# 6.3000: Signal Processing

## Discrete Fourier Transform 1

- Discrete Fourier Transform (DFT)
- Relations to Discrete-Time Fourier Series (DTFS)
- Relations to Discrete-Time Fourier Transform (DTFT)

*March 31, 2026*

# Yet Another Fourier Representation

Why do we need another Fourier Representation?

**Fourier series** represent signals as sums of sinusoids. They provide insights that are not obvious from time representations, but Fourier series are only defined for periodic signals.

$$X[k] = \sum_{n=\langle N \rangle} x[n] e^{-j2\pi kn/N} \qquad \text{(summed over a period)}$$

**Fourier transforms** have no periodicity constaint:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \qquad \text{(summed over all samples } n\text{)}$$

but are functions of continuous domain ($\Omega$).
$\rightarrow$ not convenient for numerical computations

**Discrete Fourier Transform:** discrete frequencies for aperiodic signals.

# Discrete Fourier Transform

Definition and comparison to other Fourier representations.

| | **analysis** | **synthesis** |
|---|---|---|
| **DTFS:** | $X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\frac{2\pi k}{N}n}$ | $x[n] = \sum_{k=\langle N \rangle} X[k] e^{j\frac{2\pi k}{N}n}$ |
| **DTFT:** | $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$ | $x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$ |
| **DFT:** | $X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$ | $x[n] = \sum_{k=\langle N \rangle} X[k] e^{j\frac{2\pi k}{N}n}$ |

**Major differences:**

DTFS: $x[n]$ is presumed to be periodic in $N$

DTFT: $x[n]$ is arbitrary

DFT: only a portion of an arbitrary $x[n]$ is considered

# Relation Between DFT and DTFS

If we compute the DFT with a window length $N$ that is equal to the period of a periodic signal, then the DFT and DTFS coefficients are equal.

Let $x_1[n] = \cos\frac{2\pi n}{64}$. Then the DFT coefficients computed with $N$=64 are

$$X_1[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} = \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k-63]$$

as plotted below.


These DFT coefficients are the same as the Fourier series coefficients.

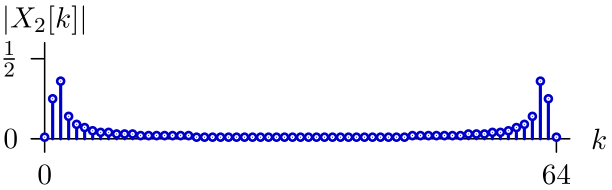
# Relation Between DFT and DTFS

If a signal is not periodic in the DFT window length $N$, then there are no Fourier series coefficients to compare.

Let $x_2[n] = \cos\frac{3\pi n}{64}$. Then if $N{=}64$, the DFT coefficients are

$$X_2[k] = \frac{1}{N}\sum_{n=0}^{N-1} x_2[n]e^{-j\frac{2\pi}{N}kn}$$

are plotted below.

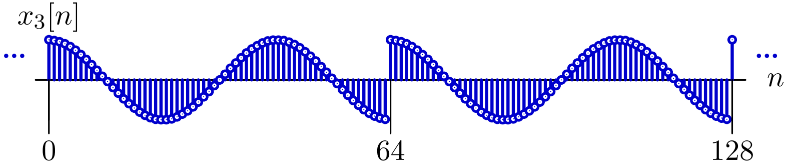



Even though $x_2[n]$ contains a single frequency $\Omega = 3\pi/64$, there are large coefficients at many different frequencies $k$.

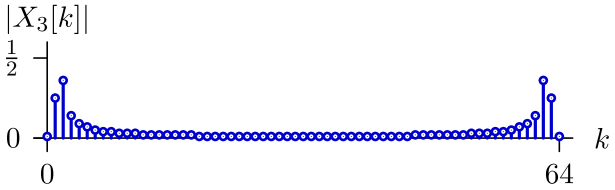
The reason is that $x_2[n]$ is not periodic in $n$ with period $N = 64$.

# Relation Between DFT and DTFS

Although $x_2[n] = \cos\frac{3\pi n}{64}$ is not periodic in $N{=}64$, we can define a signal $x_3[n]$ that is equal to $x_2[n]$ for $0 \le n < 64$ and that is periodic in $N{=}64$.




The DFT coefficients for this signal are the same as those for $x_2[n]$:




Furthermore, the DFT coefficients of $x_3[n]$ equal the DTFS coefficients of $x_3[n]$. The large number of non-zero coefficients are necessary to produce the step discontinuity at $n = 64$.

## Two Ways to Think About the DFT

We just compared the DFT to the DTFS:

1. The DFT of a signal $x[n]$ is equal to the **DTFS** of a version of $x[n]$ that is periodically extended so that it is periodic in $N$.

$\rightarrow$ emphasizes the importance of **periodicity** in time.

2. We can also gain intuition for the DFT by comparing it to the DTFT.

## Relation Between DFT and DTFT

The DFT can also be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $\frac{1}{N}$.

Let $x_w[n] = x[n] \times w[n]$ represent a **windowed** version of $x[n]$ where

$$w[n] = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Then the Fourier transform of $x_w[n]$ is

$$X_w(\Omega) = \sum_{n=-\infty}^{\infty} x_w[n] e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n]w[n] e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\Omega n}$$

**Sample** the resulting function of $\Omega$ at $\Omega = \frac{2\pi k}{N}$:

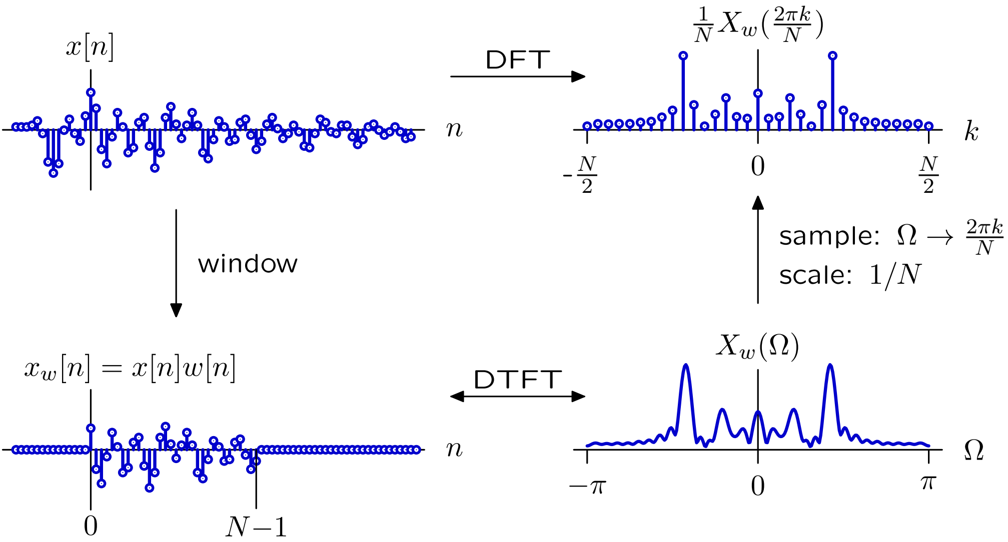
$$X_w\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2\pi k}{N}\right)n}$$

**Divide** both sides by $N$:

$$\frac{1}{N} X_w\left(\frac{2\pi k}{N}\right) = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2\pi k}{n}\right)n} = X[k], \quad \text{which is the DFT of } x[n]$$

# Relation Between DFT and DTFT

Graphical depiction of relation between DFT and DTFT.




While sampling and scaling are important, it is the **windowing** that most affects frequency content.

# Effect of Windowing on Fourier Representations

Determine effects of windowing on signals with a single frequency $\Omega_o$.

Step 1: Find $X(\Omega)$, the DTFT of a complex exponential signal:
$x[n] = e^{j\Omega_0 n}$

Step 2: Find $X_w(\Omega)$, the DTFT of a windowed version of $x[n]$:
$x_w[n] = x[n]w[n]$

Step 3: Compare $X_w(\Omega)$ to $X(\Omega)$.

# Effect of Windowing on Fourier Representations

Step 1: Find $X(\Omega)$, the DTFT of a complex exponential signal:

$$x[n] = e^{j\Omega_0 n}$$

# Effect of Windowing on Fourier Representations

Step 2: Find $X_w(\Omega)$, the DTFT of a windowed version of $x[n]$:

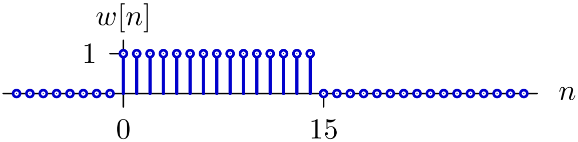
$$x_w[n] = x[n]w[n] = e^{j\Omega_o n}w[n]$$

# Effect of Windowing on Fourier Representations

Simplest window is rectangular, with width of $N$ (length of DFT analysis)

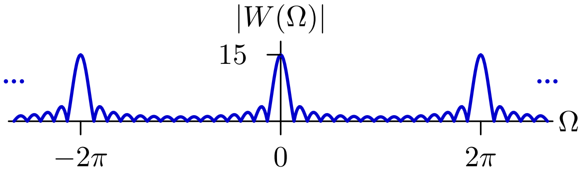
$$w[n] = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

as shown below for $N = 15$.




The DTFT of $w[n]$ is

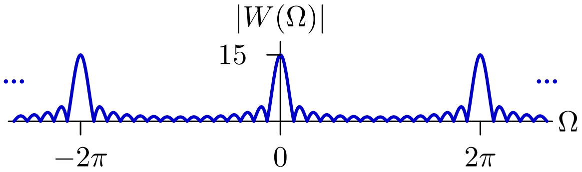
$$W(\Omega) = \sum_{n=-\infty}^{\infty} w[n] e^{-j\Omega n} = \sum_{n=0}^{N-1} e^{-j\Omega n} = \frac{1 - e^{-j\Omega N}}{1 - e^{-j\Omega}} = \frac{\sin \frac{N\Omega}{2}}{\sin \frac{\Omega}{2}} e^{-j\Omega \frac{N-1}{2}}$$

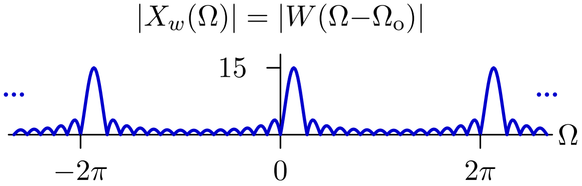


# Effect of Windowing on Fourier Representations

The DTFT of a windowed version of a complex exponential signal is a shifted version of the DTFT of the window signal.

$$x_w[n] = e^{j\Omega_o n} w[n] \quad \overset{\text{DTFT}}{\Longrightarrow} \quad X_w(\Omega) = W(\Omega - \Omega_o)$$

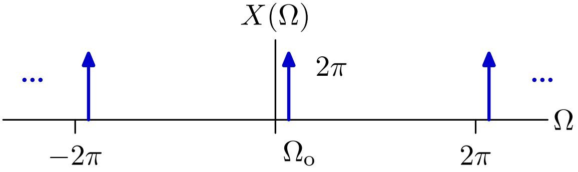


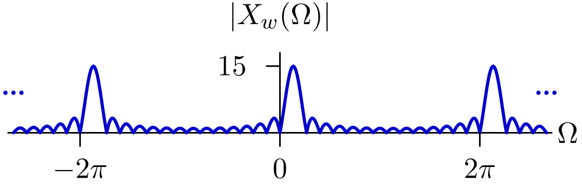
# Effect of Windowing on Fourier Representations

Step 3: Compare $X_w(\Omega)$ to $X(\Omega)$.

$$x[n] = e^{j\Omega_o n} \quad \stackrel{\text{DTFT}}{\Longrightarrow}$$




$$x_w[n] = x[n]w[n] \quad \stackrel{\text{DTFT}}{\Longrightarrow}$$

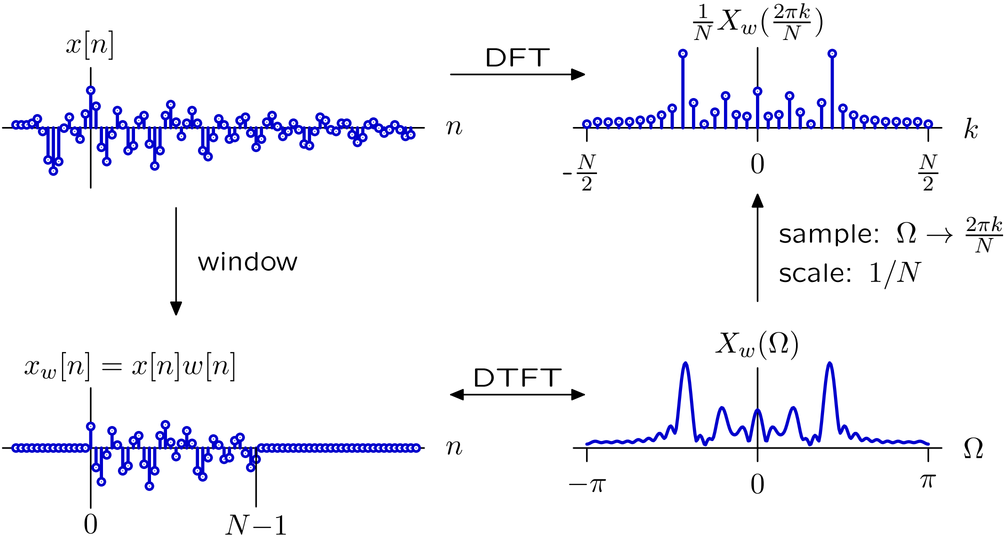



The frequency content of $X(\Omega)$ is at discrete frequencies $\Omega = \Omega_o + 2\pi m$.

The frequency content of $X_w(\Omega)$ is most dense at these same frequencies, but is spread out over almost all other frequencies as well.

# Relation Between DFT and DTFT

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$.



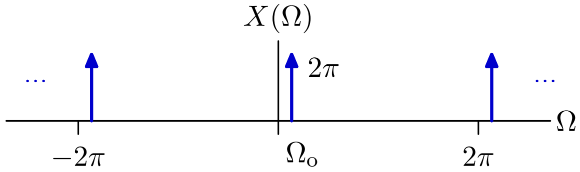

Next: apply these steps to a sinusoidal input.

# Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$. Here $\Omega_o = \frac{2\pi}{15}$.

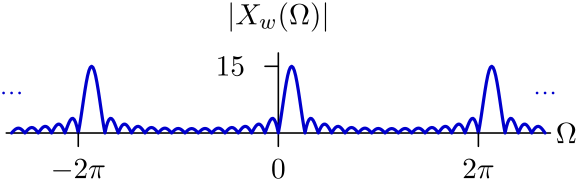
**original signal**

$$x[n] = e^{j\Omega_o n} \overset{\text{DTFT}}{\Longrightarrow}$$



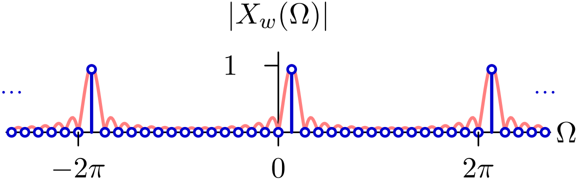

**windowed**

$$x_w[n] = x[n]w[n] \overset{\text{DTFT}}{\Longrightarrow}$$




**sampled and scaled**

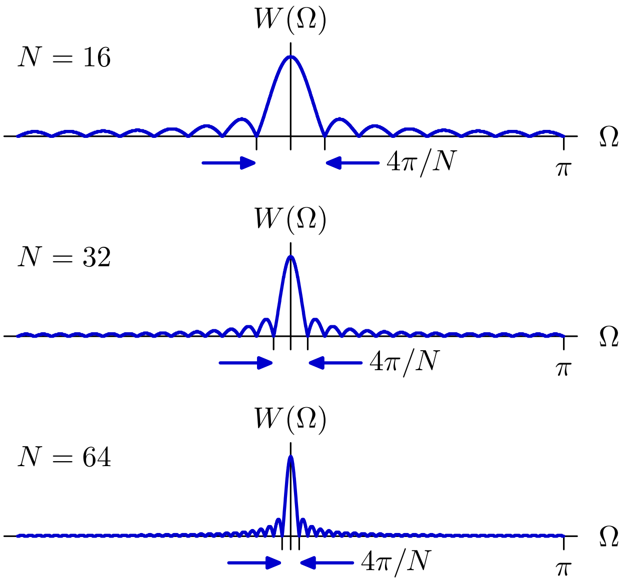
$$x_w[n] = x[n]w[n] \overset{\text{DFT}}{\Longrightarrow}$$




One sample is taken at the peak, and the others fall on zeros.

# Spectral Blurring introduces a Time/Frequency Tradeoff

Longer windows provide finer frequency resolution.

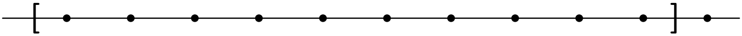



The width of the central lobe is inversely related to window length.

# Frequency Resolution

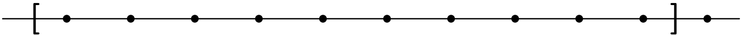
The DFT analysis period $N$ determines both the window in time that is analyzed and the frequency resolution of the result.

The time window is divided into $N$ samples numbered $n = 0$ to $N{-}1$.




| | | | | |
|---|---|---|---|---|
| $0$ | $1$ | $N{-}1$ | $N$ | $n$ [samples] |
| $0$ | $\frac{1}{fs}$ | $\frac{N-1}{fs}$ | $\frac{N}{fs}$ | $t$ [seconds] |

Discrete frequencies are similarly numbered as $k = 0$ to $N{-}1$.




| | | | | |
|---|---|---|---|---|
| $0$ | $1$ | $N{-}1$ | $N$ | $k$ |
| $0$ | $\frac{2\pi}{N}$ | | $2\pi$ | $\Omega$ [rad/sample] |
| $0$ | $\frac{f_s}{N}$ | | $f_s$ | $f$ [Hz] |

As the analysis length $N$ increases, both temporal duration and spectral resolution increase.

## Two Ways to Think About the DFT

Compare to DTFS:

1. The DFT of a signal $x[n]$ is equal to the **DTFS** of a version of $x[n]$ that is periodically extended so that it is periodic in $N$.
$\rightarrow$ emphasizes the importance of **periodicity** in time.

Compare to DTFT:

2. The DFT is equal to samples of the **DTFT** of a windowed version of the original signal.
$\rightarrow$ emphasizes the importance of **spectral smear** in frequency.

The DTFS and DTFT offer different and complementary

- **rules** for constructing all of the components of the DFT, and
- **intuition** for understanding the origin of "extra" components of DFT.

## Summary

Today we introduced a new Fourier representation for DT signals: the Discrete Fourier Transform (DFT).

The DFT has a number of features that make it particularly convenient.

- It is not limited to periodic signals.
- It has discrete domain ($k$ instead of $\Omega$) and finite length: convenient for numerical computation.

The finite analysis window of the DFT can smear the resulting spectral representation.

- The DFT is equivalent to the DTFS of a periodically extended version of the input signal. The smear results because of discontinuities introduced by periodic extension.
- The DFT is equivalent to the DTFT of a windowed version of the input signal that is then sampled and scaled in amplitude. The windowing smears the spectral representation because of discontinuities introduced by the windowing.