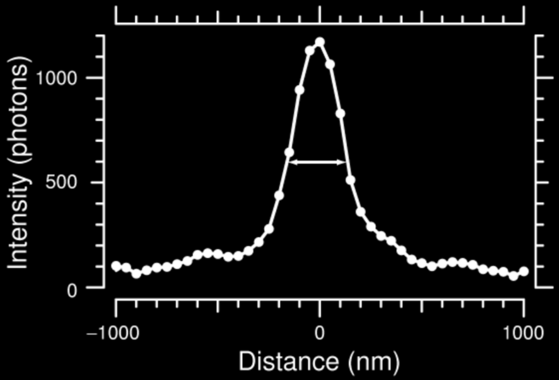
Measurement of PSF



Measurement of PSF

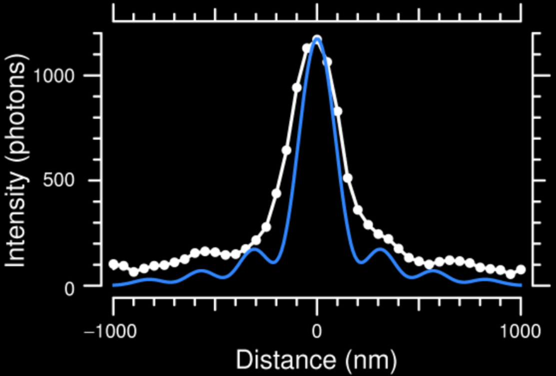


Measurement of PSF



Measured diameter = 290 nm

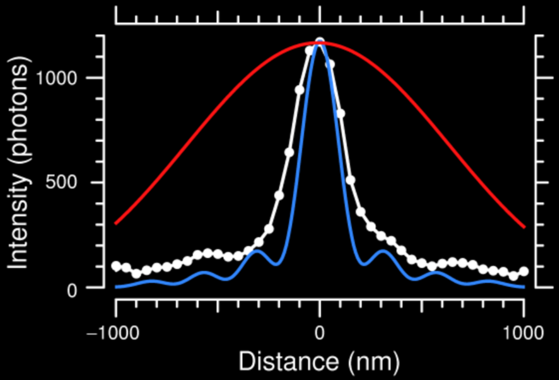
Measurement of PSF



Measured diameter = 290 nm

Predicted diameter = 250 nm

Measurement of PSF

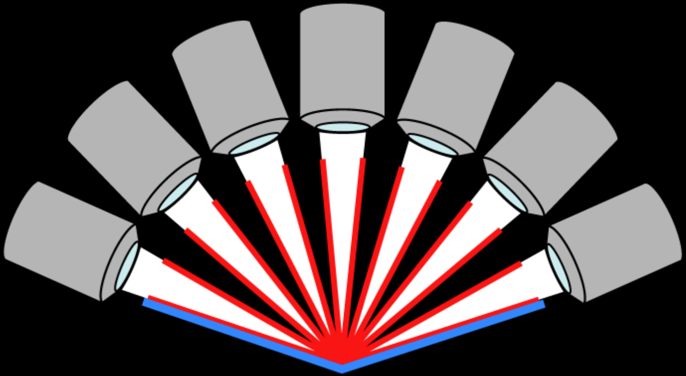


Measured diameter = 290 nm

Predicted diameter = 250 nm

Diameter lens alone = 1,500 nm

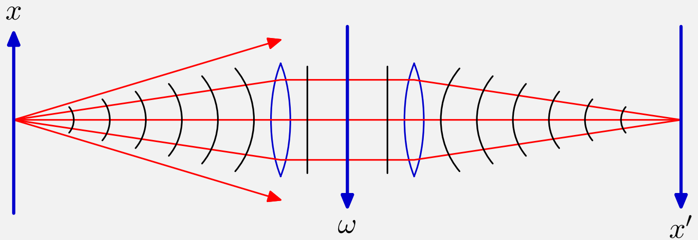
Aperture synthesis



Combine multiple low-NA optics to *synthesize* high NA

Question of the Day

Because the diameter of a lens limits the frequency content of an image, we can think of the transformation from $f(x)$ to $f(x')$ as lowpass.



$$F(\omega) = \int f(x) e^{-j\omega x} dx; \qquad f'(x') = \frac{1}{2\pi} \int F(\omega) e^{j\omega x'} d\omega$$

Specify a simple modification to such an imaging system so that the transformation is bandpass (i.e., having not only a high-frequency cutoff but also a low-frequency cutoff).