6.3000: Signal Processing

DT Fourier Transform

- Definition
- Examples
- Properties
- Relations between Fourier series and transforms (DT and CT)

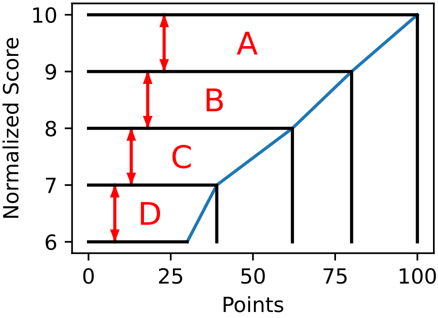
March 05, 2026

Points, Normalized Scores, and Letter Grade

Two-part grading procedure:

- We grade the exams on a **point** basis (out of 100 points for Quiz 1).
- We convert the point score to a 10-point **normalized** score using MIT's definitions of letter grades.

total points	10-point score	letter grade
100%	10	
		A
A/B boundary	9	
		B
B/C boundary	8	
		C
C/D boundary	7	
		D
D/F boundary	6	
		F
0%	0	



- Your final grade in 6.3000 will be a **weighted sum** of your 10-point **normalized scores** for participation, homeworks, quizzes, bfinal exam.

MIT's Definitions of Grades

A: Exceptionally good performance demonstrating a superior understanding of the subject matter, a foundation of extensive knowledge, and a skillful use of concepts and/or materials.

B: Good performance demonstrating capacity to use the appropriate concepts, a good understanding of the subject matter, and an ability to handle the problems and materials encountered in the subject.

C: Adequate performance demonstrating an adequate understanding of the subject matter, an ability to handle relatively simple problems, and adequate preparation for moving on to more advanced work in the field.

D: Minimally acceptable performance demonstrating at least partial familiarity with the subject matter and some capacity to deal with relatively simple problems, but also demonstrating deficiencies serious enough to make it inadvisable to proceed further in the field without additional work.

From Periodic to Aperiodic

This course focuses primarily on **frequency representations** of signals and how they can be useful in signal processing.

We started with **periodic signals** because they are relatively easy:

- **countable** number of basis functions and
- simple **sifting property** for finding weights of each component.

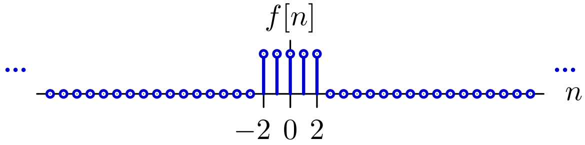
However, most real-world signals are not periodic.

Last lecture: introduced the **Fourier transform** representation for CT signals that are not necessarily periodic.

Today: generalize the Fourier transform idea to DT signals.

Fourier Representions of Aperiodic Signals

How can we represent an aperiodic signal as a sum of sinusoids?



Fourier Representions of Aperiodic Signals

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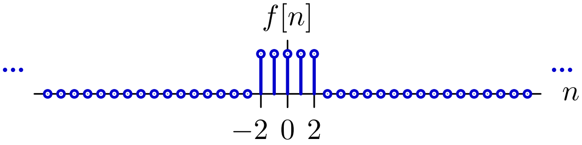
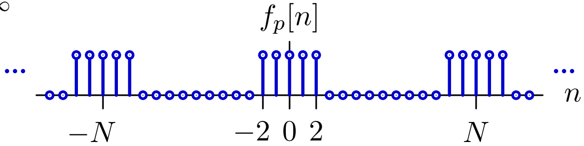


Strategy: make a periodic version of $f[n]$ by summing shifted copies:

$$f_p[n] = \sum_{m=-\infty}^{\infty} f[n - mN]$$



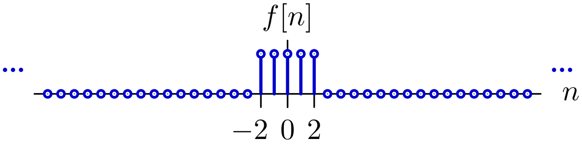
Since $f_p[n]$ is periodic, it has a Fourier series (which depends on N).

Find Fourier series coefficients $F_p[k]$ and take the limit of $F_p[k]$ as $N \to \infty$.

As $N \to \infty$, $f_p[n] \to f[n]$, and the Fourier series $\to$ Fourier transform.

Fourier Representions of Aperiodic Signals

Example.



Strategy: make a periodic version of $f[n]$ by summing shifted copies:

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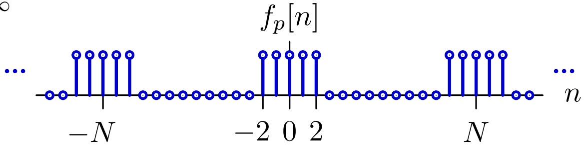


Calculate the Fourier series coefficients $F_p[k]$:

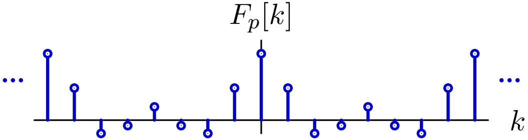
$$F_p[k] = \frac{1}{N} \sum_{n=<N>} f_p[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} + \frac{2}{N}\cos\frac{2\pi k}{N} + \frac{2}{N}\cos\frac{4\pi k}{N}$$

Fourier Representations of Aperiodic Signals

Calculate the Fourier series coefficients $F_p[k]$:

$$F_p[k] = \frac{1}{N} \sum_{n=\langle N \rangle} f_p[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} + \frac{2}{N} \cos\frac{2\pi k}{N} + \frac{2}{N} \cos\frac{4\pi k}{N}$$

Plot the resulting Fourier coefficients for $N{=}8$.



Notice that the Fourier series coefficients $F_p[k]$ are periodic in k: period N.
This periodicity is a fundamental property of the Fourier kernel functions:

$$e^{-j\frac{2\pi}{N}(k+N)n} = e^{-j\frac{2\pi}{N}kn} \times e^{-j\frac{2\pi}{N}Nn} = e^{-j\frac{2\pi}{N}kn} \times e^{-j2\pi n} = e^{-j\frac{2\pi}{N}kn}$$

since n is an integer.

Such a periodicity is not seen in CT
$\rightarrow$ important difference between CT and DT.

Fourier Representions of Aperiodic Signals

Calculate the Fourier series coefficients $F_p[k]$:

$$F_p[k] = \frac{1}{N}\sum_{n=\langle N\rangle} f_p[n]e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} + \frac{2}{N}\cos\frac{2\pi k}{N} + \frac{2}{N}\cos\frac{4\pi k}{N}$$

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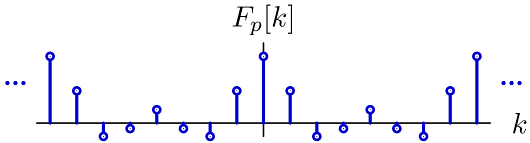


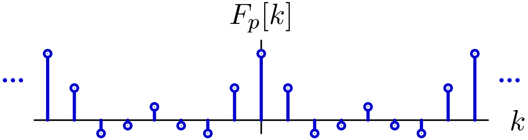
What happens if you double the period N?

Fourier Representions of Aperiodic Signals

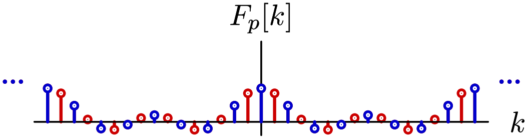
Calculate the Fourier series coefficients $F_p[k]$:

$$F_p[k] = \frac{1}{N} \sum_{n=\langle N \rangle} f_p[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} + \frac{2}{N} \cos \frac{2\pi k}{N} + \frac{2}{N} \cos \frac{4\pi k}{N}$$

Plot the resulting Fourier coefficients for $N{=}8$.



What happens if you double the period N? Make a plot for $N{=}16$.



There are now twice as many samples per period. (The red samples are at new intermediate frequencies.) The amplitude is halved.

Fourier Representions of Aperiodic Signals

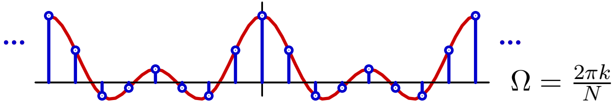
Define a new function $F(\Omega) = NF_p[k]$ where $\Omega = k\Omega_o = 2\pi k/N$.

$$NF_p[k] = 1 + 2\cos\frac{2\pi k}{N} + 2\cos\frac{4\pi k}{N} = 1 + 2\cos(\Omega) + 2\cos(2\Omega) = F(\Omega)\Big|_{\Omega=\frac{2\pi k}{N}}$$

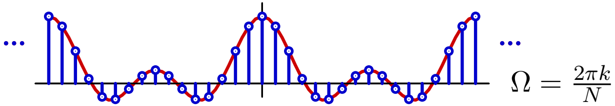
Then $NF_p[k]$ represents samples of $F(\Omega)$ with increasing resolution in Ω.

$$NF_p[k] = F(\Omega)$$

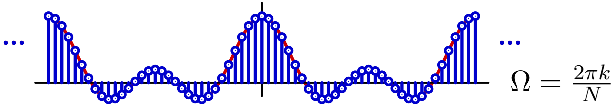
N=8:



N=16:



N=32:



The discrete function $NF_p[k]$ is a sampled version of the function $F(\Omega)$.

Fourier Representations of Aperiodic Signals

From $f[n]$ to $F(\Omega)$:

$$f[n] \underbrace{\rightarrow}_{\text{periodic extension}} f_p[n] \underbrace{\rightarrow}_{\text{Fourier series}} F_p[k] \underbrace{\rightarrow}_{\text{interpolation}} F(\Omega)$$

The limiting behaviors as $N \rightarrow \infty$ define the Fourier transform:

$$\begin{aligned}
F(\Omega) &= \lim_{N\rightarrow\infty} N F_p[k]\Big|_{k=\frac{N}{2\pi}\Omega} \\
&= \lim_{N\rightarrow\infty} N \left[\frac{1}{N}\sum_{n=<N>} f_p[n] e^{-j\frac{2\pi k}{N}n}\right]_{k=\frac{N}{2\pi}\Omega} \\
&= \lim_{N\rightarrow\infty} \sum_{n=<N>} f_p[n] e^{-j\Omega n} \\
F(\Omega) &= \sum_{n=-\infty}^{\infty} f[n] e^{-j\Omega n}
\end{aligned}$$

This analysis equation **defines** the Fourier transform.

Fourier Representations of Aperiodic Signals

The **synthesis equation** follows from a piecewise constant approximation.

$$f[n] = \lim_{N\to\infty} f_p[n] = \lim_{N\to\infty} \sum_{k=\langle N\rangle} F_p[k]\, e^{j\frac{2\pi}{N}kn}$$

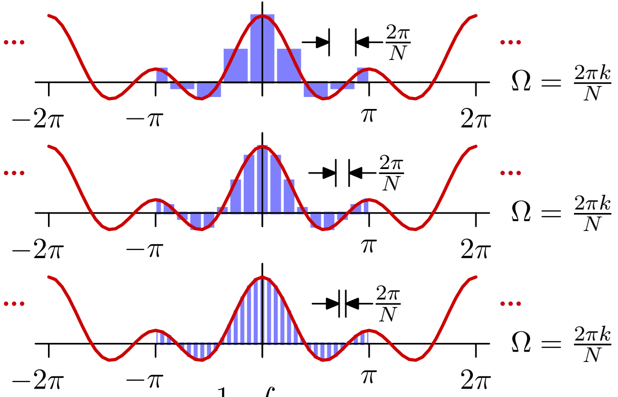
$$= \lim_{N\to\infty} \left(\frac{1}{2\pi}\right) \sum_{k=\langle N\rangle} N F_p[k]\, e^{j\frac{2\pi}{N}kn} \left(\frac{2\pi}{N}\right) = \frac{1}{2\pi}\int_{2\pi} F(\Omega)\, e^{j\Omega n}\, d\Omega$$

$$N F_p[k] = F(\Omega)$$

$N{=}8$:

$N{=}16$:

$N{=}32$:



Synthesis Equation: $f[n] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)\, e^{j\Omega n}\, d\Omega$

Fourier Series and Fourier Transform

Fourier series and transforms are similar:
both represent signals by their frequency content.

Discrete-Time Fourier Series

$$F[k] = F[k+N] = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-jk\Omega_o n}$$ **analysis equation**

$$f[n] = f[n+N] = \sum_{k=\langle N\rangle} F[k]e^{jk\Omega_o n}$$ **synthesis equation**

where $\Omega_o = \frac{2\pi}{N}$

Discrete-Time Fourier Transform

$$F(\Omega) = F(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n}$$ **analysis equation**

$$f[n] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega n}\,d\Omega$$ **synthesis equation**

Fourier Series and Fourier Transform

All of the information in a periodic signal is contained in one period.
The information in an aperiodic signal can spread across all time.

Discrete-Time Fourier Series

$$F[k] = F[k+N] = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\Omega_o n}$$ **analysis equation**

$$f[n] = f[n+N] = \sum_{k=\langle N \rangle} F[k] e^{jk\Omega_o n}$$ **synthesis equation**

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$$f[n] = \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{j\Omega n} \, d\Omega$$ **synthesis equation**

Fourier Series and Fourier Transform

Harmonic frequencies $k\Omega_o$ are samples of continuous frequency Ω.

Discrete-Time Fourier Series

$$F[k] = F[k+N] = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-jk\Omega_o n}$$

analysis equation

$$f[n] = f[n+N] = \sum_{k=\langle N\rangle} F[k]e^{jk\Omega_o n}$$

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where $\Omega_o = \frac{2\pi}{N}$

Discrete-Time Fourier Transform

$$F(\Omega) = F(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n}$$

analysis equation

$$f[n] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega n}\,d\Omega$$

synthesis equation

Fourier Series and Fourier Transform

Periodic signals can be synthesized from a discrete set of k harmonics. Aperiodic signals generally require a continuous set of frequencies Ω.

Discrete-Time Fourier Series

$$F[k] = F[k+N] = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\Omega_o n}$$

analysis equation

$$f[n] = f[n+N] = \sum_{k=\langle N \rangle} F[k] e^{jk\Omega_o n}$$

synthesis equation

where $\Omega_o = \frac{2\pi}{N}$

Discrete-Time Fourier Transform

$$F(\Omega) = F(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} f[n] e^{-j\Omega n}$$

analysis equation

$$f[n] = \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{j\Omega n} \, d\Omega$$

synthesis equation

CT and DT Fourier Transforms

DT frequencies alias because adding 2π to Ω does not change $e^{-j\Omega n}$.

Continuous-Time Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$ **analysis equation**

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$ **synthesis equation**

Discrete-Time Fourier Transform

$$F(\Omega) = F(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} f[n] e^{-j\Omega n}$$ **analysis equation**

$$f[n] = \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{j\Omega n} d\Omega$$ **synthesis equation**

CT and DT Fourier Transforms

DT frequencies alias because adding 2π to Ω does not change $e^{-j\Omega n}$.
Since $F(\Omega)$ is periodic in 2π, we need only integrate $d\Omega$ over a 2π interval.

Continuous-Time Fourier Transform

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$ **analysis equation**

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$ **synthesis equation**

Discrete-Time Fourier Transform

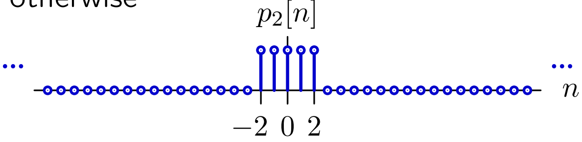
$$F(\Omega)= F(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n}$$ **analysis equation**

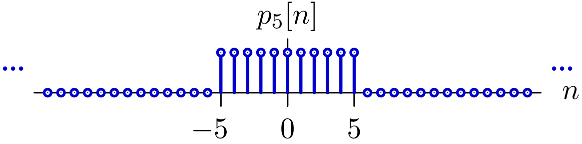
$$f[n] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega n}\,d\Omega$$ **synthesis equation**

Examples of Fourier Transforms

Find the Fourier Transform (FT) of a rectangular pulse of width $2S+1$:

$$p_S[n] = \begin{cases} 1 & -S \le n \le S \\ 0 & \text{otherwise} \end{cases}$$





$$P_S(\Omega) = \sum_{n=-\infty}^{\infty} p_S[n] e^{-j\Omega n} = \sum_{n=-S}^{S} e^{-j\Omega n}$$

$$= e^{j\Omega S} + e^{j\Omega(S-1)} + \cdots + 1 + \cdots + e^{-j\Omega(S-1)} + e^{-j\Omega S}$$

Close the sum to better identify trends across S.

Working with Sums

Closed form sums of geometric sequences.

$$A = \sum_{n=0}^{N-1} \alpha^n$$

If the series has finite length (here N terms), it will converge for finite α.

$$\begin{aligned} A &= 1 + \alpha + \alpha^2 + \cdots + \alpha^{N-1} \\ \alpha A &= \quad\;\; \alpha + \alpha^2 + \cdots + \alpha^{N-1} + \alpha^N \\ A - \alpha A &= 1 \qquad\qquad\qquad\qquad\qquad\; - \alpha^N \\ A &= \begin{cases} \dfrac{1-\alpha^N}{1-\alpha} & \text{if } \alpha \neq 1 \\ N & \text{if } \alpha = 1 \end{cases} \end{aligned}$$

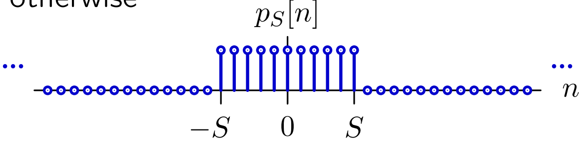
If the series has infinite length, it will converge if $|\alpha| < 1$.

$$\sum_{n=0}^{\infty} \alpha^n = \lim_{N\to\infty} \sum_{n=0}^{N-1} \alpha^n = \lim_{N\to\infty} \frac{1-\alpha^N}{1-\alpha} = \frac{1}{1-\alpha} \quad \text{if } |\alpha| < 1$$

Examples of Fourier Transforms

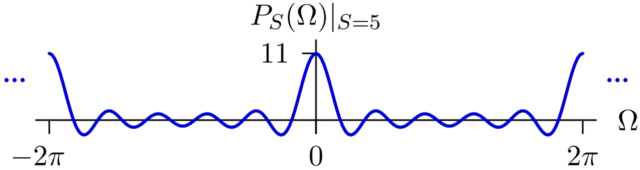
Find the Fourier Transform (FT) of a rectangular pulse of width $2S+1$:

$$p_S[n] = \begin{cases} 1 & -S \le n \le S \\ 0 & \text{otherwise} \end{cases}$$



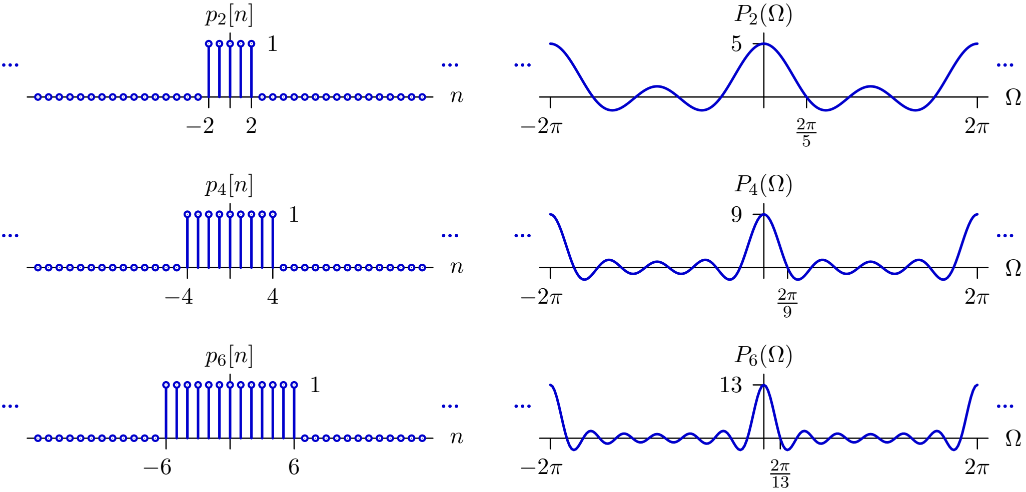
$$P_S(\Omega) = \sum_{n=-S}^{S} e^{-j\Omega n} = e^{j\Omega S} \sum_{m=0}^{2S} e^{-j\Omega m}\Bigg|_{m=n+S} = \left(\frac{e^{j\Omega(S+\frac{1}{2})}}{e^{j\Omega/2}}\right)\left(\frac{1-e^{-j\Omega(2S+1)}}{1-e^{-j\Omega}}\right)$$

$$= \frac{e^{j\Omega(S+\frac{1}{2})} - e^{-j\Omega(S+\frac{1}{2})}}{e^{j\Omega/2} - e^{-j\Omega/2}} = \frac{\sin\left(\Omega(S+\frac{1}{2})\right)}{\sin(\Omega/2)}$$



Examples of Fourier Transforms

Compare Fourier transforms of pulses with different widths.

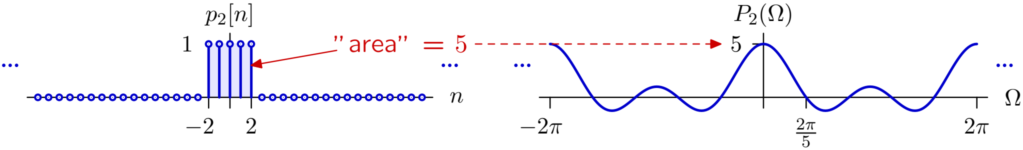


As the function widens in n the Fourier transform narrows in Ω.

Areas

Similar to CT, the value of $F(\Omega)$ at $\Omega = 0$ is the sum of $f[n]$ over time n.

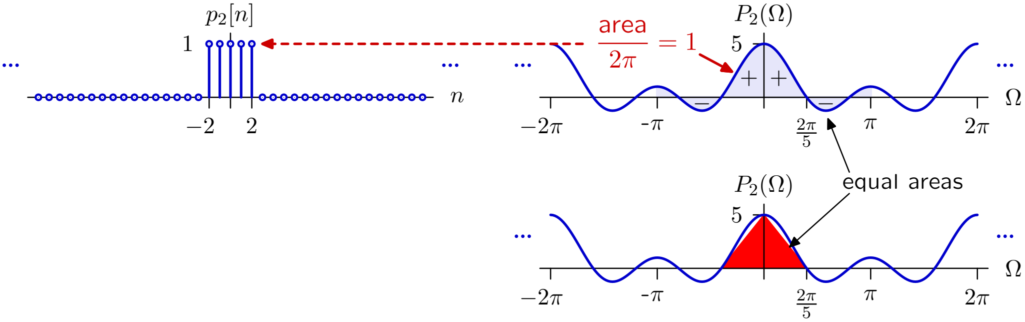
$$F(0) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} f[n]$$



Areas

Similarly, the value of $f[0]$ is $\frac{1}{2\pi}$ times the integral of $F(\Omega)$ over $\Omega = [-\pi, \pi]$.

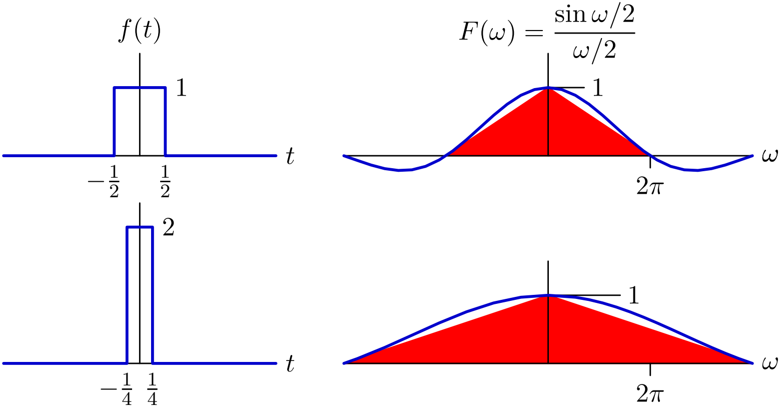
$$f[0] = \frac{1}{2\pi} \int_{2\pi} F(\Omega)\, e^{j\Omega n} d\Omega = \frac{1}{2\pi} \int_{2\pi} F(\Omega)\, d\Omega$$



The inscribed triangle relation that worked in CT works in DT as well.

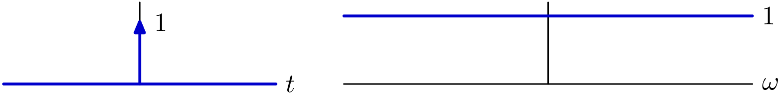
Extreme Cases

The **CT Fourier transform** of the **shortest** possible CT signal $f(t) = \delta(t)$ is the widest possible CT transform $F(\omega) = 1$.



In the limit, the pulse has zero width but area 1!

We represent this limit with the delta (or impulse) function: $\delta(t)$.



Extreme Cases

The **CT Fourier transform** of the **shortest** possible CT signal $f(t) = \delta(t)$ is the widest possible CT transform $F(\omega) = 1$.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t} = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega 0} = 1$$

A similar result holds in DT.

$$F(\Omega) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\Omega 0} = 1$$

Extreme Cases

The Fourier transform of the **widest** possible CT signal $f(t)=1$ is the narrowest possible CT transform $F(\omega)=2\pi\delta(\omega)$.[1]

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}2\pi\delta(\omega)e^{j\omega t}d\omega=\int_{-\infty}^{\infty}\delta(\omega)e^{j0t}d\omega=1$$

A similar result holds in DT.

$$f[n]=\frac{1}{2\pi}\int_{2\pi}F(\Omega)e^{j\Omega n}d\Omega=\frac{1}{2\pi}\int_{2\pi}2\pi\delta(\Omega)e^{j\Omega n}d\Omega=\int_{2\pi}\delta(\Omega)e^{j0n}d\Omega=1$$

1 the factors of 2π are needed to cancel the 2π in the synthesis equations.

More Math With Impulses

A similar construction reveals the transform of a complex exponential.

$$
\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_o)e^{j\omega t}d\omega \\
&= \int_{-\infty}^{\infty} \delta(\omega-\omega_o)e^{j\omega_o t}d\omega \\
&= e^{j\omega_o t}\int_{-\infty}^{\infty} \delta(\omega-\omega_o)d\omega \\
&= e^{j\omega_o t}
\end{aligned}
$$

Thus, the Fourier transform of a complex exponential is a delta function at the frequency of the complex exponential:

$$e^{j\omega_o t} \quad \overset{\text{CTFT}}{\Longrightarrow} \quad 2\pi\delta(\omega-\omega_o)$$

The impulse in frequency has infinite value at $\omega = \omega_o$ and is zero at all other frequencies.

More Math With Impulses

A similar construction applies in DT.

$$
\begin{aligned}
f[n] &= \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\int_{2\pi} 2\pi\delta(\Omega-\Omega_o)e^{j\Omega n}d\Omega \\
&= \int_{2\pi} \delta(\Omega-\Omega_o)e^{j\Omega_o n}d\Omega \\
&= e^{j\Omega_o n}\int_{2\pi} \delta(\Omega-\Omega_o)d\Omega \\
&= e^{j\Omega_o n}
\end{aligned}
$$

Thus, the Fourier transform of a complex exponential is a delta function at the frequency of the complex exponential:

$$e^{j\Omega_o n} \quad \stackrel{\text{DTFT}}{\Longrightarrow} \quad 2\pi\delta(\Omega-\Omega_o)$$

The impulse in frequency shows that the transform is infinite at $\Omega = \Omega_o$ and is zero at all other frequencies.

Relations Between Fourier Series and Fourier Transforms

Continuous Time:

$$e^{j\omega_o t} \quad \stackrel{\text{CTFT}}{\Longrightarrow} \quad 2\pi\delta(\omega-\omega_o)$$

$$f(t) = f(t+T) \quad \stackrel{\text{CTFS}}{\Longrightarrow} \quad F[k]$$

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k]e^{j\omega_o t} \quad \stackrel{\text{CTFT}}{\Longrightarrow} \quad \sum_{k=-\infty}^{\infty} 2\pi F[k]\delta(\omega-\omega_o)$$

Discrete Time:

$$e^{j\Omega_o n} \quad \stackrel{\text{DTFT}}{\Longrightarrow} \quad 2\pi\delta(\Omega-\Omega_o)$$

$$f[n] = f[n+N] \quad \stackrel{\text{DTFS}}{\Longrightarrow} \quad F[k]$$

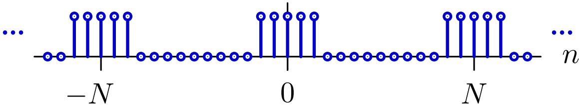
$$f[n] = f[n+N] = \sum_{k=\langle N\rangle} F[k]e^{j\Omega_o n} \quad \stackrel{\text{DTFT}}{\Longrightarrow} \quad \sum_{k=\langle N\rangle} 2\pi F[k]\delta(\Omega-\Omega_o)$$

Periodic DT signals that have Fourier series representations also have Fourier transform representations.

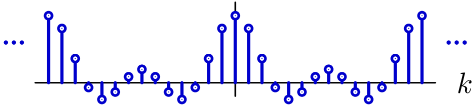
Relations Between Fourier Series and Fourier Transforms

Each Fourier series term is replaced by an impulse in the Fourier transform.

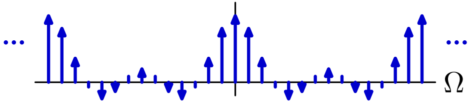
$$f[n] = f[n-N] \text{ for all integers } n$$



$$F[k] = F[k-N] \text{ for all integers } k$$



$$F(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi F[k]\delta\left(\Omega - k\frac{2\pi}{N}\right)$$



Periodic DT signals that have Fourier series representations also have Fourier transform representations.

Question of the Day

Let $F(\Omega)$ represent the Fourier transform of a discrete-time signal:

$$f[n] = \delta[n-1] + \delta[n] + \delta[n+1]\,.$$

Determine the numerical value of $F(2\pi)$.

Please enter your response at the following url:

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