6.3000: Signal Processing

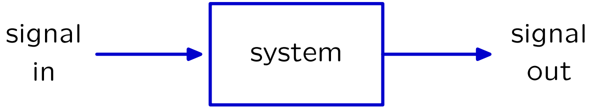
Frequency Response and Filtering

- Discrete-Time Frequency Response
- Continuous-Time Frequency Response

March 17, 2026

Context: The System Abstraction

Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



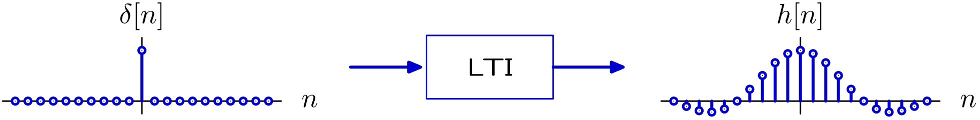
This abstraction is particularly powerful for **linear and time-invariant** systems, which are both **prevalent** and **mathematically tractable**.

Representations of systems:

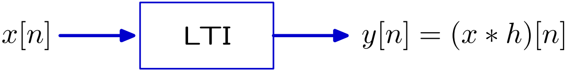
- **Difference/Differential Eq:** constraints on inputs and outputs $\surd$
- **Convolution:** represent system by unit-sample/impulse response $\surd$
- **Filter:** represent a system by its frequency response

Last Time: Unit-Sample Response and Convolution

Responses of an LTI system are completely chracterized by a **single signal**.

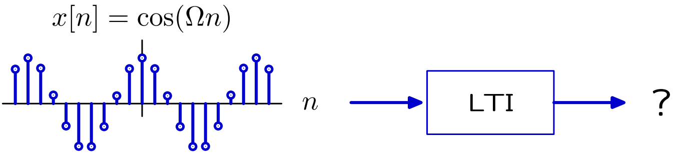


If the response of a system to a unit-sample signal $\delta[n]$ is $h[n]$, then the response of system to an arbitrary input $x[n]$ is given by convolution of that input with the unit sample response $h[n]$:



Today: Frequency Response and Filtering

An alternative representation is based on the system's response to sinusoids.



Is there an efficient way to think about responses to sinusoids?

Responses to Sinusoids

One way to find the response to a sinusoid is to use convolution.

$x[n] = \cos(\Omega n)$

$y[n] = (x * h)[n]$

$$y[n] = (x * h)[n] = \sum_{m=-\infty}^{\infty} x[n-m]h[m] = \sum_{m=-\infty}^{\infty} \cos(\Omega(n-m))h[m]$$

Each term in the sum is a cosine function of n with the same frequency Ω.
A sum of sinusoids at a single frequency is a sinusoid at that same frequency.
Only the magnitudes and phases differ.

$$y[n] = \sum_{m=-\infty}^{\infty} h[m] \cos(\Omega n - \phi[m]) = A \cos(\Omega n - \phi)$$

We'd like to find the net amplitude A and net phase ϕ.

Response to Complex Exponentials

Using complex exponentials is easier than using trigonometric functions.

$$x[n] = e^{j\Omega n}$$

$$y[n] = (x * h)[n]$$

$$y[n] = (x * h)[n] = \sum_{m=-\infty}^{\infty} x[n-m]h[m] = \sum_{m=-\infty}^{\infty} e^{j\Omega(n-m)}h[m]$$

$$= e^{j\Omega n} \sum_{m=-\infty}^{\infty} h[m]e^{-j\Omega m} = H(\Omega)\, e^{j\Omega n}$$

Response to Complex Exponentials

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$$x[n] = e^{j\Omega n} \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = (x * h)[n]$$

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$$= e^{j\Omega n} \sum_{m=-\infty}^{\infty} h[m]e^{-j\Omega m} = H(\Omega)\, e^{j\Omega n}$$

$H(\Omega)$ = discrete-time Fourier transform of the unit sample response $h[n]$:

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$$ **analysis equation**

$$h[n] = \frac{1}{2\pi}\int_{2\pi} H(\Omega)e^{j\Omega n}\, d\Omega$$ **synthesis equation**

Response to Complex Exponentials

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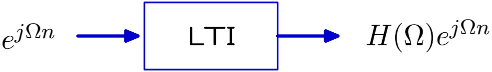
The response to a complex exponential is a complex exponential with the **same frequency (Ω)** but with amplitude and phase given by $H(\Omega)$.



The map for how a system modifies the amplitude and phase of a complex exponential input is the **Fourier transform of the unit-sample response.**

Responses to Sinusoids

The response to a cosine function follows directly from Euler's formula.

$$x[n] = \cos(\Omega n)$$

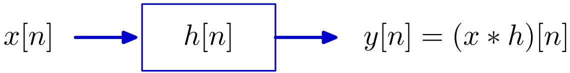
$$y[n] = A\cos(\Omega n - \phi)$$

$$\begin{aligned}
e^{j\Omega n} &\to H(\Omega)e^{j\Omega n} \\
e^{-j\Omega n} &\to H^*(\Omega)e^{-j\Omega n} \\
\cos(\Omega n) &\to \frac{1}{2}\left(H(\Omega)e^{j\Omega n} + H^*(\Omega)e^{-j\Omega n}\right) \\
&\to \text{Re}(H(\Omega)e^{j\Omega n}) \\
&\to \text{Re}\left(|H(\Omega)|e^{j\angle H(\Omega)}e^{j\Omega n}\right) \\
&\to |H(\Omega)|\,\text{Re}\left(e^{j(\angle H(\Omega)+\Omega n)}\right) \\
&\to |H(\Omega)|\cos\left(\Omega n + \angle H(\Omega)\right)
\end{aligned}$$

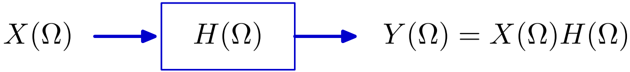
The response of an LTI system to a cosine input is a cosine output with amplitude equal to the magnitude of $H(\Omega)$ and phase given by $\angle H(\Omega)$.

Filtering

Convolution in time is equivalent to multiplication in frequency.



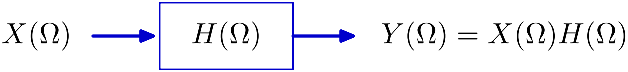
$$
\begin{aligned}
Y(\Omega) &= \sum_n y[n] e^{-j\Omega n} \\
&= \sum_n \left(\sum_m x[m] h[n-m] \right) e^{-j\Omega n} \\
&= \sum_m x[m] \sum_n h[n-m] e^{-j\Omega n} \\
&= \sum_m x[m] \sum_l h[l] e^{-j\Omega(l+m)} \quad \text{where } l = n - m \\
&= \left(\sum_m x[m] e^{-j\Omega m} \right) \left(\sum_l h[l] e^{-j\Omega l} \right) = X(\Omega) H(\Omega)
\end{aligned}
$$



An LTI system scales the Fourier transform of its input by the Fourier transform of its unit-sample response.

Filtering

The non-zero frequency components of the output of an LTI system are a subset of those at its input.



We say that the system "filters" the input:

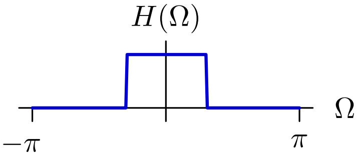
- A filter cannot introduce frequency components that were not already present in the input.
- A filter can selectively amplify or attenuate frequency components that are already present in its input.

Intuitive View of Filtering

The frequency response can be an **insightful** description of a system.

Example:

A low-pass filter passes frequencies near 0 and rejects those near π.



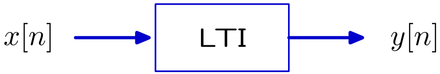
Very natural way to describe audio components:

- microphones
- loudspeakers
- audio equalizers

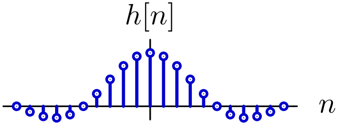
Many other examples in last half of 6.3000.

Unit-Sample Response and Frequency Response

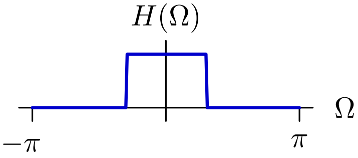
Two complete representations for linear, time-invariant systems.



Unit-Sample Response: responses across time for a unit-sample input.



Frequency Response: responses across frequencies for sinusoidal inputs.



The **frequency response** is Fourier transform of **unit-sample response**!

Example

Find the frequency response of a causal system that starts at rest:

$$y[n] - \alpha y[n-1] = x[n]$$

A causal system is one in which an input at time $n = n_0$ cannot affect the output at times that are less than $n = n_0$.

Example

Find the frequency response of a causal system that starts at rest:

$$y[n] - \alpha y[n-1] = x[n]$$

Method 1:

Find the unit-sample response and take its Fourier transform.

$$h[n] - \alpha h[n-1] = \delta[n]$$

Solve the difference equation for $h[n]$.

$$h[n] = \delta[n] + \alpha h[n-1]$$

Causality + Rest $\rightarrow$ $h[n] = 0$ for $n < 0$.

$$\begin{aligned} h[0] &= \delta[0] + \alpha h[-1] = 1 \\ h[1] &= \delta[1] + \alpha h[0] = \alpha \\ h[2] &= \delta[2] + \alpha h[1] = \alpha^2 \\ h[3] &= \delta[3] + \alpha h[2] = \alpha^3 \end{aligned}$$

$$h[n] = \alpha^n u[n]$$

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\Omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(\alpha e^{-j\Omega}\right)^n = \frac{1}{1-\alpha e^{-j\Omega}}$$

Example

Find the frequency response of a causal system that starts at rest:

$$y[n] - \alpha y[n-1] = x[n]$$

Method 2:

Find the response to $e^{j\Omega n}$ directly.

$$x[n] = e^{j\Omega n}$$

Because the system is linear and time-invariant, the output will have the same frequency as the input, but possibly different amplitude and phase.

$$y[n] = H(\Omega)e^{j\Omega n}$$

$$y[n-1] = H(\Omega)e^{j\Omega(n-1)} = H(\Omega)e^{-j\Omega}e^{j\Omega n}$$

Substitute into the difference equation.

$$H(\Omega)e^{j\Omega n} - \alpha H(\Omega)e^{-j\Omega}e^{j\Omega n} = H(\Omega)(1-\alpha e^{-j\Omega})e^{j\Omega n} = e^{j\Omega n}$$

Since $e^{j\Omega n}$ is never 0, we can divide it out.

$$H(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}}$$

Same answer as method 1.

Example

Find the frequency response of a causal system that starts at rest:

$$y[n] - \alpha y[n-1] = x[n]$$

Method 3:

Take the Fourier transform of the difference equation.

$$Y(\Omega) - \alpha e^{-j\Omega} Y(\Omega) = X(\Omega)$$

Solve for $Y(\Omega)$.

$$Y(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}} X(\Omega)$$

Since $Y(\Omega) = H(\Omega)X(\Omega)$,

$$H(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}}$$

Same answer as methods 1 and 2.

Check Yourself

Plot the frequency response.

$$H(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}}$$

Assume $0 \leq \alpha \leq 1$.

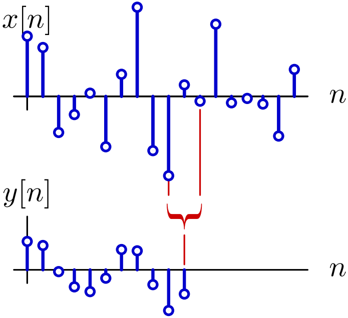
Which of the following describes the frequency response?

- Low baseband frequencies are amplified.
- High baseband frequencies are amplified.
- All baseband frequencies are amplified.
- Low baseband frequencies are delayed.
- High baseband frequencies are delayed.

Check Yourself

Find the frequency response of a three-point averager:

$$y[n] = \frac{1}{3}\Big(x[n-1] + x[n] + x[n+1]\Big)$$



Can we think of this as a low-pass filter?
Does it pass low frequences and block high frequencies?

Frequency Response of a Continuous-Time System

The frequency approach works in continuous-time systems as well.

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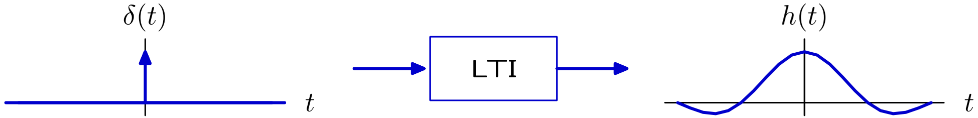
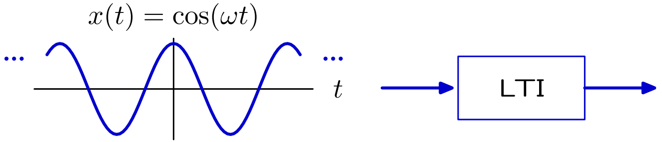
The response of a CT LTI system to the Dirac delta function $\delta(t)$ is the impulse response $h(t)$ of the system.



The response $y(t)$ to a sinusoid $x(t) = \cos(\omega t)$ is $y(t) = (x * h)(t)$.



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Frequency Response

Use complex exponentials to characterize the frequency response.

$x(t) = e^{j\omega t}$

$y(t) = (x * h)(t)$

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} e^{j\omega(t-\tau)}h(\tau)d\tau$$

$$= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau = H(\omega)\, e^{j\omega t}$$

$H(\omega)$ = continuous-time Fourier transform of the impulse response $h(t)$.

Frequency Response

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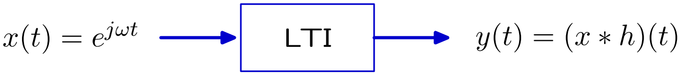

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau = \int_{-\infty}^{\infty} e^{j\omega(t-\tau)} h(\tau) d\tau$$

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$H(\omega)$ = continuous-time Fourier transform of the impulse response $h(t)$.

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$ **analysis equation**

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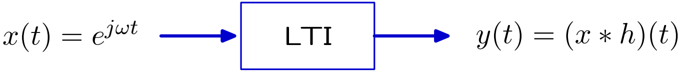

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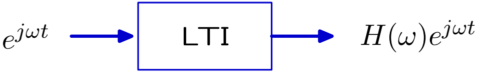

The map for how a system modifies the amplitude and phase of a complex exponential input is the **Fourier transform of the impulse response.**

Example

Find the frequency response of a system described by the following:

$$y(t) + \alpha \frac{dy(t)}{dt} = 2x(t)$$

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Method 1:

Find the response to $e^{j\omega t}$ directly.

$$x(t) = e^{j\omega t}$$

Because the system is linear and time-invariant, the output will have the same frequency as the input, but possibly different amplitude and phase.

$$y(t) = H(\omega)e^{j\omega t}$$

$$\frac{dy(t)}{dt} = j\omega H(\omega)e^{j\omega t}$$

Substitute into the differential equation.

$$H(\omega)e^{j\omega t} + j\omega\alpha H(\omega)e^{j\omega t} = (1 + j\omega\alpha)H(\omega)e^{j\omega t} = 2e^{j\omega t}$$

Since $e^{j\omega t}$ is never 0, we can divide it out.

$$H(\omega) = \frac{2}{1 + j\omega\alpha}$$

Example

Find the frequency response of a system described by the following:

$$y(t) + \alpha \frac{dy(t)}{dt} = 2x(t)$$

Method 2:

Take the Fourier transform of the differential equation.

$$Y(\omega) + j\omega\alpha Y(\omega) = 2X(\omega)$$

Solve for $Y(\omega)$.

$$Y(\omega) = \frac{1}{1 + j\omega\alpha} 2X(\omega)$$

Since $Y(\omega) = H(\omega)X(\omega)$,

$$H(\omega) = \frac{2}{1 + j\omega\alpha}$$

Same answer as method 1.

Check Yourself

Plot the frequency response of the following system:

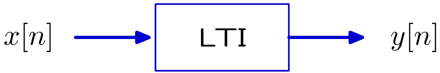
$$y(t) + \alpha \frac{dy(t)}{dt} = 2x(t)$$

Which of the following describes the frequency response?

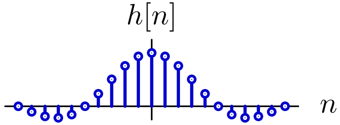
- Low frequencies are attenuated.
- High frequencies are attenuated.
- All frequencies are attenuated.
- Low frequencies are delayed.
- High frequencies are delayed.

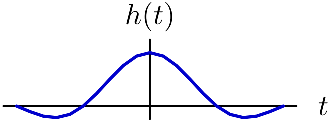
Time and Frequency Representations

Two complete representations for linear, time-invariant systems.

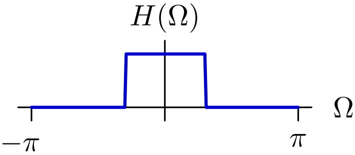


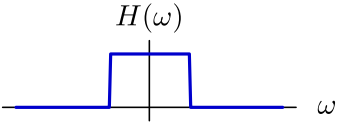
Convolution: responses calculated in the time domain.





Filtering: responses calculated in the frequency domain.





The representation in **frequency** is the Fourier transform of that in **time**!