6.3000: Signal Processing

Sinusoids and Series

- Relations between time and frequency representations of signals
- Mathematical perspectives: continuous and discontinuous functions
- Physics perspectives: Fourier representations of a vibrating string

Homework 1 is posted and is due next Thursday (Feb 12) at 2pm.

February 05, 2026

Last Time

Signals are functions that contain and convey information.

Examples:

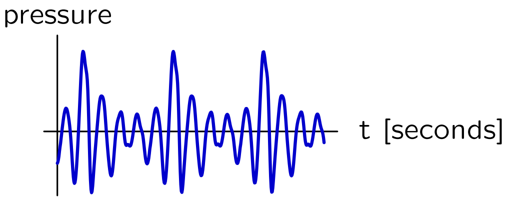
- the MP3 representation of a sound
- the JPEG representation of a picture
- an MRI image of a brain

Signal Processing develops the use of signals as abstractions:

- **identifying** signals in physical, mathematical, computation contexts,
- **analyzing** signals to understand the information they contain, and
- **manipulating** signals to modify the information they contain.

Musical Sounds as Signals

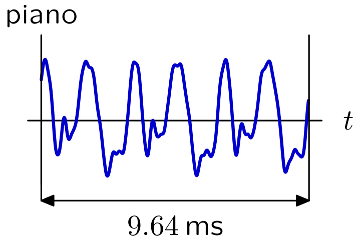
Example signal: a musical sound can be represented as a function of time.



Although this time function is a complete description of the sound, it does not expose many of the important properties of the sound.

Check Yourself

Even determining pitch can be tricky. Consider a single piano note:



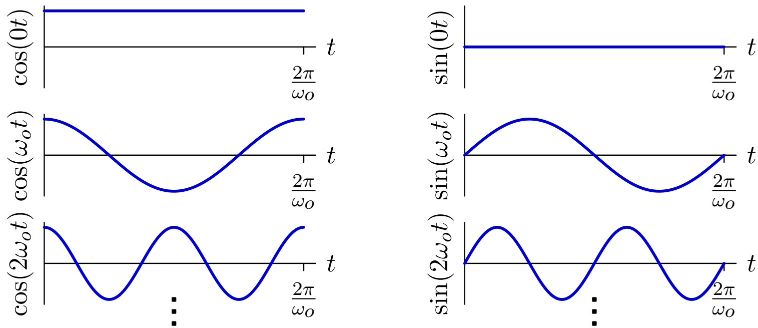
What is the fundamental frequency (pitch) of this waveform?

1. 104 Hz
2. 156 Hz
3. 311 Hz
4. 652 Hz
5. none of the above

Last Time: Signals as Sums of Sinusoids

Fourier series express explicit functions of time (e.g., $f(t)$) as explicit sums of single-frequency components ($\sin(k\omega_o t)$ and $\cos(k\omega_o t)$).

$$f(t) = f(t+T) = \sum_{k=0}^{\infty} \left(c_k \cos k\omega_o t + d_k \sin k\omega_o t\right)$$

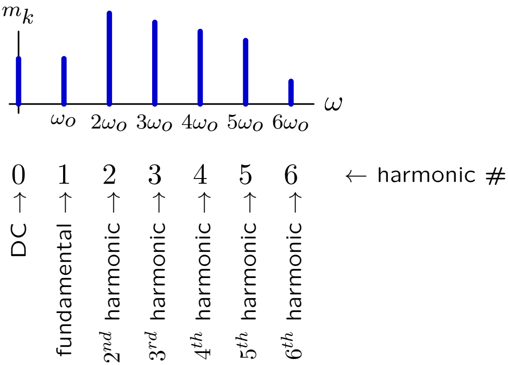


Fourier Series Representation

The original function of time ($f(t)$) is replaced by a sum of discrete, single-frequency components.

$$f(t) = f(t{+}T) = \sum_{k=0}^{\infty} (c_k \cos k\omega_o t + d_k \sin k\omega_o t) = \sum_{k=0}^{\infty} m_k \cos(k\omega_o t + \phi_k)$$

where $m_k^2 = c_k^2 + d_k^2$ and $\tan\phi_k = \dfrac{d_k}{c_k}$.

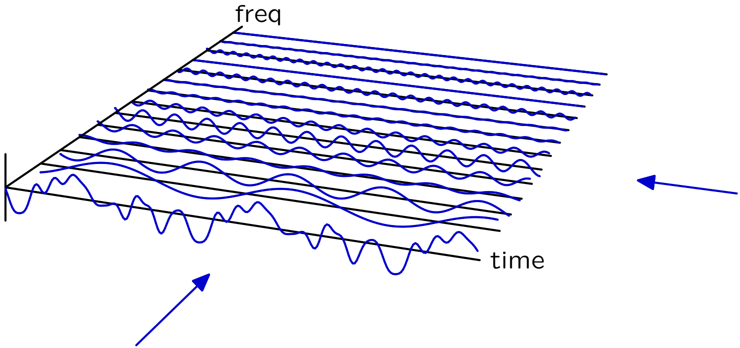


Viewing a Signal as a Fourier Series

Fourier Series provide an alternative view of information contained in $f(t)$.

$$f(t) = \sum_{k=0}^{\infty} m_k \cos(k\omega_o t + \phi_k)$$

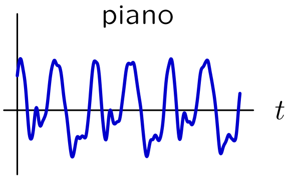
$$= m_1 \cos(\omega_o t+\phi_1) + m_2 \cos(2\omega_o t+\phi_2) + m_3 \cos(3\omega_o t+\phi_3) + \cdots$$

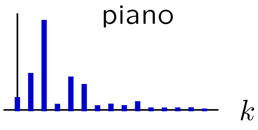


Two views: as a function of time and as a function of frequency

Fourier Series Representation of Piano Note

Viewing a signal as a function of time or as a Fourier series highlights different (complementary) information.





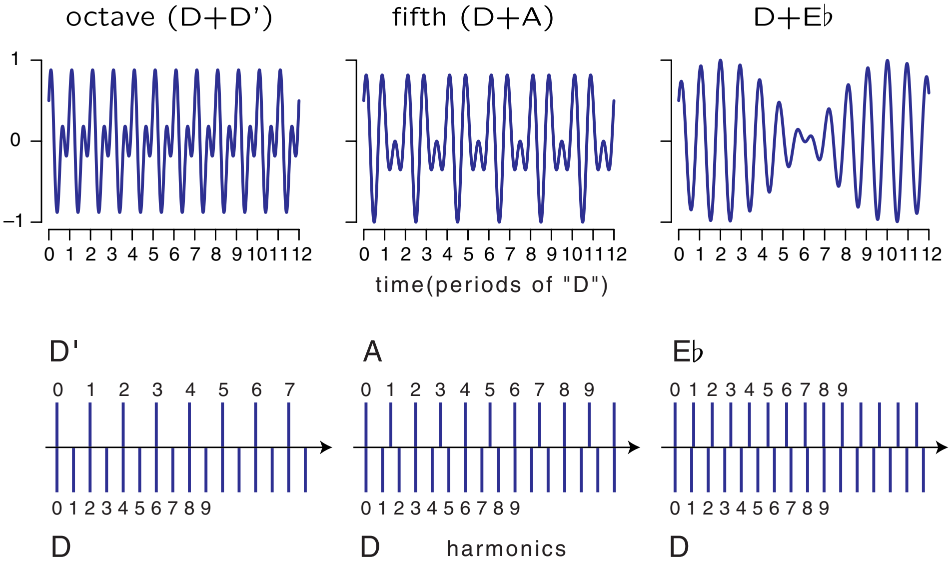
For example, the time function highlights periodicity, while the Fourier series highlights the distribution of energy across frequency.

Differences in apparent periodicity of the positive and negative peaks in time show up as separate components at $k=1$ and $k=2$ in frequency.

Higher order frequency omponents are more apparent in the Fourier series than in time. Notice for example that the $k=3$ term is especially small.

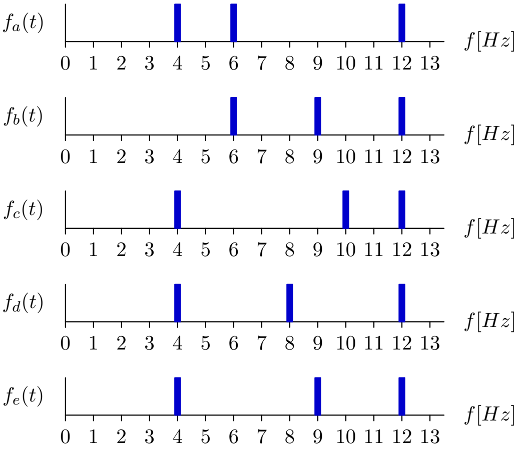
Last Time

Dissonance is more easily recognized in the Fourier representation.



Check Yourself

The Fourier series for each of five periodic signals ($f_a(t)$–$f_e(t)$) contain just three nonzero components at frequencies shown by bars below:



Which Fourier series has the largest fundamental period?

1. $f_a(t)$ 2. $f_b(t)$ 3. $f_c(t)$ 4. $f_d(t)$ 5. $f_e(t)$

Last Time: Determining Fourier Series

How do we find the coefficients c_k and d_k?

Key idea: Sift out the component of interest by

- multiplying by the corresponding basis function, and then
- integrating over a period.

This results in the following expressions for the Fourier series coefficients:

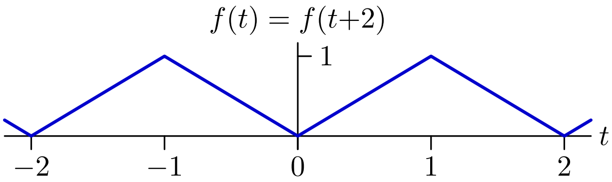
$$c_0 = \frac{1}{T}\int_T f(t)\,dt$$

$$c_k = \frac{2}{T}\int_T f(t)\cos(k\omega_o t)\,dt;\ \ k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T}\int_T f(t)\sin(k\omega_o t)\,dt;\ \ k = 1, 2, 3, \ldots$$

Example of Analysis

Find the Fourier series coefficients for the following triangle wave:



$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

$$c_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi k t}{T}\,dt = 2\int_0^1 t\cos(\pi k t)\,dt = \begin{cases} -\frac{4}{\pi^2 k^2} & k \text{ odd} \\ 0 & k = 2, 4, 6, \ldots \end{cases}$$

$d_k = 0$ (by symmetry)

Fourier Synthesis

The previous example shows that the sum of an infinite number of sinusoids can approximate a piecewise linear function **with discontinuous slope**!

This result is a bit surprising since none of the basis functions have discontinuous slopes.

What about signals **with discontinuous values**?

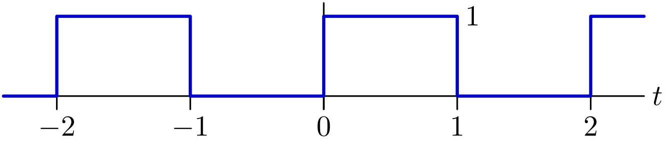
Fourier defended the idea that such a series is meaningful.
Lagrange ridiculed the idea that discontinuities could be generated from a sum of continuous signals.

We can test this idea empirically – using computation.

Fourier Analysis of a Square Wave

Find the Fourier series coefficients for the following square wave:

$$f(t) = f(t+2)$$



$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

$$c_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega_o t)\,dt = \int_0^1 \cos(k\pi t)dt = \left.\frac{\sin(k\pi t)}{k\pi}\right|_0^1 = 0 \text{ for } k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega_o t)\,dt = \int_0^1 \sin(k\pi t)dt = -\left.\frac{\cos(k\pi t)}{k\pi}\right|_0^1 = \begin{cases} \frac{2}{k\pi} & k = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases}$$

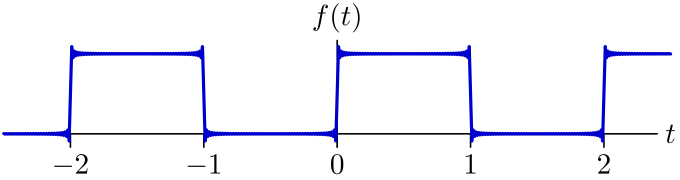
Fourier Synthesis of a Square Wave

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)$$

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{99} \frac{2}{k\pi} \sin(k\pi t)$$

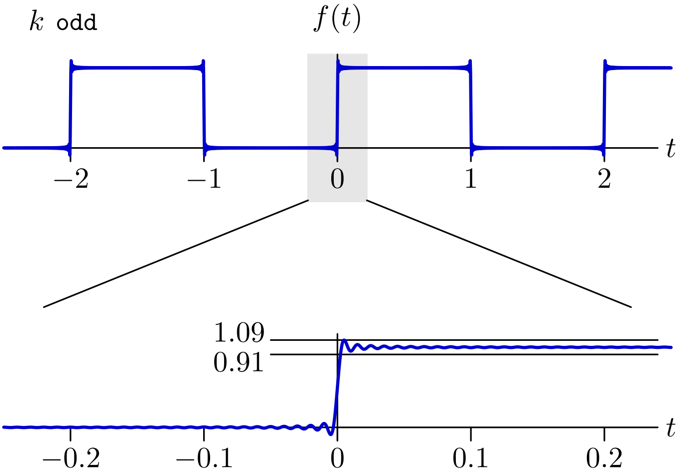


The synthesized function approaches original as number of terms increases.

Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{199} \frac{2}{k\pi} \sin(k\pi t)$$



Increasing the number of terms does not decrease the peak overshoot, but it does shrink the region of time that is occupied by the overshoot.

Convergence of Fourier Series

If there is a **step discontinuity** in $f(t)$ at $t = t_0$, then the Fourier series for $f(t_0)$ converges to the average of the limits of $f(t)$ as t approaches t_0 from the left and from the right.

Let $f_K(t)$ represent the **partial sum** of the Fourier series using just N terms:

$$f_K(t) = a_0 + \sum_{k=0}^{K} \Big(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big)$$

As $K \to \infty$,

- the maximum difference between $f(t)$ and $f_K(t)$ converges to $\approx 9\%$ of $|f(t_0^+) - f(t_0^-)|$ and
- the region over which the absolute value of the difference exceeds any small number ϵ shrinks to zero.

We refer to this type of overshoot as **Gibb's Phenomenon**.

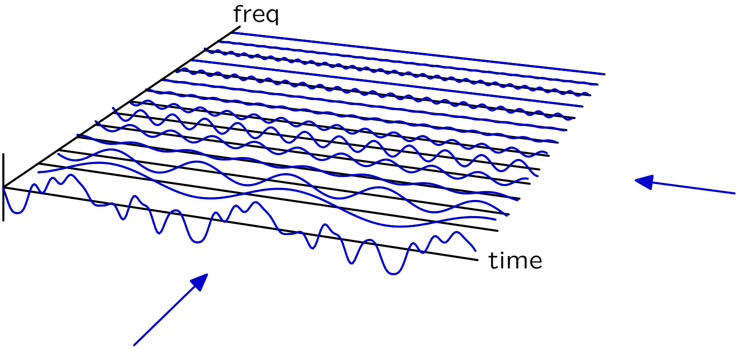
So who was right? Fourier or Lagrange? **Both!**

The Fourier series of a discontinuous function converges, but not uniformly.

Using Fourier Series

The time and Fourier series representations of a signal are equivalent. Therefore we can do signal processing tasks using either representation.

$$f(t) = f(t+T) = \sum_{k=0}^{\infty} (c_k \cos k\omega_o t + d_k \sin k\omega_o t) = \sum_{k=0}^{\infty} m_k \cos(k\omega_o t + \phi_k)$$

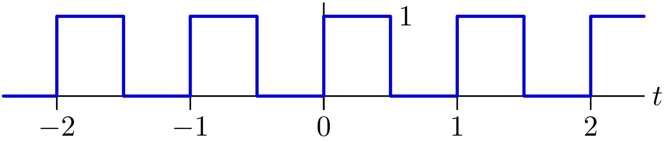


Furthermore, we can use **properties** of Fourier series to develop operations on Fourier series that are equivalent to operations on time functions.

Properties of Fourier Series: Scaling Time

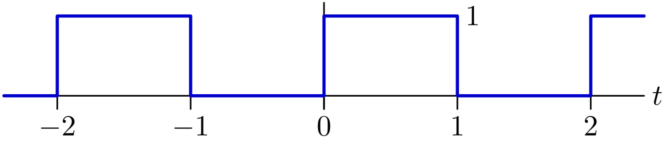
Find the Fourier series coefficients for the following square wave:

$$g(t) = g(t+1)$$



We could repeat the process used to find the Fourier coefficients for $f(t)$.

$$f(t) = f(t+2)$$



Alternatively, we can take advantage of the relation between $f(t)$ and $g(t)$:

$$g(t) = f(2t)$$

Scaling Time

We already know the Fourier series expansion of $f(t)$:

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi} \sin(k\pi t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi} \sin(k\omega_o t)$$

$$c_k = \begin{cases} \frac{1}{2} & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$d_k = \begin{cases} \frac{2}{k\pi} & k = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases}$$

where $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$.

Check Yourself

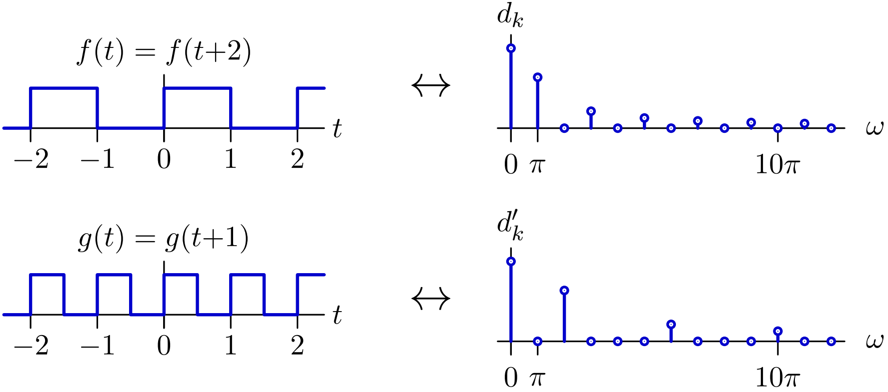
Let d_k represent the Fourier series coefficients for $f(t)$ and let d'_k represent those for $g(t) = f(2t)$.

Which of the following relations are true?

- $d'_k = 2d_k$: amplitudes double
- $d'_k = d_{2k}$: harmonic indicies half
- $d'_k = d_{k/2}$: harmonic indicies double
- $d'_k = 2d_{k/2}$: amplitudes and harmonic indices double
- $d'_k = d_k$: no change

Scaling Time

Plot the Fourier series coefficients on a frequency scale.



Compressing the time axis has stretched the ω axis.

Check Yourself

What is the effect of **shifting** time?

Assume that $f(t)$ is periodic in time with period T:

$$f(t) = f(t+T) \,.$$

Let $g(t)$ represent a version of $f(t)$ shifted by half a period:

$$g(t) = f(t-T/2) \,.$$

How many of the following statements correctly describe the effect of this shift on the Fourier series coefficients.

- cosine coefficients c_k are negated
- sine coefficients d_k are negated
- odd-numbered coefficients $c_1, d_1, c_3, d_3, \ldots$ are negated
- sine and cosine coefficients are swapped: $c_k \rightarrow d_k$ and $d_k \rightarrow c_k$

Why Focus on Fourier Series?

What's so special about sines and cosines?

Sinusoidal functions have interesting **mathematical properties**.
$\rightarrow$ harmonically related sinusoids are **orthogonal** to each other over $[0, T]$.

Orthogonality: $f(t)$ and $g(t)$ are orthogonal over $0 \leq t \leq T$ if

$$\int_T f(t)g(t)\,dt = 0$$

Example: Calculate this integral for the k^{th} and l^{th} harmonics of $\cos(\omega_o t)$.

$$\int_T \cos(k\omega_o t)\cos(l\omega_o t)\,dt$$

We can use trigonometry to express the product of the two cosines as the sum of cosines of the sum and difference frequencies:

$$\int_T \Big(\frac{1}{2}\cos((k{+}l)\omega_o t) + \frac{1}{2}\cos((k{-}l)\omega_o t)\Big)\,dt$$

The sum and difference frequencies are also harmonics of ω_o, so their integral over T is zero (provided $k \neq l$).

Why Focus on Fourier Series?

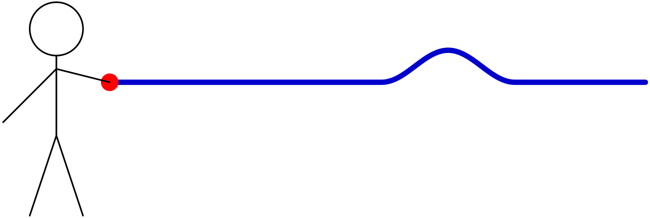
What's so special about sines and cosines?

Sinusoidal functions have interesting **mathematical properties**.
$\rightarrow$ harmonically related sinusoids are **orthogonal** to each other over $[0, T]$.

Sines and cosines also play important roles in **physics** –
especially the physics of waves.

Physical Example: Vibrating String

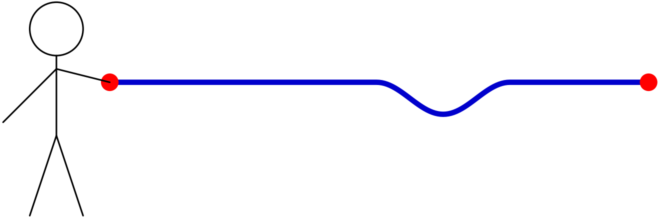
A taut string supports wave motion.



The speed of the wave depends on the tension on and mass of the string.

Physical Example: Vibrating String

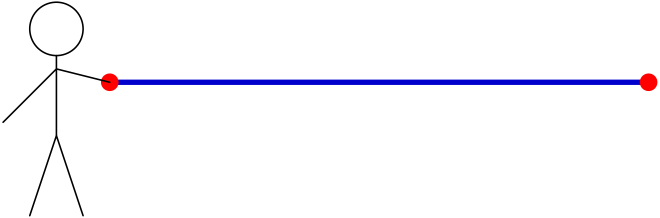
The wave will reflect off a rigid boundary.



The amplitude of the reflected wave is opposite that of the incident wave.

Physical Example: Vibrating String

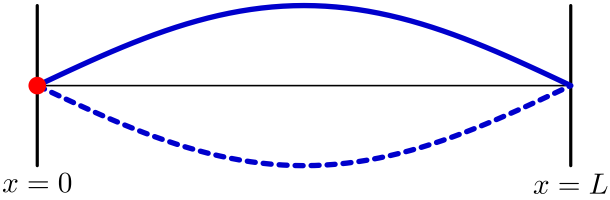
Reflections can interfere with excitations.



The interference can be constructive or destructive depending on the frequency of the excitation.

Physical Example: Vibrating String

We get constructive interference if round-trip travel time equals the period.

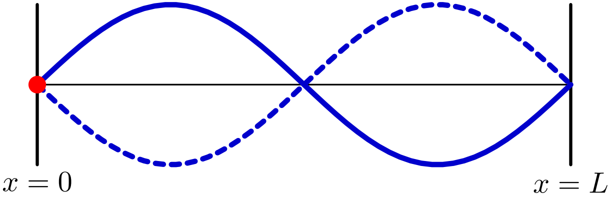


Round-trip travel time $= \dfrac{2L}{v} = T$

$$\omega_o = \frac{2\pi}{T} = \frac{2\pi}{2L/v} = \frac{\pi v}{L}$$

Physical Example: Vibrating String

We also get constructive interference if round-trip travel time is $2T$.

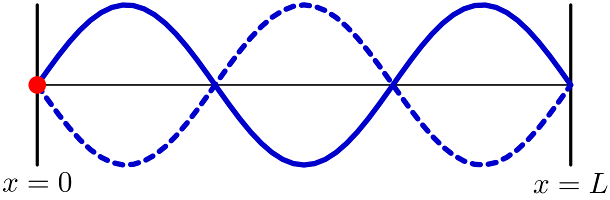


Round-trip travel time $= \dfrac{2L}{v} = 2T$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{L/v} = \frac{2\pi v}{L} = 2\omega_o$$

Physical Example: Vibrating String

In fact, we also get constructive interference if round-trip travel time is kT.



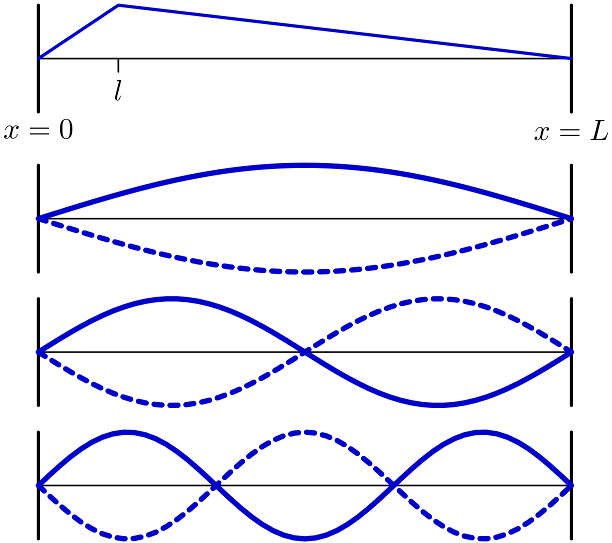
Round-trip travel time $= \dfrac{2L}{v} = kT$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2L/kv} = \frac{k\pi v}{L} = k\omega_o$$

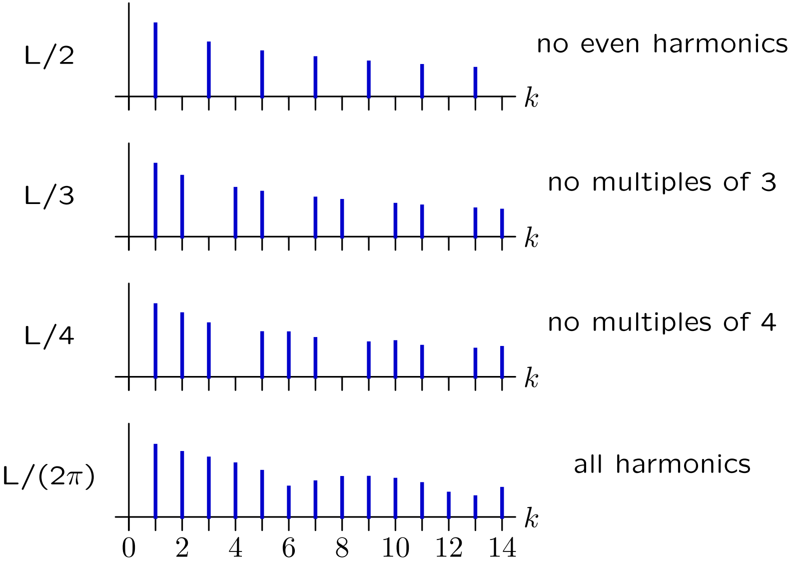
Only certain frequencies (harmonics of $\omega_o = \pi v/L$) persist.
This is the basis of stringed instruments.

Physical Example: Vibrating String

More complicated motions can be expressed as a sum of normal modes using Fourier series. Here the string is "plucked" at $x = l$.



Physical Example: Vibrating String



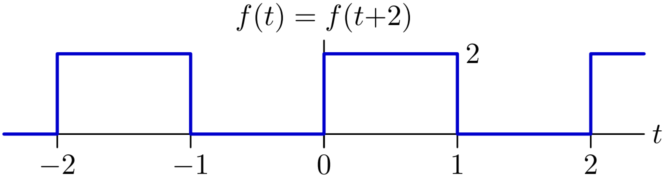
Differences in harmonic structure generate differences in timbre.

Summary

- We examined the convergence of Fourier series.
  - Functions with discontinuous slopes well represented.
  - Functions with discontinuous values generate ripples
    $\rightarrow$ Gibb's phenomenon.
- We investigated several **properties** of Fourier series.
  - scaling time
  - shifting time
  - We will find that there are **many** others
- We saw how Fourier series are useful for modeling a vibrating string.

Question of the Day

Let $f(t)$ represent the following periodic square wave:



How many of the following orthogonality conditions are true?

- $f(t) \perp f(2t)$?
- $(f(t)-1) \perp (f(2t)-1)$?
- $(f(t)-2) \perp (f(2t)-2)$?
- $(f(t)-1) \perp (f(2t)-2)$?

Trig Table

```
sin(a+b) = sin(a) cos(b) + cos(a) sin(b)
sin(a-b) = sin(a) cos(b) - cos(a) sin(b)
cos(a+b) = cos(a) cos(b) - sin(a) sin(b)
cos(a-b) = cos(a) cos(b) + sin(a) sin(b)
tan(a+b) = (tan(a)+tan(b))/(1-tan(a) tan(b))
tan(a-b) = (tan(a)-tan(b))/(1+tan(a) tan(b))

sin(A) + sin(B) =  2 sin((A+B)/2) cos((A-B)/2)
sin(A) - sin(B) =  2 cos((A+B)/2) sin((A-B)/2)
cos(A) + cos(B) =  2 cos((A+B)/2) cos((A-B)/2)
cos(A) - cos(B) = -2 sin((A+B)/2) sin((A-B)/2)

sin(a+b) + sin(a-b) =  2 sin(a) cos(b)
sin(a+b) - sin(a-b) =  2 cos(a) sin(b)
cos(a+b) + cos(a-b) =  2 cos(a) cos(b)
cos(a+b) - cos(a-b) = -2 sin(a) sin(b)

2 cos(A) cos(B) = cos(A-B) + cos(A+B)
2 sin(A) sin(B) = cos(A-B) - cos(A+B)
2 sin(A) cos(B) = sin(A+B) + sin(A-B)
2 cos(A) sin(B) = sin(A+B) - sin(A-B)
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