6.3000: Signal Processing

Fourier Series

- complex numbers
- complex exponentials and their relation to sinusoids
- complex exponential form of Fourier series
- delay property of Fourier series

Homework 1 is due this Thursday (Feb 12) at 2pm.
The (optional) "Check-In" for Lab 1 is due tonight at 9:30pm.

February 10, 2026

Fourier Series

Previously: Representing periodic signals as weighted sums of sinusoids.

Synthesis Equation

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t) \quad \text{where } \omega_o = \frac{2\pi}{T}$$

Analysis Equations

$$c_0 = \frac{1}{T}\int_T f(t)\,dt$$

$$c_k = \frac{2}{T}\int_T f(t)\cos(k\omega_o t)\,dt$$

$$d_k = \frac{2}{T}\int_T f(t)\sin(k\omega_o t)\,dt$$

Today: Simplifying the math with complex numbers.

Simplifying Math By Using Complex Numbers

Complex numbers simplify thinking about roots of numbers / polynomials:

- all numbers have two square roots, three cube roots, etc.
- all polynomials of order n have n roots (some of which may be repeated).

$\rightarrow$ much simpler than the rules that govern purely real-valued formulations. For example, a cubic equation with real-valued coefficients can have 1 or 3 real-valued roots; a quartic equation can have 0, 2, or 4.

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Complex exponentials simplify working with trigonometric functions (Euler's formula, Leonhard Euler, 1748):

$$e^{j\theta} = \cos\theta + j\sin\theta$$

This single equation virtually eliminates a need for trig tables. Richard Feynman called this "the most remarkable formula in mathematics."

The special case $\theta = \pi$ leads to Euler's Identity:

$$e^{j\pi} + 1 = 0$$

which relates five fundamental constants in a single equation.

Complex numbers/exponentials simply working with Fourier series.

Where Does Euler's Formula Come From?

Euler showed the relation between complex exponentials and sinusoids by solving the following differential equation two ways.

$$\frac{d^2 f(\theta)}{d\theta^2} + f(\theta) = 0$$

Let $f_1(\theta) = A\cos(\alpha\theta) + B\sin(\beta\theta)$

$$\frac{df_1(\theta)}{d\theta} = -\alpha A\sin(\alpha\theta) + \beta B\cos(\beta\theta)$$

$$\frac{d^2 f_1(\theta)}{d\theta^2} = -\alpha^2 A\cos(\alpha\theta) - \beta^2 B\sin(\beta\theta)$$

Let $\alpha = \beta = 1$

$$f_1(\theta) = A\cos\theta + B\sin\theta$$

Let $f_2(\theta) = Ce^{\gamma\theta}$

$$\frac{df_2(\theta)}{d\theta} = \gamma Ce^{\gamma\theta}$$

$$\frac{d^2 f_2(\theta)}{d\theta^2} = \gamma^2 Ce^{\gamma\theta}$$

Let $\gamma^2 = -1$

$$f_2(\theta) = Ce^{\pm j\theta}$$

If we arbitrarily take $f_2(\theta) = e^{j\theta}$, then $f_2(0) = 1$ and $f_2'(0) = j$.
To make $f_1(\theta) = f_2(\theta)$, A must be 1 and B must be j:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

This argument presumes the existance of a constant j whose square is -1 and that can be manipulated as an ordinary algebraic constant.

Where Does Euler's Formula Come From?

Euler's formula also follows from Maclaurin expansion of the exponential function, assuming the j behaves like any other algebraic constant.

Start with the expansion of the real-valued function:

$$e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} + \frac{\theta^6}{6!} + \frac{\theta^7}{7!} + \cdots$$

Assume that the same expansion holds for complex-valued arguments:

$$\begin{aligned} e^{j\theta} &= 1 + j\theta + \frac{j^2\theta^2}{2!} + \frac{j^3\theta^3}{3!} + \frac{j^4\theta^4}{4!} + \frac{j^5\theta^5}{5!} + \frac{j^6\theta^6}{6!} + \frac{j^7\theta^7}{7!} + \cdots \\ &= 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{j\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{j\theta^7}{7!} + \cdots \\ &= \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)}_{\cos\theta} + j\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)}_{\sin\theta} \end{aligned}$$

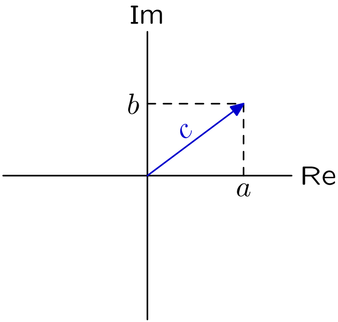
Euler's formula results by splitting the even and odd powers of θ.

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Geometric Interpretation

In 1799, Caspar Wessel was the first to describe complex numbers as points in the complex plane. Imaginary numbers had been in use since the 1500's.

$$c = a + jb$$



Complex numbers are fundamentally two dimensional. Unlike other constants (such as π), $j = \sqrt{-1}$ defines an entirely new (imaginary) dimension – and a new way to think about operations that involve complex numbers.

Algebraic Addition

Addition: the real part of a sum is the sum of the real parts, and the imaginary part of a sum is the sum of the imaginary parts.

Let c_1 and c_2 represent complex numbers:

$$c_1 = a_1 + jb_1$$

$$c_2 = a_2 + jb_2$$

Then

$$c_1 + c_2 = (a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$$

Geometric Addition

Rules for adding complex numbers are same as those for adding vectors.

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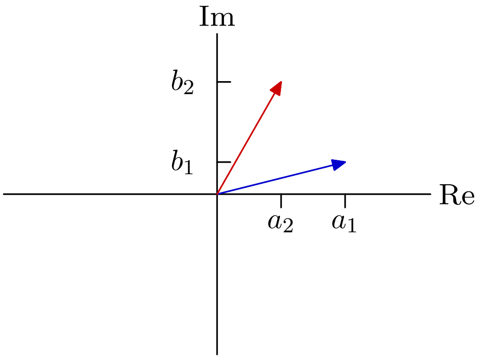

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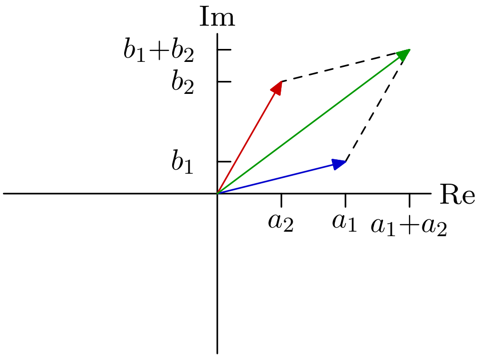


Algebraic Multiplication

Multiplication is more complicated.

Let c_1 and c_2 represent complex numbers:

$$c_1 = a_1 + jb_1$$

$$c_2 = a_2 + jb_2$$

Then

$$\begin{aligned} c_1 \times c_2 &= (a_1 + jb_1) \times (a_2 + jb_2) \\ &= a_1 \times a_2 + a_1 \times jb_2 + jb_1 \times a_2 + jb_1 \times jb_2 \\ &= (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + b_1 a_2) \end{aligned}$$

Although the rules of algebra still apply, the result is complicated:

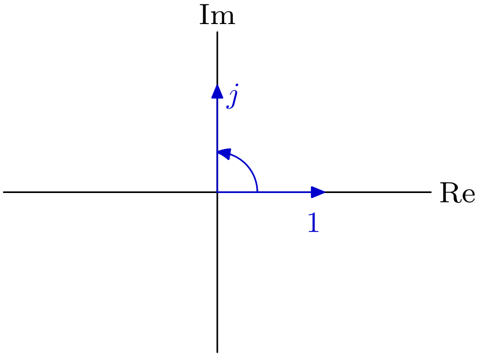
- the real part of a product is NOT the product of the real parts, and
- the imaginary part is NOT the product of the imaginary parts.

Geometric Multiplication

The two-dimensional view of complex numbers allows us to think about multiplication by an imaginary number as a **rotation**.

Multiplying by j

- **rotates 1 to j**,
- rotates j to -1,
- rotates -1 to $-j$, and
- rotates $-j$ to 1.



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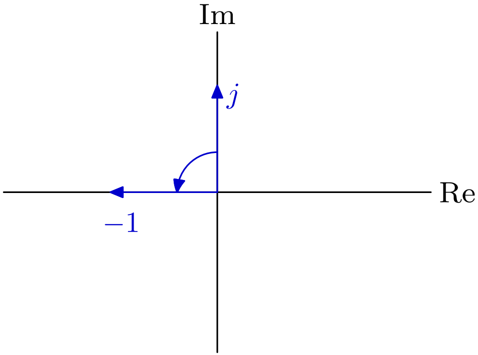

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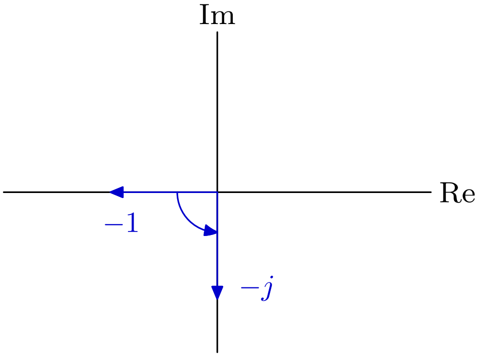

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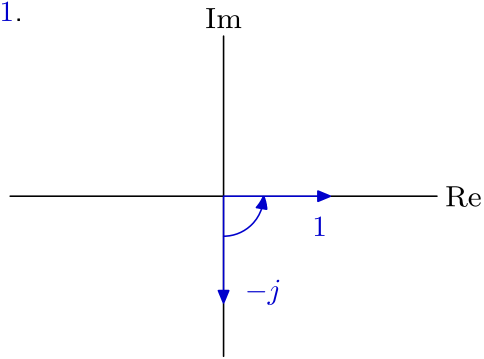

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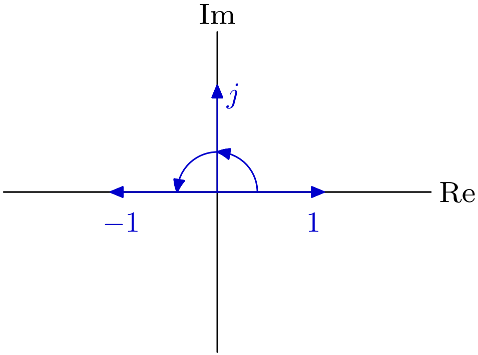


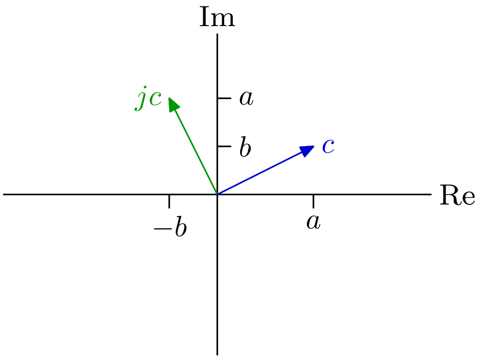
Multiplying by j rotates a vector by $\pi/2$.

Multiplying by $j^2 = -1$ rotates a vector by π.

Geometric Multiplication

Multiplying by j rotates an arbitrary complex number by $\pi/2$.

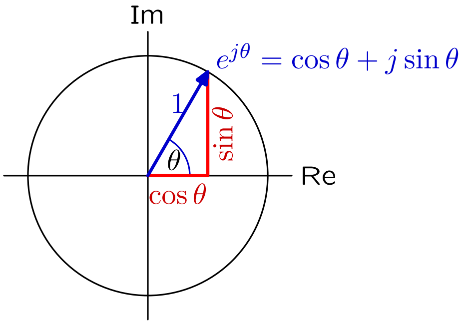
$$c = a + jb$$
$$jc = ja - b$$



Geometric Interpretation of Euler's Formula

Euler's formula equates polar and rectangular descriptions of a unit vector at angle θ.

$$e^{j\theta} = \cos\theta + j\sin\theta$$

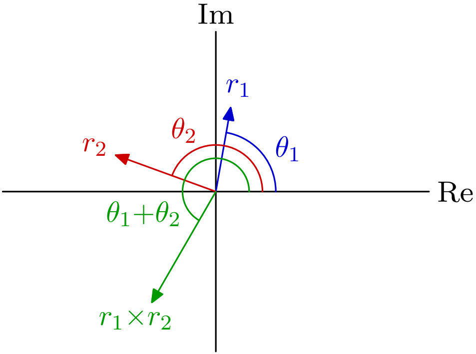


This construction provides

- a direct link between Euler's formula and the planar representation of complex numbers
- a new **polar** representation of complex numbers

Geometric Approach: Polar Form

The magnitude of the product of complex numbers is the **product** of their magnitudes. The angle of a product is the **sum** of the angles.



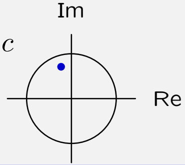
$$r_1 e^{j\theta_1} \times r_2 e^{j\theta_2} = r_1(\cos\theta_1 + j\sin\theta_1) \times r_2(\cos\theta_2 + j\sin\theta_2)$$

$$= r_1 r_2(\underbrace{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}_{\cos(\theta_1+\theta_2)} + \underbrace{j\cos\theta_1\sin\theta_2 + j\sin\theta_1\cos\theta_2}_{j\sin(\theta_1+\theta_2)})$$

$$= r_1 r_2 e^{j(\theta_1+\theta_2)}$$

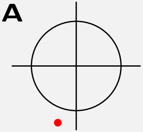
Check Yourself

Let c represent the complex number shown by a filled dot in the complex plane below, where the circle has radius 1.



Which if any of the following figures shows the value of jc?

A



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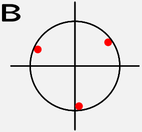

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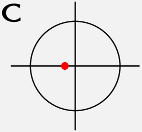

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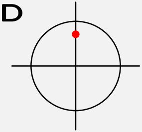

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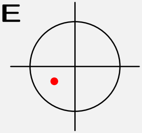

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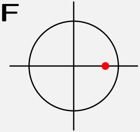
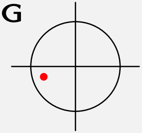
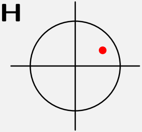


G

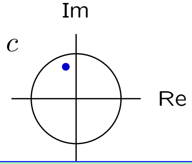


H

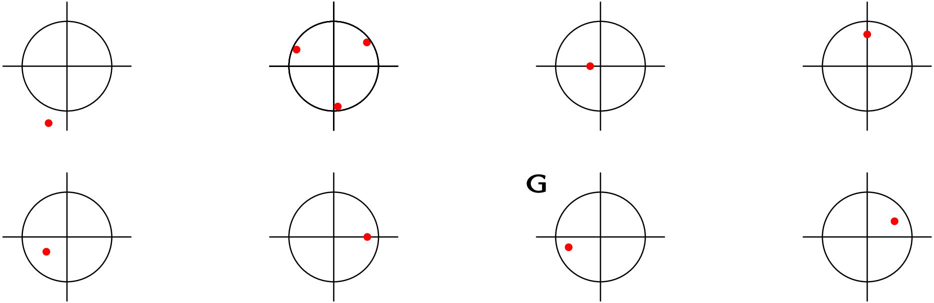


Check Yourself

Let c represent the complex number shown by a filled dot in the complex plane below, where the circle has radius 1.



Which if any of the following figures shows the value of jc? **G**



$|jc| = |c|$ and $\angle(jc) = \angle c + \pi/2$

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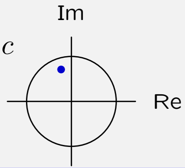
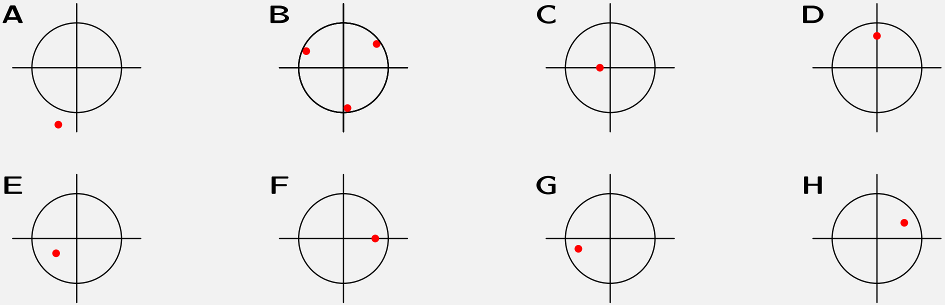


Which if any of the following figures shows the value of Im(c)?



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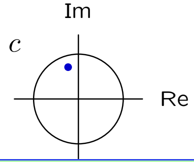
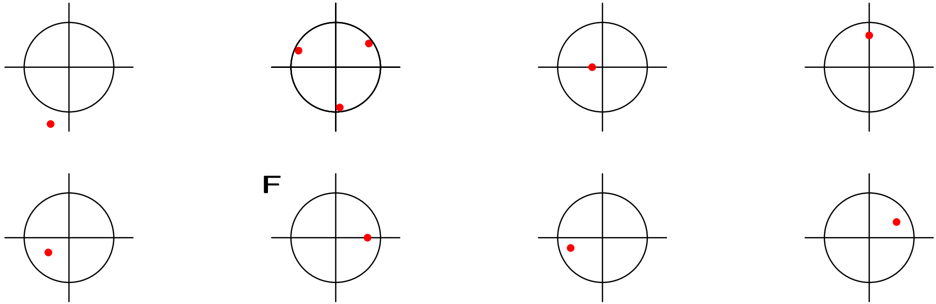


Which if any of the following figures shows the value of Im(c)? **F**



Im(c)= $\frac{c-c^*}{j2}$, which is a real number.

Check Yourself

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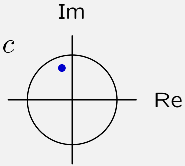
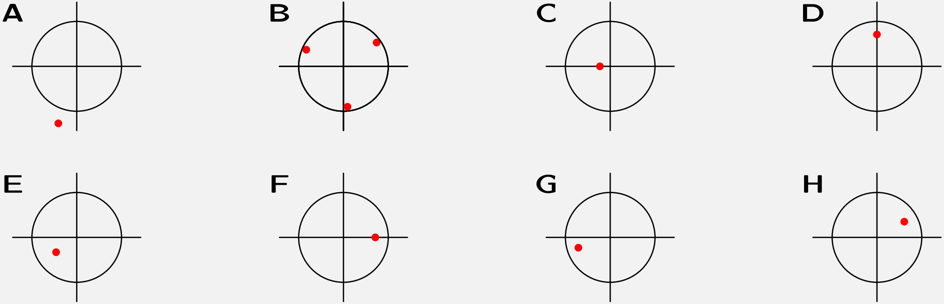


Which if any of the following figures shows the value of $1/c$?



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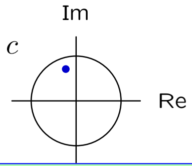

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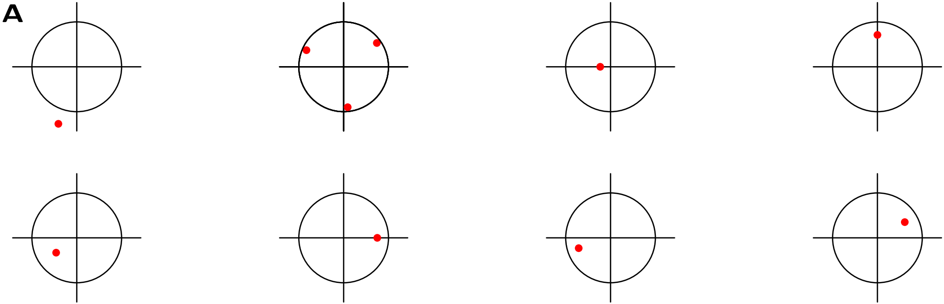

$|1/c| = 1/|c|$ and $\angle(1/c) = -\angle c$

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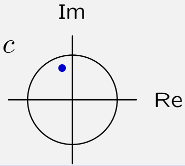
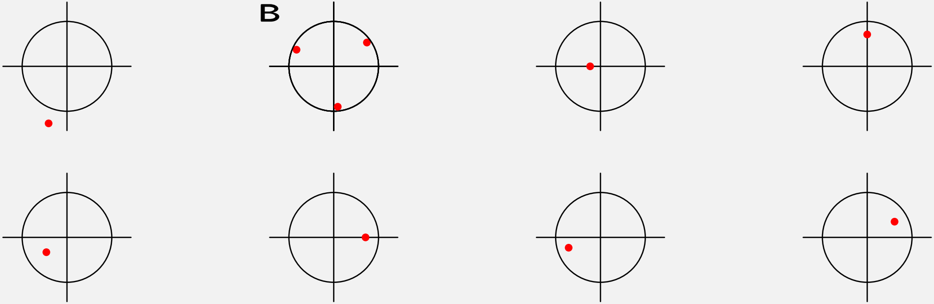


What simple function of c is shown in **B**?



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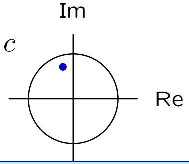

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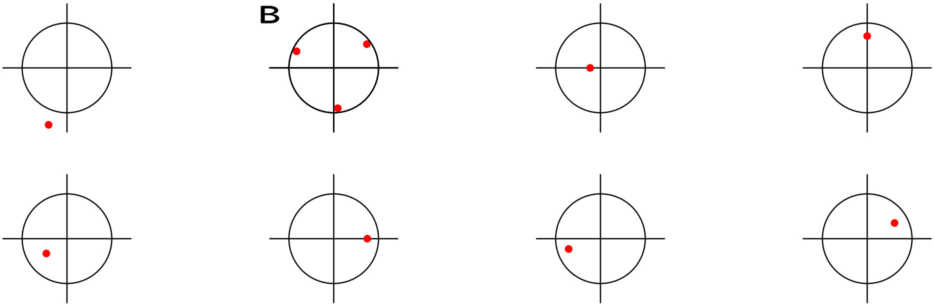

$\sqrt[3]{c}$

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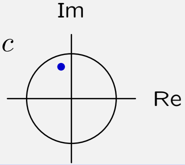

Which if any of the following figures shows the value of $1/(1-c)$?

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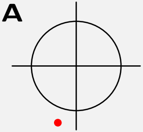

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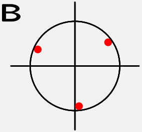

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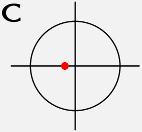

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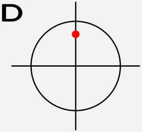

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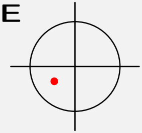

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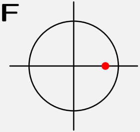

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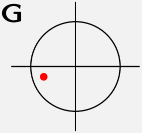

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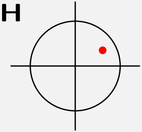

Check Yourself

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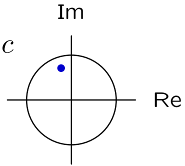
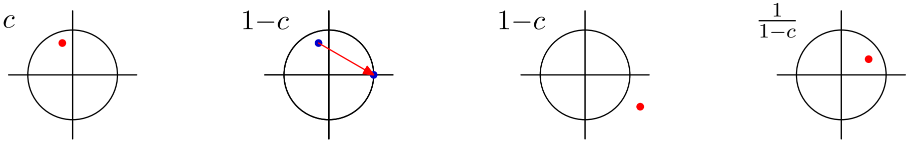


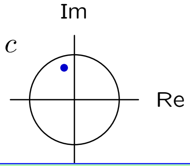
Which if any of the following figures shows the value of $1/(1-c)$?



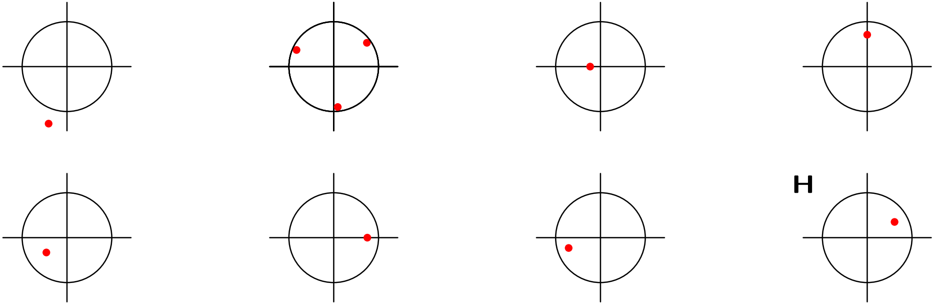
The reciprocal of $1-c$ has a magnitude of $\frac{1}{|1-c|}$ and an angle that is the negative of that of $1-c$, as shown in the right panel above.

Check Yourself

Let c represent the complex number shown by a filled dot in the complex plane below, where the circle has radius 1.



Which if any of the following figures shows the value of $1/(1-c)$? **H**



First find $1-c$ then take the reciprocal.

Using Complex Numbers to Simplify Fourier Series

We have reviewed complex numbers and complex exponentials.

- complex numbers $\surd$
- complex exponentials and their relation to sinusoids $\surd$
- complex exponential form of Fourier series
- delay property of Fourier series

Next: develop a complex exponential form for Fourier series.

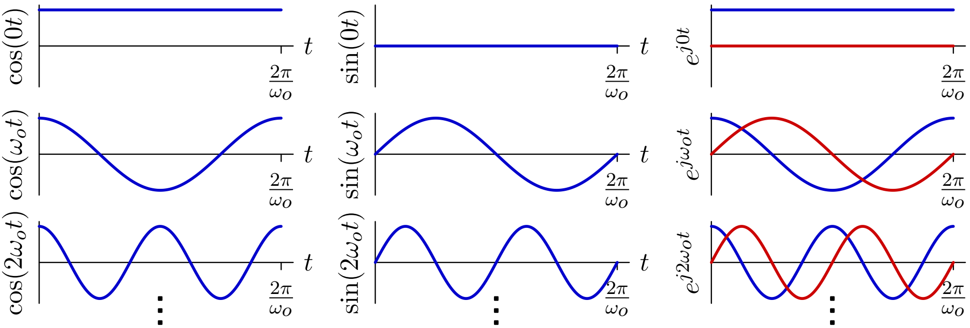
Develop a Complex Exponenial Form for Fourier Series.

Euler's formula allows us to represent both sine and cosine basis functions with a single complex exponential:

$$f(t) = \sum \Big(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big) = \sum a_k e^{jk\omega_o t}$$

Real-valued basis functions

Complex basis functions



This halves the number of coefficients, but each is now complex-valued. More importantly, it replaces the trig functions with an exponential.

Fourier Series Directly From Complex Exponential Form

Assume that $f(t)$ is periodic in T and is composed of a weighted sum of harmonically related complex exponentials.

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_o kt}$$

We can "sift" out the component at $l\omega_o$ by multiplying both sides by $e^{-jl\omega_o t}$ and integrating over a period.

$$\int_T f(t)e^{-j\omega_o lt}dt = \int_T \left(\sum_{k=-\infty}^{\infty} a_k e^{j\omega_o kt}dt\right) e^{-j\omega_o lt}dt = \int_T \sum_{k=-\infty}^{\infty} a_k e^{j\omega_o (k-l)t}dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_T e^{j\omega_o (k-l)t}dt = \begin{cases} Ta_l & \text{if } k = l \\ 0 & \text{otherwise} \end{cases}$$

Solving for a_l provides an explicit formula for the coefficients:

$$a_l = \frac{1}{T}\int_T f(t)e^{-j\omega_o lt}dt\,; \quad \text{where } \omega_o = \frac{2\pi}{T}\,.$$

Negative k

The complex exponential form of the series has positive and negative k's.

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

Only positive values of k are used in the trig form.

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t)$$

The negative k's are required by Euler's formula:

$$e^{jk\omega_o t} = \cos(k\omega_o t) + j\sin(k\omega_o t)$$

Representing $\cos(k\omega_o t)$ and $\sin(k\omega_o t)$ requires not only $e^{jk\omega_o t}$ but also $e^{-jk\omega_o t}$:

$$\cos(k\omega_o t) = \mathrm{Re}\{e^{jk\omega_o t}\} = \frac{1}{2}\left(e^{jk\omega_o t} + e^{-jk\omega_o t}\right)$$

$$\sin(k\omega_o t) = \mathrm{Im}\{e^{jk\omega_o t}\} = \frac{1}{2j}\left(e^{jk\omega_o t} - e^{-jk\omega_o t}\right)$$

The negative k do not indicate negative frequencies. They are the mathematical result of representing sinusoids with complex exponentials.

Fourier Series

Comparison of trigonometric and complex exponential forms.

Complex Exponential Form

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

$$a_k = \frac{1}{T}\int_T f(t) e^{-jk\omega_o t}\, dt\,; \qquad\qquad k = -\infty \ldots \infty$$

Trigonometric Form

$$f(t) = f(t+T) = c_o + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t)$$

$$c_0 = \frac{1}{T}\int_T f(t)\, dt$$

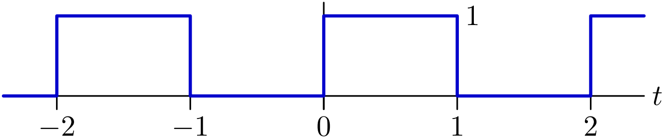
$$c_k = \frac{2}{T}\int_T f(t)\cos(k\omega_o t)\, dt\,; \qquad\qquad k = 1, 2, 3, \ldots \infty$$

$$d_k = \frac{2}{T}\int_T f(t)\sin(k\omega_o t)\, dt\,; \qquad\qquad k = 1, 2, 3, \ldots \infty$$

Fourier Analysis of a Square Wave using Trig Functions

We previously used trig functions to find the Fourier series for $f(t)$ below:

$$f(t) = f(t+2)$$



$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

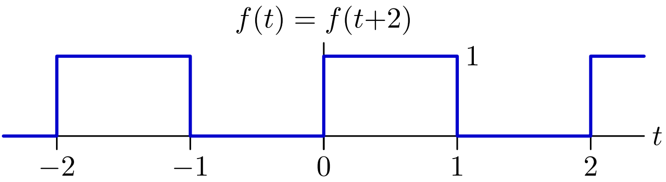
$$c_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega_o t)\,dt = \int_0^1 \cos(k\pi t)dt = \left.\frac{\sin(k\pi t)}{k\pi}\right|_0^1 = 0 \text{ for } k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega_o t)\,dt = \int_0^1 \sin(k\pi t)dt = -\left.\frac{\cos(k\pi t)}{k\pi}\right|_0^1 = \begin{cases} \frac{2}{k\pi} & k = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi}\sin(k\pi t)$$

Fourier Analysis of a Square Wave using Complex Exponentials

Now try complex exponentials.



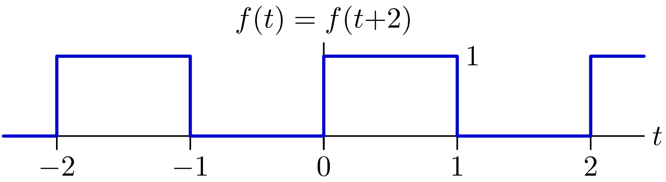
$$a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_0 t}dt = \frac{1}{2}\int_0^1 e^{-jk\pi t}dt = \frac{1}{2}\left[\frac{e^{-jk\pi t}}{-jk\pi}\right]_0^1 = \begin{cases} \frac{1}{jk\pi} & \text{if } k \text{ is odd} \\ 0/0 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a_0 = \frac{1}{T}\int_T f(t)dt = \frac{1}{2}$$

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \frac{1}{2} + \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{1}{jk\pi} e^{jk\pi t}$$

Fourier Analysis of a Square Wave using Complex Exponentials

Now try complex exponentials.



$$a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_o t}dt = \frac{1}{2}\int_0^1 e^{-jk\pi t}dt = \frac{1}{2}\left[\frac{e^{-jk\pi t}}{-jk\pi}\right]_0^1 = \begin{cases} \frac{1}{jk\pi} & \text{if } k \text{ is odd} \\ 0/0 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a_0 = \frac{1}{T}\int_T f(t)dt = \frac{1}{2}$$

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \frac{1}{2} + \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{1}{jk\pi}e^{jk\pi t} = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi}\sin(k\pi t)$$

Same answer we obtained with trig functions.

Check Yourself

Is the complex exponential form actually easier to work with than trig?

Assume that $f(t)$ is periodic in time with period T:

$$f(t) = f(t+T)\,.$$

Let $g(t)$ represent a version of $f(t)$ shifted by half a period:

$$g(t) = f(t-T/2)\,.$$

How many of the following statements correctly describe the effect of this shift on the Fourier series coefficients.

- cosine coefficients c_k are negated
- sine coefficients d_k are negated
- odd-numbered coefficients $c_1, d_1, c_3, d_3, \ldots$ are negated
- sine and cosine coefficients are swapped: $c_k \to d_k$ and $d_k \to c_k$

What is the Effect of Shifting Time?

Let c_k and c'_k represent the cosine coefficients of $f(t)$ and $g(t)$ respectively.

$$
\begin{aligned}
c_k &= \frac{2}{T}\int_0^T f(t)\cos(k\omega_o t)\,dt \\
c'_k &= \frac{2}{T}\int_0^T g(t)\cos(k\omega_o t)\,dt \\
&= \frac{2}{T}\int_0^T f(t-T/2)\cos(k\omega_o t)\,dt && \mid\ g(t) = f(t-T/2) \\
&= \frac{2}{T}\int_0^T f(s)\cos(k\omega_o(s+T/2))\,ds && \mid\ s = t-T/2 \\
&= \frac{2}{T}\int_0^T f(s)\cos(k\omega_o s+k\omega_o T/2)\,ds && \mid\ \text{distribute } k\omega_o \text{ over sum} \\
&= \frac{2}{T}\int_0^T f(s)\cos(k\omega_o s+k\pi)\,ds && \mid\ \omega_o = 2\pi/T \\
&= \frac{2}{T}\int_0^T f(s)\cos(k\omega_o s)(-1)^k\,ds && \mid\ \cos(a+b) = \cos a\cos b - \sin a \sin b \\
&= (-1)^k c_k && \mid\ \text{pull } (-1)^k \text{ outside integral}
\end{aligned}
$$

What is the Effect of Shifting Time?

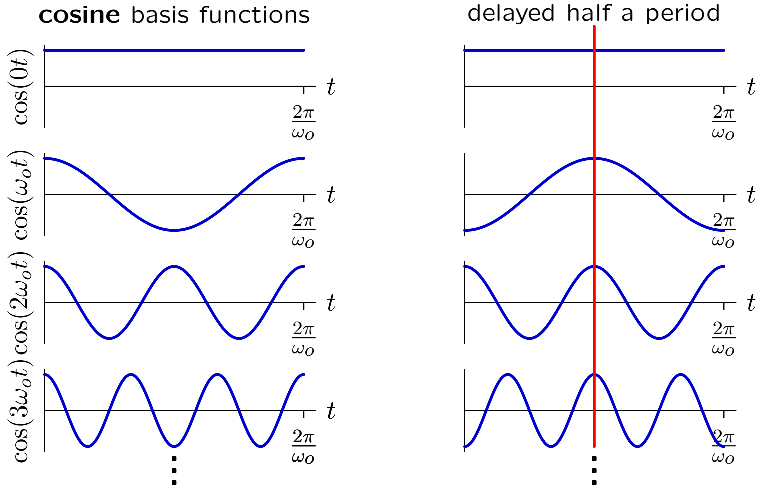
Let d_k and d'_k represent the sine coefficients of $f(t)$ and $g(t)$ respectively.

$$
\begin{aligned}
d_k &= \frac{2}{T}\int_0^T f(t)\sin(k\omega_o t)\,dt \\
d'_k &= \frac{2}{T}\int_0^T g(t)\sin(k\omega_o t)\,dt \\
&= \frac{2}{T}\int_0^T f(t-T/2)\sin(k\omega_o t)\,dt && \mid\ g(t) = f(t-T/2) \\
&= \frac{2}{T}\int_0^T f(s)\sin(k\omega_o (s+T/2))\,ds && \mid\ s = t-T/2 \\
&= \frac{2}{T}\int_0^T f(s)\sin(k\omega_o s+k\omega_o T/2)\,ds && \mid\ \text{distribute } k\omega_o \text{ over sum} \\
&= \frac{2}{T}\int_0^T f(s)\sin(k\omega_o s+k\pi)\,ds && \mid\ \omega_o = 2\pi/T \\
&= \frac{2}{T}\int_0^T f(s)\sin(k\omega_o s)(-1)^k\,ds && \mid\ \sin(a+b) = \sin a\cos b + \cos a \sin b \\
&= (-1)^k d_k && \mid\ \text{pull } (-1)^k \text{ outside integral}
\end{aligned}
$$

Check Yourself: Alternative (more intuitive) Approach

Shifting $f(t)$ shifts the underlying basis functions of it Fourier expansion.

$$f(t-T/2) = \sum_{k=0}^{\infty} c_k \cos\left(k\omega_o(t-T/2)\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\omega_o(t-T/2)\right)$$



Half-period shift inverts odd harmonics. No effect on even harmonics.

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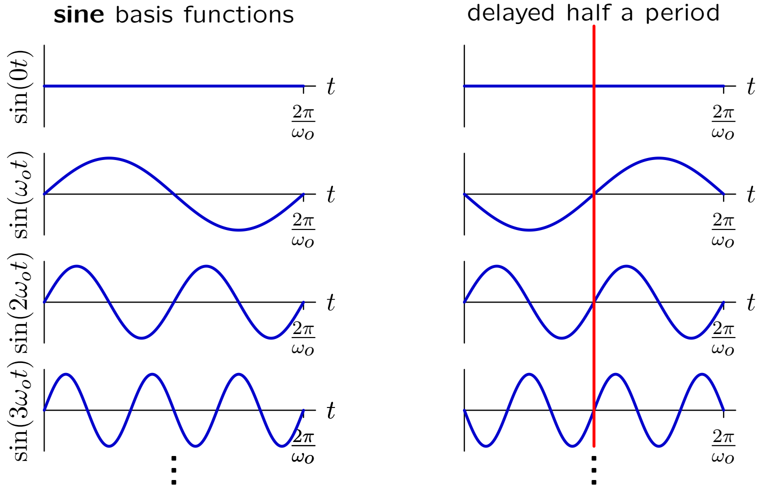

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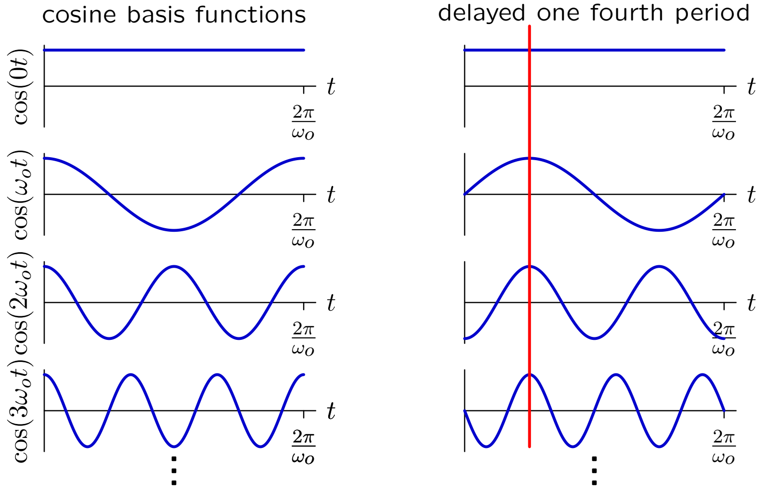
How many of the following statements correctly describe the effect of this shift on the Fourier series coefficients.

- cosine coefficients c_k are negated ✗
- sine coefficients d_k are negated ✗
- odd-numbered coefficients $c_1, d_1, c_3, d_3, \ldots$ are negated $\surd$
- sine and cosine coefficients are swapped: $c_k \to d_k$ and $d_k \to c_k$ ✗

Quarter-Period Shift

Shifting by $T/4$ is **more complicated**.

$$f(t-T/4) = \sum_{k=0}^{\infty} c_k \cos\left(k\omega_o(t-T/4)\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\omega_o(t-T/4)\right)$$

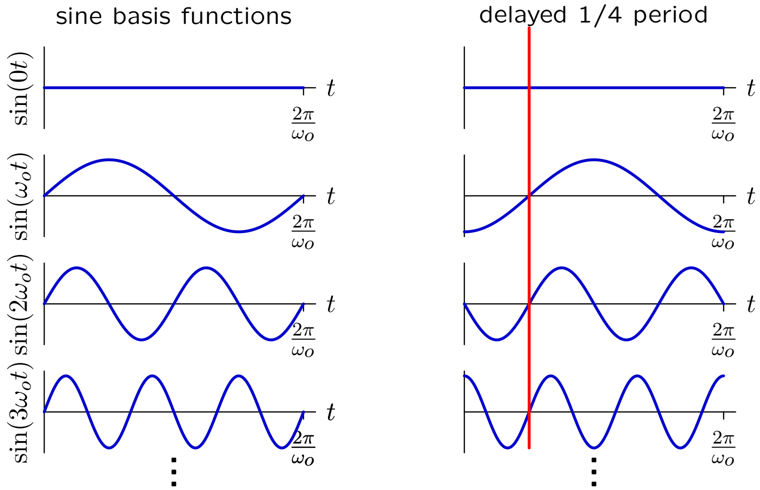


$\cos(\omega_o t) \rightarrow \sin(\omega_o t)$; $\quad \cos(2\omega_o t) \rightarrow -\cos(2\omega_o t)$; $\quad \cos(3\omega_o t) \rightarrow -\sin(3\omega_o t)$

Quarter-Period Shift

Shifting by $T/4$ is **even more complicated**.

$$f(t-T/4)=\sum_{k=0}^{\infty}c_k\cos\left(k\omega_o(t-T/4)\right)+\sum_{k=1}^{\infty}d_k\sin\left(k\omega_o(t-T/4)\right)$$



$\sin(\omega_o t)\rightarrow -\cos(\omega_o t)$; $\sin(2\omega_o t)\rightarrow -\sin(2\omega_o t)$; $\sin(3\omega_o t)\rightarrow \cos(3\omega_o t)$

Comparison of Half- and Quarter-Period Shifts

Let c_k and d_k represent the Fourier series coefficients for $f(t)$

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t)$$

and c'_k and d'_k represent those for a **half-period delay**.

$$g(t) = f(t-T/2) = c_0 + \sum_{k=1}^{\infty} c'_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d'_k \sin(k\omega_o t)$$

Then $c'_k = (-1)^k c_k$ and $d'_k = (-1)^k d_k$.

Comparison of Half- and Quarter-Period Shifts

Let c_k and d_k represent the Fourier series coefficients for $f(t)$

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t)$$

and c'_k and d'_k represent those for a **half-period delay**.

$$g(t) = f(t - T/2) = c_0 + \sum_{k=1}^{\infty} c'_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d'_k \sin(k\omega_o t)$$

Then $c'_k = (-1)^k c_k$ and $d'_k = (-1)^k d_k$.

Let c''_k and d''_k represent those for a **quarter-period delay**.

$$g(t) = f(t - T/4) = c_0 + \sum_{k=1}^{\infty} c''_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d''_k \sin(k\omega_o t)$$

Then

$$c''_k = \begin{cases} c_k & \text{if } k = 0, 4, 8, 12, \ldots \\ d_k & \text{if } k = 1, 5, 9, 13, \ldots \\ -c_k & \text{if } k = 2, 6, 10, 14, \ldots \\ -d_k & \text{if } k = 3, 7, 11, 15, \ldots \end{cases} \qquad d''_k = \begin{cases} d_k & \text{if } k = 0, 4, 8, 12, \ldots \\ -c_k & \text{if } k = 1, 5, 9, 13, \ldots \\ -d_k & \text{if } k = 2, 6, 10, 14, \ldots \\ c_k & \text{if } k = 3, 7, 11, 15, \ldots \end{cases}$$

Other Shifts Yield Even More Complicated Results

Let c_k and d_k represent the Fourier series coefficients for $f(t)$

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t)$$

and c_k''' and d_k''' represent those for an **eighth-period delay**.

$$g(t) = f(t - T/8) = c_0 + \sum_{k=1}^{\infty} c_k''' \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k''' \sin(k\omega_o t)$$

$$c_k''' = \begin{cases} c_k & \text{if } k = 0, 8, 16, 24, \ldots \\ \frac{\sqrt{2}}{2}(c_k + d_k) & \text{if } k = 1, 9, 17, 25, \ldots \\ d_k & \text{if } k = 2, 10, 18, 26, \ldots \\ \frac{\sqrt{2}}{2}(-c_k + d_k) & \text{if } k = 3, 11, 19, 27, \ldots \\ -c_k & \text{if } k = 4, 12, 20, 28, \ldots \\ \frac{\sqrt{2}}{2}(-c_k - d_k) & \text{if } k = 5, 13, 21, 29, \ldots \\ -d_k & \text{if } k = 6, 14, 22, 30, \ldots \\ \frac{\sqrt{2}}{2}(c_k - d_k) & \text{if } k = 7, 15, 23, 31, \ldots \end{cases} \qquad d_k''' = \ldots$$

Effects of Time Shifts on Complex Exponential Series

Delaying time by τ multiplies the complex exponential coefficients of a Fourier series by a constant $e^{-jk\omega_o\tau}$.

Let a_k represent the complex exponential series coefficients of $f(t)$ and a_k' represent the complex exponential series coefficents of $g(t) = f(t-\tau)$.

$$
\begin{aligned}
a_k' &= \frac{1}{T}\int_T g(t)e^{-jk\omega_o t}dt \\
&= \frac{1}{T}\int_T f(t-\tau)e^{-jk\omega_o t}dt \\
&= \frac{1}{T}\int_T f(s)e^{-jk\omega_o (s+\tau)}ds \\
&= e^{-jk\omega_o\tau}\frac{1}{T}\int_T f(s)e^{-jk\omega_o s}ds \\
&= e^{-jk\omega_o\tau}a_k
\end{aligned}
$$

Each coefficient a_k' in the series for $g(t)$ is a constant $e^{-jk\omega_o\tau}$ times the corresponding coefficient a_k in the series for $f(t)$.

Summary

We introduced the complex exponential form of Fourier series.

- complex numbers
- complex exponentials and their relation to sinusoids
- analysis and synthesis with complex exponentials
- delay property: much simpler with complex exponentials

Question of the Day

Determine a complex number c whose square is $j = \sqrt{-1}$.

Is your answer unique?

Trig Table

```
sin(a+b) = sin(a) cos(b) + cos(a) sin(b)
sin(a-b) = sin(a) cos(b) - cos(a) sin(b)
cos(a+b) = cos(a) cos(b) - sin(a) sin(b)
cos(a-b) = cos(a) cos(b) + sin(a) sin(b)
tan(a+b) = (tan(a)+tan(b))/(1-tan(a) tan(b))
tan(a-b) = (tan(a)-tan(b))/(1+tan(a) tan(b))

sin(A) + sin(B) =  2 sin((A+B)/2) cos((A-B)/2)
sin(A) - sin(B) =  2 cos((A+B)/2) sin((A-B)/2)
cos(A) + cos(B) =  2 cos((A+B)/2) cos((A-B)/2)
cos(A) - cos(B) = -2 sin((A+B)/2) sin((A-B)/2)

sin(a+b) + sin(a-b) =  2 sin(a) cos(b)
sin(a+b) - sin(a-b) =  2 cos(a) sin(b)
cos(a+b) + cos(a-b) =  2 cos(a) cos(b)
cos(a+b) - cos(a-b) = -2 sin(a) sin(b)

2 cos(A) cos(B) = cos(A-B) + cos(A+B)
2 sin(A) sin(B) = cos(A-B) - cos(A+B)
2 sin(A) cos(B) = sin(A+B) + sin(A-B)
2 cos(A) sin(B) = sin(A+B) - sin(A-B)
```