6.3000: Signal Processing

DT Fourier Series

- Fourier series representations for discrete-time signals
- Comparison of Fourier series for CT and DT signals
- Properties of DT Fourier series
- Applications of Fourier analysis

February 19, 2026

Recall: Continuous-Time Fourier Series

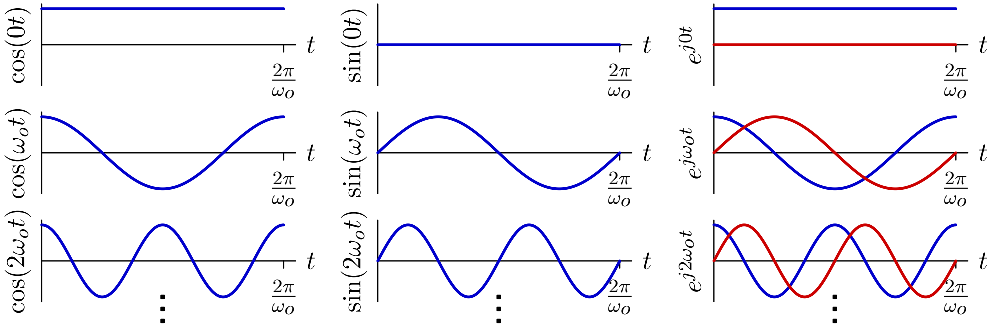
Fourier series: expansions in terms of harmonic **basis functions**.

$$f(t) = f(t+T) \;=\; c_0 + \sum_{k=1}^{\infty} c_k \cos k\omega_o t + \sum_{k=1}^{\infty} d_k \sin k\omega_o t \;=\; \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

where $\omega_o = \frac{2\pi}{T}$ represents the fundamental frequency.

Real-valued basis functions

Complex-valued basis functions



Today: Develop an analogous representation for discrete-time signals.

Check Yourself

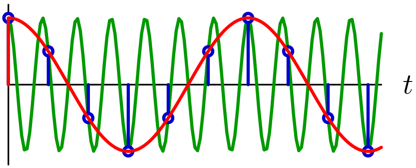
What is the fundamental (shortest) period of each of the following DT signals?

1. $f_1[n] = \cos\left(\frac{\pi n}{12}\right)$
2. $f_2[n] = \cos\left(\frac{\pi n}{12}\right) + 3\cos\left(\frac{\pi n}{15}\right)$
3. $f_3[n] = \cos(n)$

Aliasing

Recall that the same sequence of samples can result when two CT sinusoids with **different** frequencies are sampled with the **same** sampling interval Δ.

Example:



Samples (blue) of the original high-frequency signal (green) could just as easily have come from a much lower frequency signal (red).

Aliasing

In fact, many CT frequencies ω **alias** to the same DT frequency Ω.

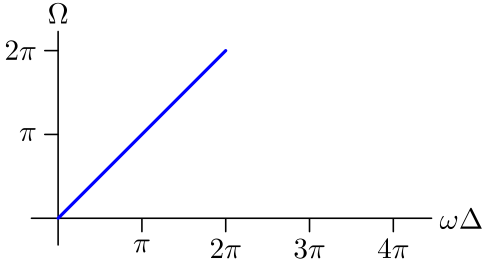
If we sample a CT sinusoid that has frequency ω

$$f(t) = \cos(\omega t)$$

at integer multiples of the sampling period Δ, we generate a DT signal:

$$f[n] = f(t)|_{t=n\Delta} = f(n\Delta) = \cos(\omega n\Delta) = \cos(\Omega n)$$

where the discrete frequency $\Omega = \omega\Delta$ grows linearly with ω as shown by the blue line below.



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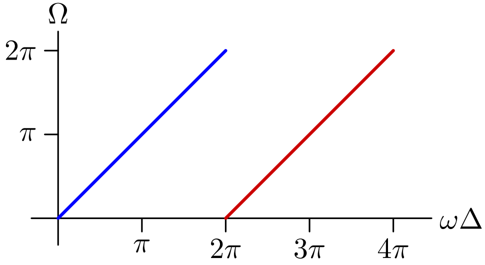

However, the cosine function is periodic with period 2π. Therefore Ω could also be $\omega\Delta - 2\pi$ as shown by the red line above.

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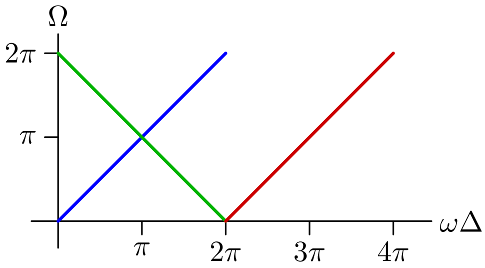

Since the cosine function is symmetric about 2π, Ω could also be $2\pi - \omega\Delta$ as shown by the green line above.

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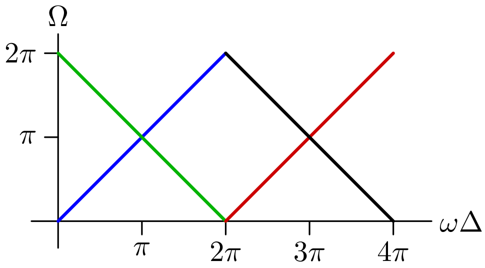

Since the cosine function is symmetric about 2π, Ω could also be $2\pi - \omega\Delta$ as shown by the green line or by the black line above.

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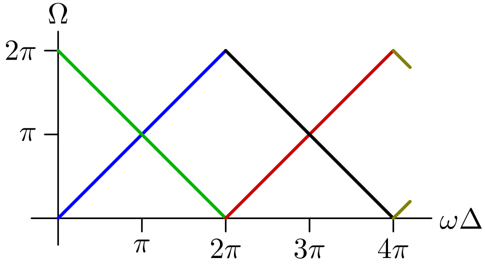

Since the cosine function is symmetric about 2π, Ω could also be $2\pi - \omega\Delta$ as shown by the green line or by the black line above, etc.

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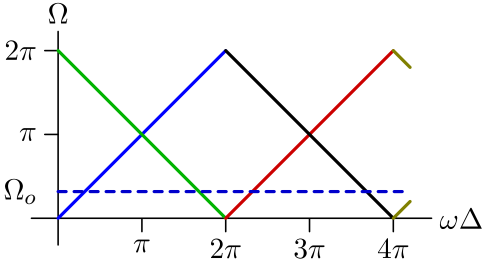

Many continuous frequencies ω produce the same sequence of samples.

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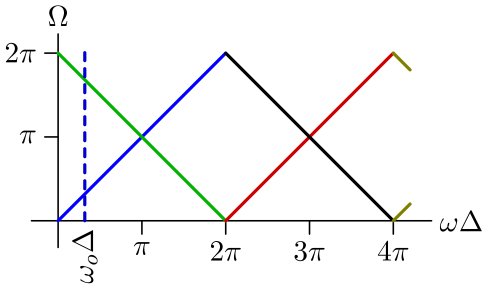
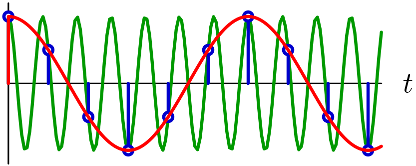


Many discrete frequencies Ω could represent the samples produced by ω_o.

Aliasing

Aliasing is an intrinsic property of DT sinusoids.



Aliasing has important implications for representing DT signals as Fourier series.

Discrete-Time Sinusoids

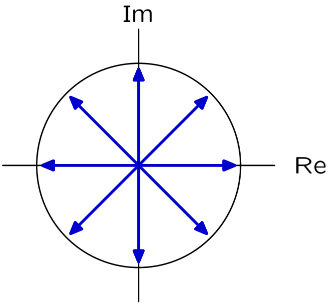
There are (only) N distinct complex exponentials with integer period N.

If $f[n] = e^{j\Omega n}$ is periodic in N then

$$f[n] = e^{j\Omega n} = f[n+N] = e^{j\Omega(n+N)} = e^{j\Omega n}e^{j\Omega N}$$

and $e^{j\Omega N}$ must be 1. Therefore $e^{j\Omega}$ must be one of the N^{th} roots of 1.

Example: $N = 8$



There are only 8 distinct complex exponentials with period $N = 8$:

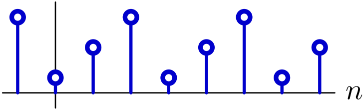
$$e^{j0\pi/4}, \quad e^{j1\pi/4}, \quad e^{j2\pi/4}, \quad e^{j3\pi/4}, \quad e^{j4\pi/4}, \quad e^{j5\pi/4}, \quad e^{j6\pi/4}, \quad e^{j7\pi/4}.$$

There are an infinite number of complex exponentials with period T in CT!

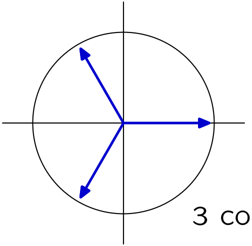
Discrete-Time Sinusoids

There are (only) N distinct complex exponentials with integer period N.

Example: periodic in N=3

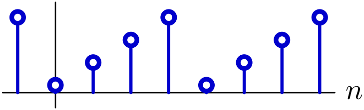


3 samples repeated in time

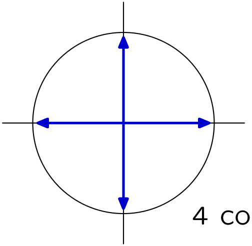


3 complex exponentials

Example: periodic in N=4



4 samples repeated in time



4 complex exponentials

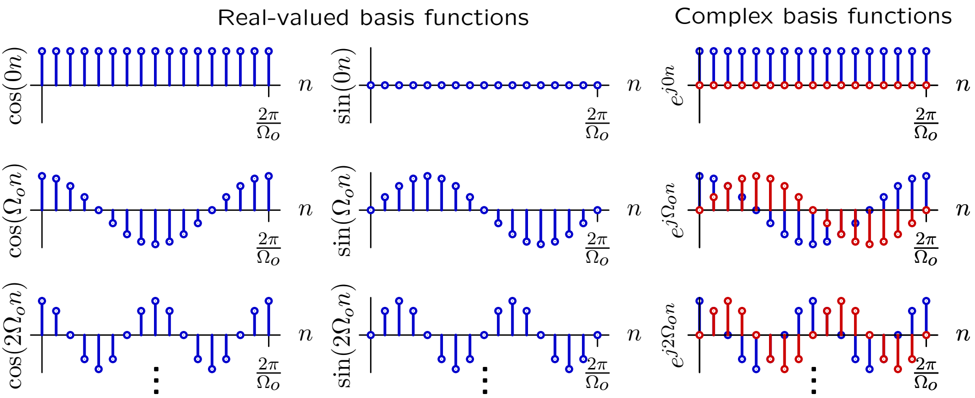
If a DT signal is periodic with period N,
then its Fourier series will contain just N terms.

Discrete-Time Fourier Series

A DT Fourier Series has just N harmonic frequencies $k\Omega_o$.

$$f[n] = f[n+N] = \sum_{k=\langle N\rangle} c_k \cos(k\Omega_o n) + \sum_{k=\langle N\rangle} d_k \sin(k\Omega_o n) = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_o n}$$

where Ω_o represents the fundamental frequency **(radians/sample)**.
Otherwise, DT Fourier series are similar to CT Fourier series.



Recall: Continuous-Time Fourier Series

We found the Fourier series coefficients using **two key insights**.

1. Multiplying complex harmonics of ω_o yields a complex harmonic of ω_o:

$$e^{jk\omega_o t} \times e^{jl\omega_o t} = e^{j(k+l)\omega_o t}$$

2. Integrating a complex harmonic over a period T yields zero unless the harmonic is at DC:

$$\int_{t_0}^{t_0+T} e^{jk\omega_o t} dt \equiv \int_T e^{jk\omega_o t} dt = \begin{cases} T & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

$$= T\delta[k]$$

where $\delta[k]$ is the Kronecker delta function

$$\delta[k] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow$ Fourier components are **orthogonal**.

Discrete-Time Fourier Series

The same two key insights apply to **DT analysis**.

1. Multiplying complex **DT** harmonics of Ω_o yields a new harmonic of Ω_o:

$$e^{jk\Omega_o n} \times e^{jl\Omega_o n} = e^{j(k+l)\Omega_o n}$$

2. **Summing** a complex harmonic over a period N is zero unless the harmonic is at DC:

$$\sum_{n=n_0}^{n_0+N-1} e^{jk\Omega_o n} \equiv \sum_{n=\langle N\rangle} e^{jk\Omega_o n} = \begin{cases} N & \text{if } k=0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

$$= N\delta[k]$$

$\rightarrow$ **DT** Fourier components are **orthogonal**.

Recall: Continuous-Time Fourier Series

Use orthogonality to find the Fourier series coefficients.

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

Multiply $f(t)$ by the complex conjugate of the basis function of interest, and then integrate over T.

$$\begin{aligned}\int_T f(t)e^{-jl\omega_o t}dt &= \int_T \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}\right) e^{-jl\omega_o t}dt \\ &= \sum_{k=-\infty}^{\infty} a_k \int_T e^{j(k-l)\omega_o t}dt \\ &= \sum_{k=-\infty}^{\infty} a_k T\delta[k-l] = a_l T\end{aligned}$$

Solving for a_l and then substituting k for l yields

$$a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_o t}dt$$

Discrete-Time Fourier Series

Using orthogonality to find the DT Fourier series coefficients.

$$f[n] = f[n+N] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_o n}$$

Multiply $f[n]$ by the complex conjugate of the basis function of interest, and then sum over N.

$$\begin{aligned}
\sum_{n=\langle N\rangle} f[n] e^{-jl\Omega_o n} &= \sum_{n=\langle N\rangle} \left(\sum_{k=\langle N\rangle} a_k e^{jk\Omega_o n} \right) e^{-jl\Omega_o n} \\
&= \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-l)\Omega_o n} \\
&= \sum_{k=\langle N\rangle} a_k N\delta[k-l] = a_l N
\end{aligned}$$

Solving for a_l and then substituting k for l yields

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} f[n] e^{-jk\Omega_o n}$$

Fourier Series Summary

CT and DT Fourier series are similar, but DT Fourier series require just N components while CT Fourier series require an infinite number.

Continuous-Time Fourier Series

$$a_k = \frac{1}{T}\int_T f(t) e^{-jk\omega_o t} dt$$

analysis equation

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

synthesis equation

where $\omega_o = \frac{2\pi}{T}$

Discrete-Time Fourier Series

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} f[n] e^{-jk\Omega_o n}$$

analysis equation

$$f[n] = f[n+N] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_o n}$$

synthesis equation

where $\Omega_o = \frac{2\pi}{N}$

Properties of Discrete-Time Fourier Series

Operations on the time representation of a signal can often be interpreted as equivalent (but easier) operations on the series coefficients.

We will discuss four (of many) properties of Fourier series.

- linearity
- time shift
- time reversal
- conjugate symmetry

Linearity

The Fourier series coefficients of a linear combination of two signals is the linear combination of their Fourier series coefficients.

Let

$$f[n] = af_1[n] + bf_2[n] \quad \text{where} \quad f_1[n] = f_1[n+N] \text{ and } f_2[n] = f_2[n+N]$$

then the Fourier series coefficents for $f[n]$ are given by

$$F[k] = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=\langle N\rangle}\Big(af_1[n] + bf_2[n]\Big)e^{-jk\frac{2\pi}{N}n}$$

$$= \underbrace{a\,\frac{1}{N}\sum_{n=\langle N\rangle} f_1[n]e^{-jk\frac{2\pi}{N}n}}_{F_1[k]} + \underbrace{b\,\frac{1}{N}\sum_{n=\langle N\rangle} f_2[n]e^{-jk\frac{2\pi}{N}n}}_{F_2[k]}$$

$$= aF_1[k] + bF_2[k]$$

where $F_1[k]$ and $F_2[k]$ are Fourier series coefficients for $f_1[n]$ and $f_2[n]$.

Time Shift

Shifting time changes the phases of a signal's Fourier coefficients.

Let

$$g[n] = f[n-n_0] \quad \text{where} \quad f[n] = f[n+N]$$

If

$$F[k] = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\frac{2\pi}{N}n}$$

then

$$G[k] = \frac{1}{N} \sum_{n=\langle N \rangle} g[n]\, e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} \sum_{n=\langle N \rangle} f[n-n_0]\, e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N} \sum_{m=\langle N \rangle} f[m]\, e^{-jk\frac{2\pi}{N}(m+n_0)} \quad \text{where} \quad m = n-n_0$$

$$= e^{-jk\frac{2\pi}{N}n_0}\, \frac{1}{N} \sum_{m=\langle N \rangle} f[m]\, e^{-jk\frac{2\pi}{N}m} = e^{-jk\frac{2\pi}{N}n_0}\, F[k]$$

Time Reversal

Reversing time reverses frequency.

Let

$$g[n] = f[-n] \quad \text{where} \quad f[n] = f[n+N]$$

If

$$F[k] = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\frac{2\pi}{N}n}$$

then

$$G[k] = \frac{1}{N} \sum_{n=\langle N \rangle} g[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} \sum_{n=\langle N \rangle} f[-n] e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N} \sum_{m=\langle N \rangle} f[m] e^{+jk\frac{2\pi}{N}m} \quad \text{where} \quad m = -n$$

$$= F[-k]$$

Conjugate Symmetry

If $f[n]$ is real-valued, then its Fourier coefficients have conjugate symmetry.

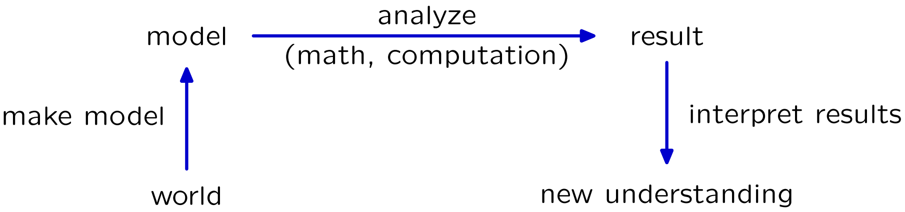
If $f[n]$ is real-valued, then $f[n] = f^*[n]$.

$$
\begin{aligned}
F[k] &= \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\frac{2\pi}{N}n} \\
F^*[k] &= \frac{1}{N} \sum_{n=\langle N \rangle} f^*[n] e^{jk\frac{2\pi}{N}n} \\
&= \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{jk\frac{2\pi}{N}n} \\
&= F[-k]
\end{aligned}
$$

Applications of Fourier Series

Signal processing is **widely used** in science and engineering to . . .

- **model** some aspect of the world,
- **analyze** the model, and
- **interpret** results to gain a new or better understanding.



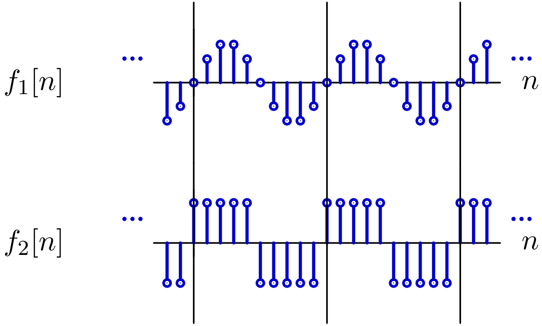
We previously touched on applications in physics, including the wave equation and how it leads directly to Fourier analysis.

Applications of Fourier analysis in **hearing**.

Applications of Fourier Analysis in Hearing

What determines the pitch of a sound? This seemingly simple question has evoked debate (sometimes fierce) for more than 150 years.

Compare two periodic signals with the same period, each played with 4000 samples per second



Different sounds, same pitch. We would like to understand why.

Applications of Fourier Analysis in Hearing

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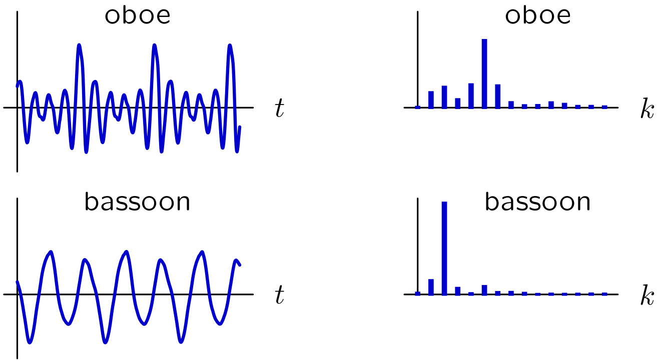

Is pitch determined by a time-based metric such as the time between peaks?
If so, is the time between the peaks shown here 1/3 or 1/6 of time shown?

Is it a frequency-based metric such as the fundamental frequency?
If so, is the first peak in the Fourier series the fundamental or just noise?

Even more fundamentally, can neurons do Fourier series?

Applications of Fourier Analysis in Hearing

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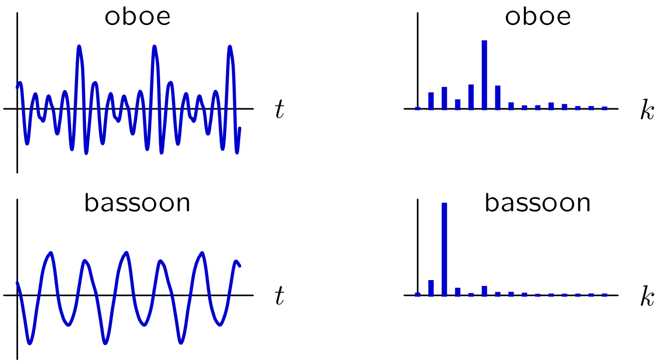


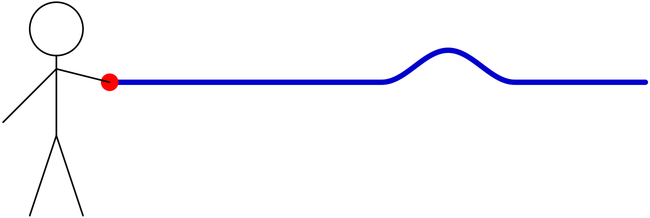
To determine if pitch is determined by time or frequency based analysis,

- we would like make changes in time without changing frequency, and
- we would like make changes in frequency without changing time, and

. . . not easy to do with the available stringed instruments and tubes.

Physical Example: Vibrating String

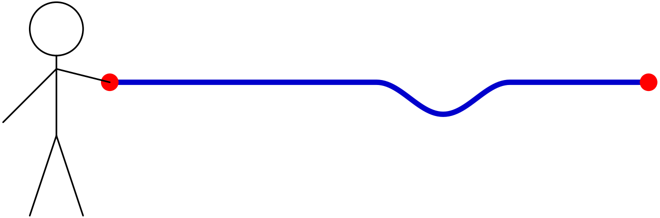
A taut string supports wave motion.



The speed of the wave depends on the tension on and mass of the string.

Physical Example: Vibrating String

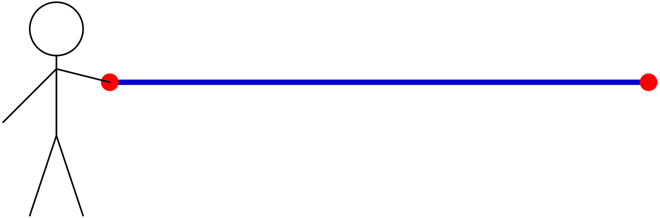
The wave will reflect off a rigid boundary.



The amplitude of the reflected wave is opposite that of the incident wave.

Physical Example: Vibrating String

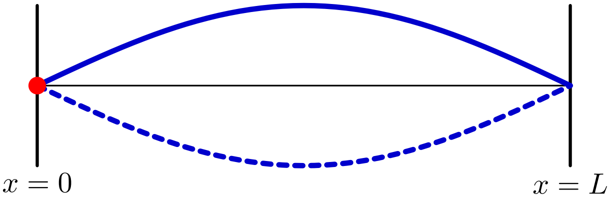
Reflections can interfere with excitations.



The interference can be constructive or destructive depending on the frequency of the excitation.

Physical Example: Vibrating String

We get constructive interference if round-trip travel time equals the period.

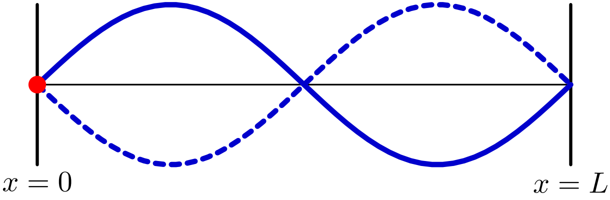


Round-trip travel time $= \dfrac{2L}{v} = T$

$$\omega_o = \frac{2\pi}{T} = \frac{2\pi}{2L/v} = \frac{\pi v}{L}$$

Physical Example: Vibrating String

We also get constructive interference if round-trip travel time is $2T$.

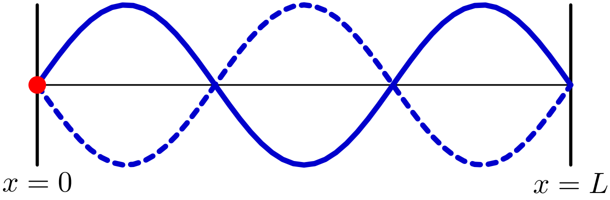


Round-trip travel time $= \dfrac{2L}{v} = 2T$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{L/v} = \frac{2\pi v}{L} = 2\omega_o$$

Physical Example: Vibrating String

In fact, we also get constructive interference if round-trip travel time is kT.



Round-trip travel time $= \dfrac{2L}{v} = kT$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2L/kv} = \frac{k\pi v}{L} = k\omega_o$$

Only certain frequencies (harmonics of $\omega_o = \pi v/L$) persist.

This is the basis of stringed instruments.

Pitch Experiments

One can change the pitch of a string by changing its length or mass or tension, but each of these manipulations affect both time and frequency-based metrics.

To determine if pitch is determined by time or frequency based analysis,

- we would like make changes in time without changing frequency, and
- we would like make changes in frequency without changing time, and

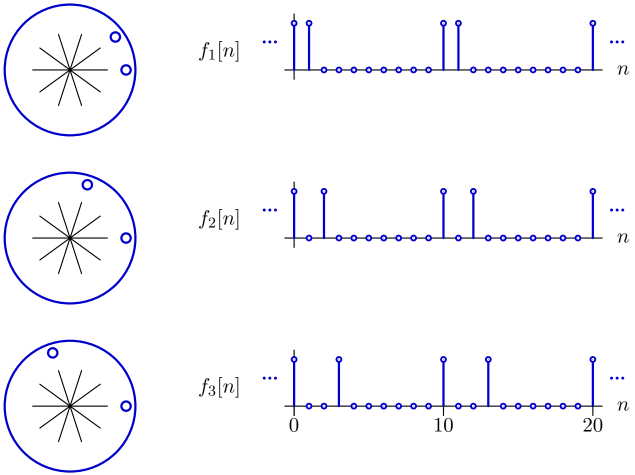
. . . not easy to do with the available stringed instruments and tubes.

A breakthrough occured with the work of Seebeck who used sirens to generate more complicated sounds.

Very clever experiment, but very controversial interpretations.

Sirens

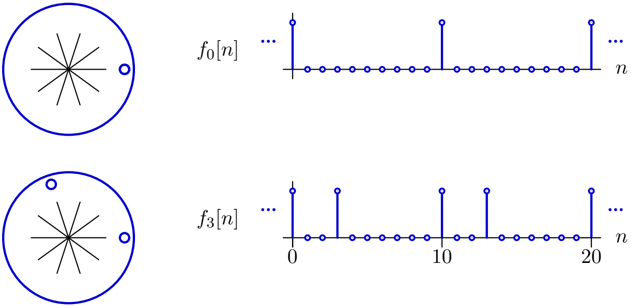
Seebeck used a siren to generate more complicated sounds (circa 1841) by passing a jet of compressed air through holes in a spinning disk.



The pattern of holes determined the pattern of pulses in each period. The speed of spinning controlled the number of periods per second.

Sirens

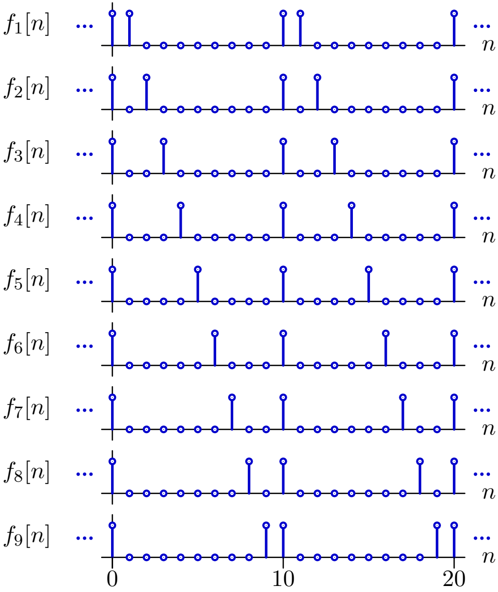
Strangely, adding a second hole per period didn't seem to affect the pitch.



Pitch should be different if it is determined by the intervals between pulses.

Sirens

There was one very interesting exception.



But hearing this exception required precise alignment of the siren's holes.

Sirens and Controversy

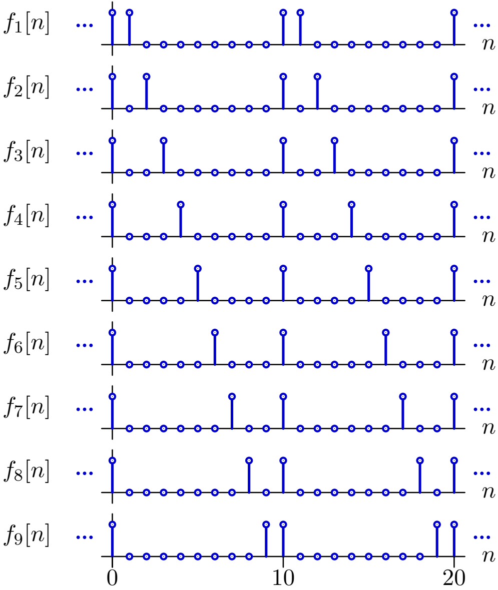
Seebeck interpreted his results in terms of the intervals between the holes. He held that pitch results from **timing** with some intervals being more important than others. As the lengths of the two intervals in his experiment converged, the pitch favored what had been the second harmonic and that frequency increasingly dominated.

Georg Ohm (already known for his work on electrical conduction) interpreted Seebeck's results using Fourier's recently described series. He held that the pulses generated by a siren contained a **fundamental** and **harmonics** that were physically present just as much as they are in a stringed instrument.

A bitter controversy ensued.

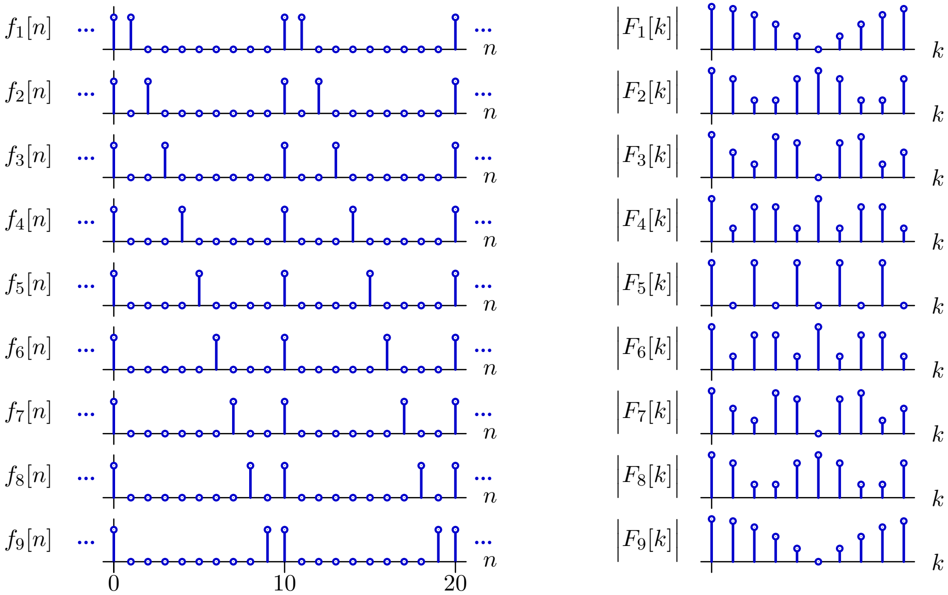
Fourier Interpretation

To understand Ohm's argument, compute the Fourier series for the siren's sound.



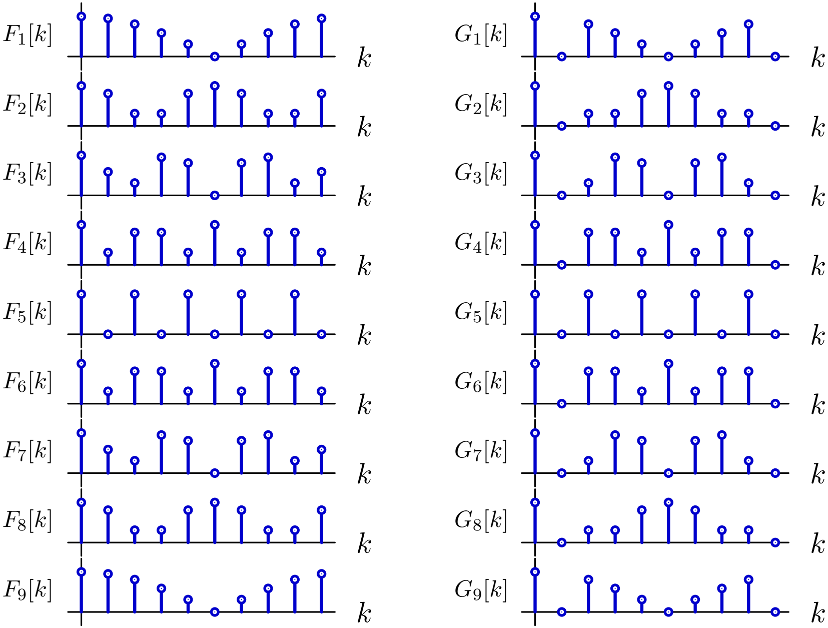
Fourier Series

Notice that $f_5[n]$ has no fundamental component!



Fourier Series With and Without the Fundamental

Resynthesize each waveform without its fundamental component.



Although perception of the fundamental is weakened, it is not gone!

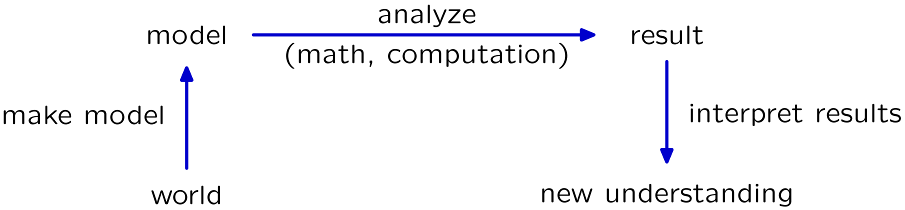
Summary

Seebeck designed an extremely clever **experiment** to test pitch perception.

Ohm analyzed an important **theory** (from Fourier) and argued that harmonics are present even in the pulsatile sounds generated by a siren.

Neither Seebeck nor Ohm could convincingly account for experimental results that demonstrated the dominance of the fundamental, even when it was weak or missing.

Progress in understanding the “missing fundamental” awaited Helmholtz, who demonstrated the importance of “combination tones” in the ear.



Summary

Today we focused on discrete-time Fourier analysis.

- We developed Fourier series for discrete-time signals.
- We compared Fourier series for CT and DT signals.
- We looked at four (of many) properties of DT Fourier series.
- We looked briefly at applications of Fourier analysis in hearing.

Next time: Fourier analysis of aperiodic signals (CT and DT).

Question of the Day

Determine the fundamental period of the following signal:

$$f[n] = \cos(-2.9\pi n + 0.1\pi)$$