

# 6.3000: Signal Processing

## CT Fourier Transform

- Definition
- Examples
- Properties

**Quiz 1:** March 3, 2-4pm, 50-340 (Walker).

- Closed book except for one page of notes (8.5" x 11" both sides).
- No electronic devices. (No headphones, cellphones, calculators, ...)
- Coverage up to and including classes on February 26 and HW 3.

More information under the **Quiz 1 Information** tab of the 6.300 website.

## From Periodic to Aperiodic

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We have been focusing on frequency representations of **periodic** signals: e.g., sounds, waves, music, ...

However, most real-world signals are not periodic.

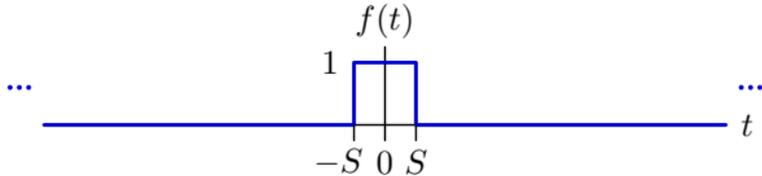
None are truly periodic since they do not have infinite duration!

Today: generalizing Fourier representations to include aperiodic signals.

# Fourier Representations of Aperiodic Signals

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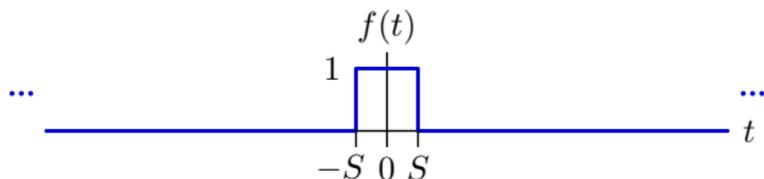
How can we represent an aperiodic signal as a sum of sinusoids?



## Fourier Representations of Aperiodic Signals

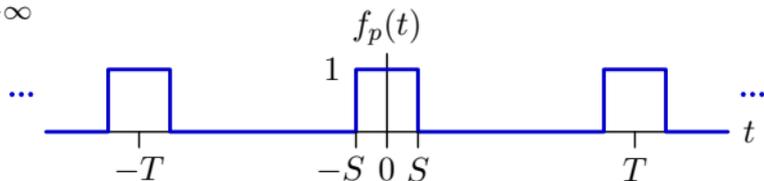
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How can we represent an aperiodic signal as a sum of sinusoids?



Strategy: make a periodic version of  $f(t)$  by summing shifted copies:

$$f_p(t) = \sum_{m=-\infty}^{\infty} f(t - mT)$$



Since  $f_p(t)$  is periodic, it has a Fourier series (which depends on  $T$ ).

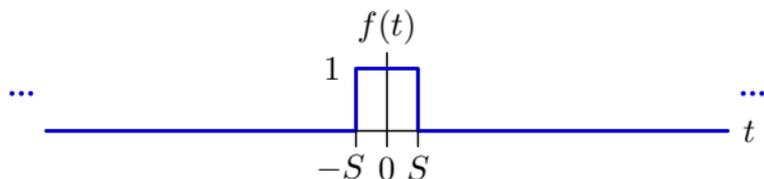
Find Fourier series coefficients  $F_p[k]$  and take the limit of  $F_p[k]$  as  $T \rightarrow \infty$ .

As  $T \rightarrow \infty$ ,  $f_p(t) \rightarrow f(t)$ , and the Fourier series  $\rightarrow$  Fourier transform.

## Fourier Representations of Aperiodic Signals

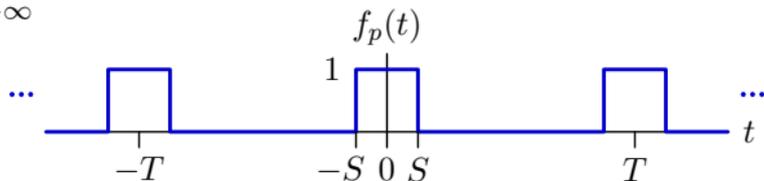
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Example.



Strategy: make a periodic version of  $f(t)$  by summing shifted copies:

$$f_p(t) = \sum_{m=-\infty}^{\infty} f(t - mT)$$



Calculate the Fourier series coefficients  $F_p[k]$  for this example:

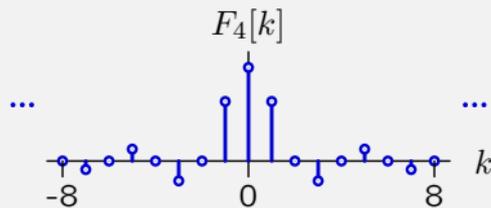
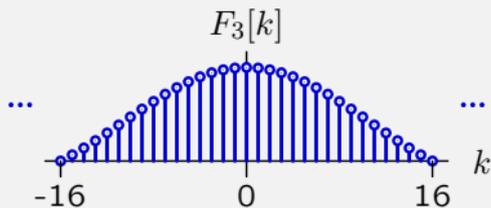
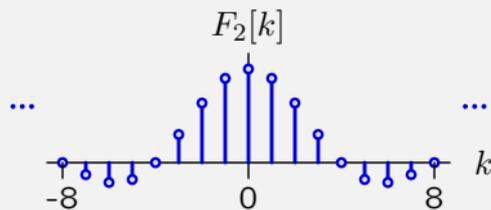
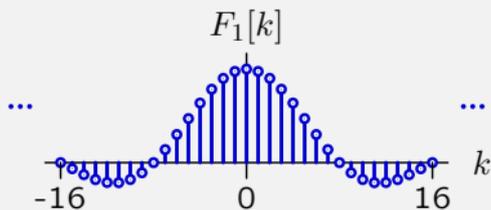
$$F_p[k] = \frac{1}{T} \int_{-S}^S e^{-j\frac{2\pi k}{T}t} dt = \frac{1}{T} \left. \frac{e^{-j\frac{2\pi k}{T}t}}{-j\frac{2\pi k}{T}} \right|_{-S}^S = \frac{2 \sin\left(\frac{2\pi k}{T}S\right)}{T \left(\frac{2\pi k}{T}\right)}$$

## Check Yourself

Calculate the Fourier series coefficients  $F_p[k]$ :

$$F_p[k] = \frac{1}{T} \int_{-S}^S e^{-j\frac{2\pi}{T}kt} dt = \frac{1}{T} \left. \frac{e^{-j\frac{2\pi}{T}kt}}{-j\frac{2\pi k}{T}} \right|_{-S}^S = \frac{2 \sin\left(\frac{2\pi k}{T}S\right)}{T\left(\frac{2\pi k}{T}\right)}$$

Which of the following shows  $F_p[k]$  for  $S = 1$  and  $T = 8$ ?



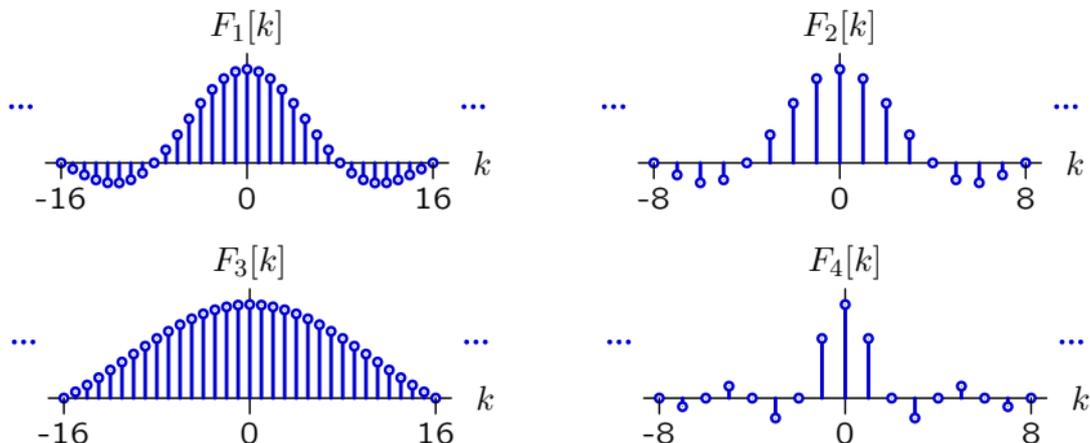
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2.  $F_2[k]$
3.  $F_3[k]$
4.  $F_4[k]$
5. none of the above

## Check Yourself

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Which of the following shows  $F_p[k]$  for  $S = 1$  and  $T = 8$ ?



According to the above formula,  $F_p[k]$  is a **sinc** function (i.e.,  $\sin(x)/x$ ).

The first zero of a sinc function is at  $x = \pi$ .

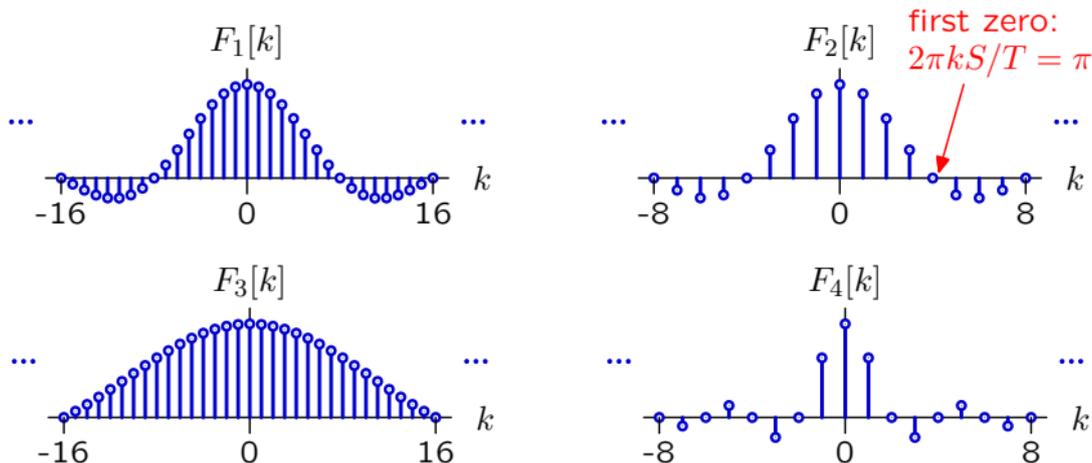
Therefore,  $x = 2\pi kS/T = \pi$ , and  $k = 4$ . Therefore  $F_p[k] = F_2[k]$ .

## Check Yourself

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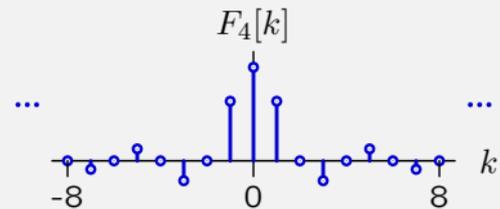
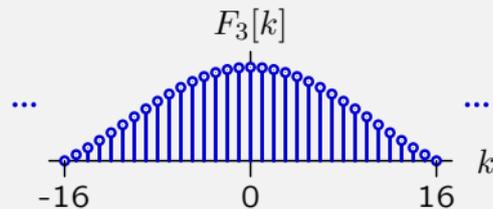
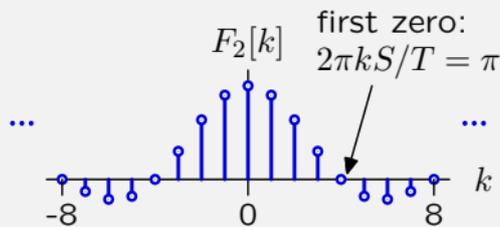
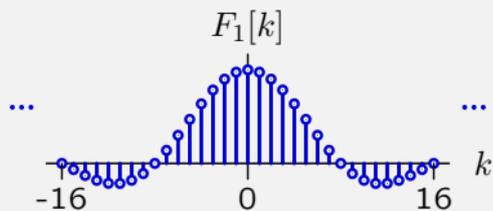
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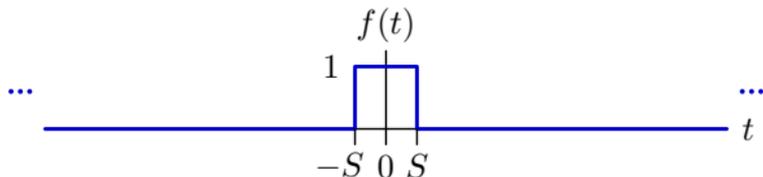


1.  $F_1[k]$     2.  $F_2[k]$     3.  $F_3[k]$     4.  $F_4[k]$     5. none of the above

## So far ...

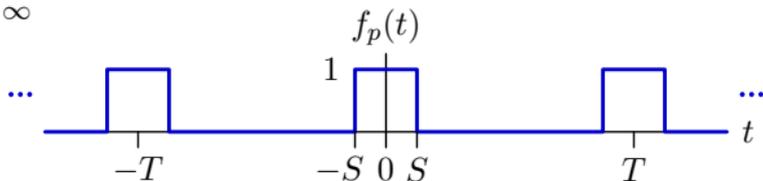
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Goal: represent an aperiodic signal as a sum of sinusoids.



Strategy: make a periodic version of  $f(t)$  by summing shifted copies: ✓

$$f_p(t) = \sum_{m=-\infty}^{\infty} f(t - mT)$$



Since  $f_p(t)$  is periodic, it has a Fourier series (which depends on  $T$ ). ✓

Find Fourier series coefficients  $F_p[k]$  ✓, take the limit of  $F_p[k]$  as  $T \rightarrow \infty$ .

As  $T \rightarrow \infty$ ,  $f_p(t) \rightarrow f(t)$ , and the Fourier series  $\rightarrow$  Fourier transform.

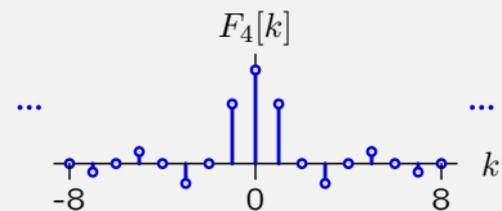
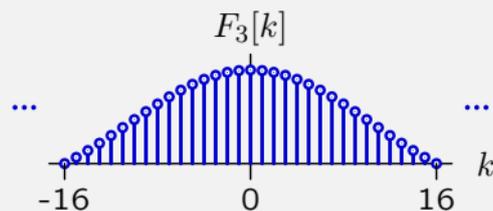
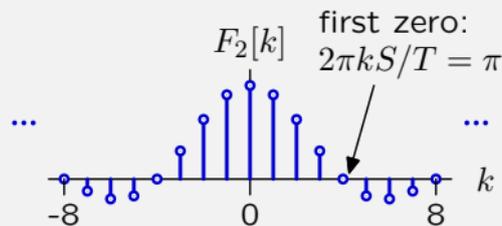
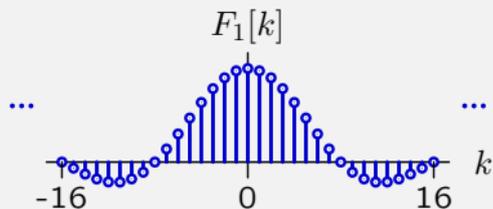
## Fourier Representation of Aperiodic Signals: Example

Calculate the Fourier series coefficients  $F_p[k]$ :

$$F_p[k] = \frac{1}{T} \int_{-S}^S e^{-j\frac{2\pi}{T}kt} dt = \frac{1}{T} \frac{e^{-j\frac{2\pi}{T}kt}}{-j\frac{2\pi k}{T}} \Bigg|_{-S}^S = \frac{2 \sin\left(\frac{2\pi k}{T}S\right)}{T\left(\frac{2\pi k}{T}\right)}$$

We just found that  $F_p[k] = F_2[k]$  when  $S=1$  and  $T=8$ .

Which (if any) of these plots shows  $F_p[k]$  when  $S=1$  and  $T=16$ ?



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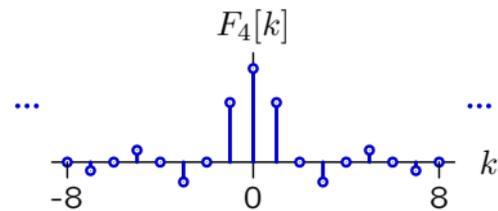
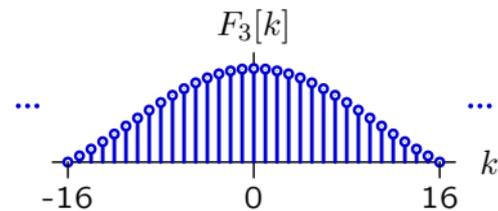
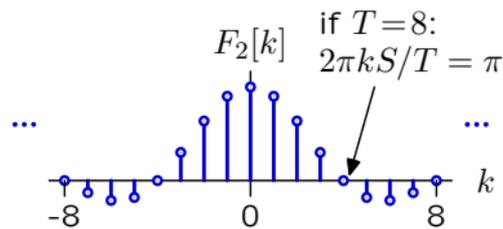
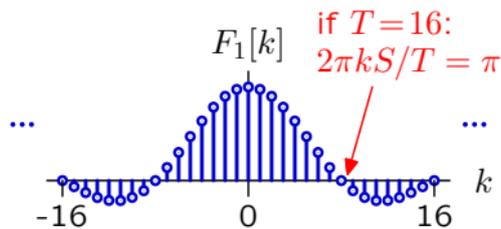
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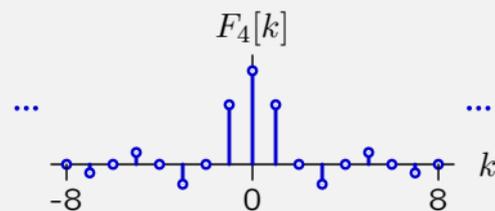
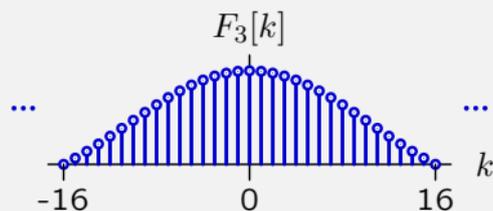
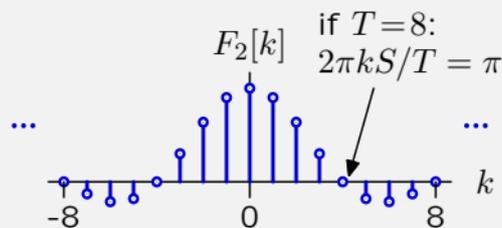
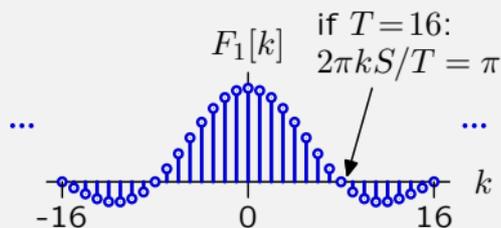
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## Fourier Representations of Aperiodic Signals

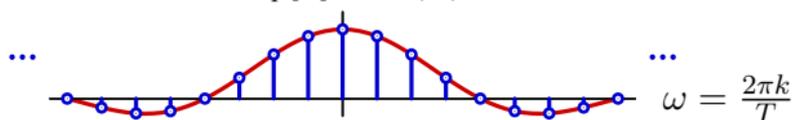
Define a new function  $F(\omega)$  where  $\omega = k\omega_o = 2\pi k/T$ .

$$TF_p[k] = \frac{2 \sin\left(\frac{2\pi k}{T} S\right)}{\frac{2\pi k}{T}} = 2 \frac{\sin(\omega S)}{\omega} \Big|_{\omega=\frac{2\pi k}{T}} = F(\omega) \Big|_{\omega=\frac{2\pi k}{T}}$$

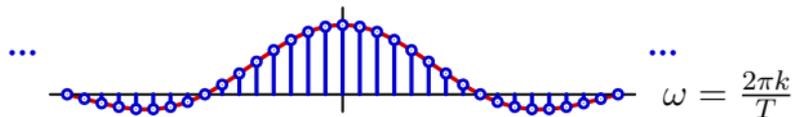
Then  $TF_p[k]$  represents samples of  $F(\omega)$  with increasing resolution in  $\omega$ .

$$TF_p[k] = F(\omega)$$

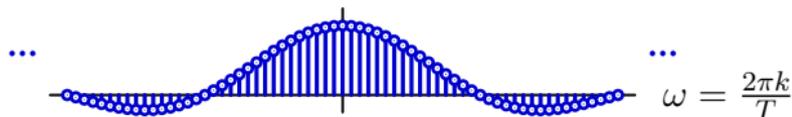
$S=1$  and  $T=8$ :



$S=1$  and  $T=16$ :



$S=1$  and  $T=32$ :



The discrete function  $TF_p[k]$  is a sampled version of the function  $F(\omega)$ .

## Fourier Representations of Aperiodic Signals

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From  $f(t)$  to  $F(\omega)$ :

$$f(t) \xrightarrow{\substack{\text{periodic} \\ \text{extension}}} f_p(t) \xrightarrow{\substack{\text{Fourier} \\ \text{series}}} F_p[k] \xrightarrow{\substack{\text{interpolation}}} F(\omega)$$

The limiting behaviors as  $T \rightarrow \infty$  define the Fourier transform:

$$\begin{aligned} F(\omega) &= \lim_{T \rightarrow \infty} T F_p[k] \Big|_{k=\frac{\omega}{\omega_0}=\frac{T}{2\pi}\omega} \\ &= \lim_{T \rightarrow \infty} T \left[ \frac{1}{T} \int_T f_p(t) e^{-j\frac{2\pi k}{T}t} dt \right]_{k=\frac{T}{2\pi}\omega} \\ &= \lim_{T \rightarrow \infty} \int_T f_p(t) e^{-j\omega t} dt \\ F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \end{aligned}$$

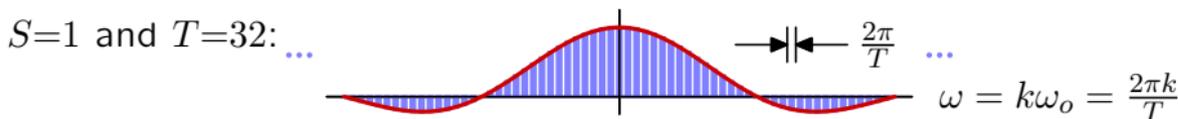
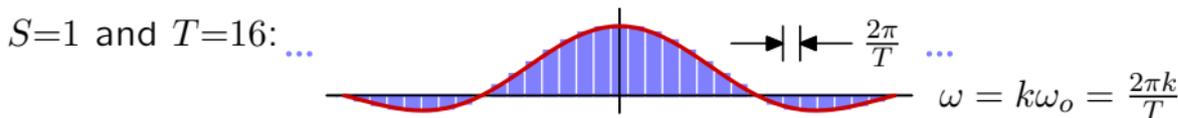
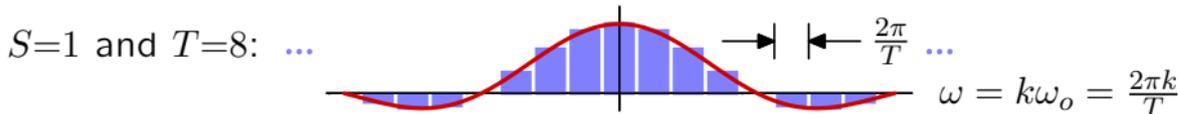
This **analysis** equation **defines** the Fourier transform.

## Fourier Representations of Aperiodic Signals

The **synthesis equation** follows from a piecewise constant approximation.

$$\begin{aligned} f(t) &= \lim_{T \rightarrow \infty} f_p(t) = \lim_{T \rightarrow \infty} \sum_{k=-\infty}^{\infty} F_p[k] e^{j \frac{2\pi}{T} kt} \\ &= \lim_{T \rightarrow \infty} \left( \frac{1}{2\pi} \right) \sum_{k=-\infty}^{\infty} T F_p[k] e^{j \frac{2\pi}{T} kt} \left( \frac{2\pi}{T} \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \end{aligned}$$

$$T F_p[k] = F(\omega)$$



**Synthesis Equation:**  $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$

## Continuous-Time Fourier Representations

---

Fourier series and transforms are similar:  
both represent signals by their frequency content.

### Continuous-Time Fourier Series

$$F[k] = \frac{1}{T} \int_T f(t) e^{-jk\omega_0 t} dt$$

**analysis equation**

$$f(t) = f(t + T) = \sum_{k=-\infty}^{\infty} F[k] e^{jk\omega_0 t}$$

**synthesis equation**

$$\text{where } \omega_0 = \frac{2\pi}{T}$$

### Continuous-Time Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

**analysis equation**

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

**synthesis equation**

## Continuous-Time Fourier Representations

---

All of the information in a periodic signal is contained in one period.  
The information in an aperiodic signal is spread across all time.

### Continuous-Time Fourier Series

$$F[k] = \frac{1}{T} \int_T f(t) e^{-jk\omega_0 t} dt$$

analysis equation

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$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

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synthesis equation

## Continuous-Time Fourier Representations

---

Periodic signals can be synthesized from a discrete set of harmonics.  
Aperiodic signals generally require all possible frequencies.

### Continuous-Time Fourier Series

$$F[k] = \frac{1}{T} \int_T f(t) e^{-jk\omega_0 t} dt$$

analysis equation

$$f(t) = f(t + T) = \sum_{k=-\infty}^{\infty} F[k] e^{jk\omega_0 t}$$

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### Continuous-Time Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

analysis equation

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

synthesis equation

## Continuous-Time Fourier Representations

---

Harmonic frequencies  $k\omega_o$  are samples of continuous frequency  $\omega$ .

### Continuous-Time Fourier Series

$$F[k] = \frac{1}{T} \int_T f(t) e^{-jk\omega_o t} dt$$

analysis equation

$$f(t) = f(t + T) = \sum_{k=-\infty}^{\infty} F[k] e^{jk\omega_o t}$$

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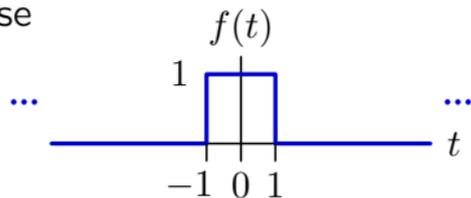
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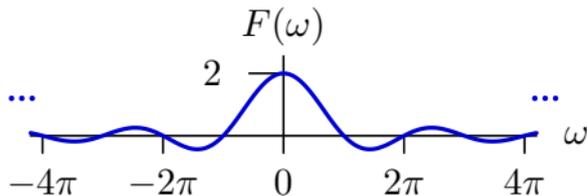
## Example

Find the Fourier Transform (FT) of a rectangular pulse:

$$f(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt = \int_{-1}^1 e^{-j\omega t} dt = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-1}^1 = 2 \frac{\sin \omega}{\omega}$$



$F(\omega)$  provides a recipe for constructing  $f(t)$  from sinusoidal components:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega$$

A square pulse contains (almost) all frequencies  $\omega$  (missing just  $\pi, 2\pi, \dots$ ).

## Properties of Fourier Transforms

---

Fourier transforms offer an alternative view of an **aperiodic** signal.

A signal and its Fourier transform contain exactly the same information, but some information is more easily seen in one domain than in the other.

There are many **properties** of Fourier transforms. These properties summarize systematic relations between time and frequency representations.

## Properties of Fourier Transforms

---

Time delay maps to linear phase delay of the Fourier transform.

$$\begin{aligned} \text{If} \quad & f(t) \xrightarrow{\text{FT}} F(\omega) \\ \text{then} \quad & f(t - \tau) \xrightarrow{\text{FT}} e^{-j\omega\tau} F(\omega) \end{aligned}$$

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$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt \\ G(\omega) &= \int_{-\infty}^{\infty} f(t - \tau)e^{-j\omega t} dt \end{aligned}$$

Let  $u = t - \tau$  (and therefore  $du = dt$  since  $\tau$  is a constant)

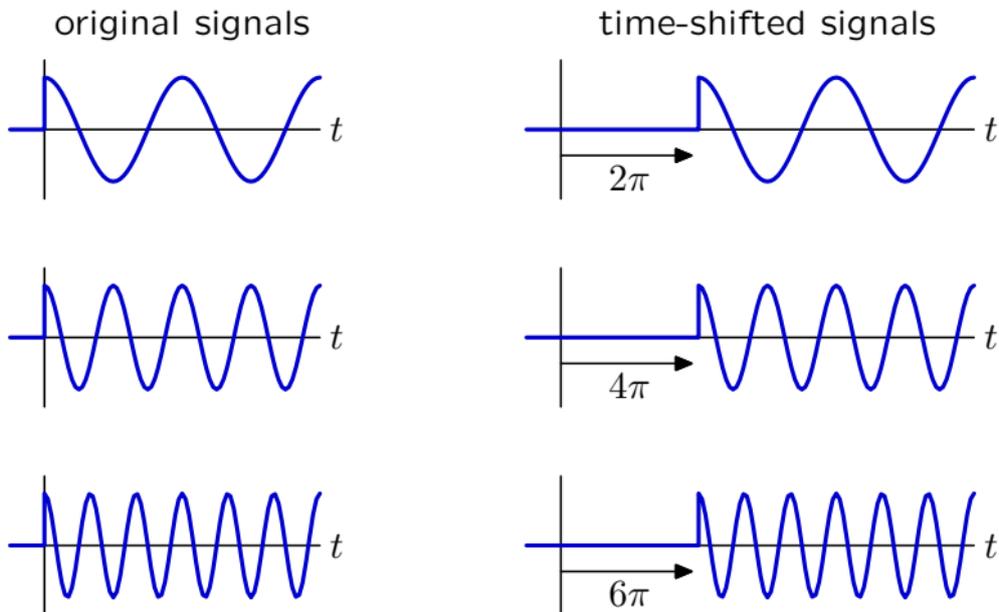
$$G(\omega) = \int_{-\infty}^{\infty} f(u)e^{-j\omega(u+\tau)} du = e^{-j\omega\tau} \int_{-\infty}^{\infty} f(u)e^{-j\omega u} du = e^{-j\omega\tau} F(\omega)$$

The angle of  $e^{-j\omega\tau} = -\omega\tau$ .

Why does time delay change phase by an amount proportional to frequency?

## Properties of Fourier Transforms

Why does time delay change phase by an amount proportional to frequency?



Doubling the frequency of a sinusoid doubles the change in phase associated with a given time delay.

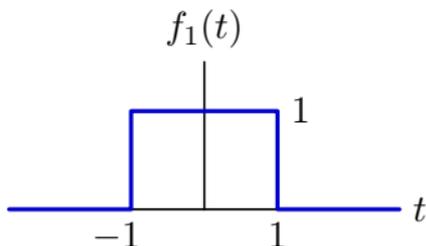
## Properties of Fourier Transforms

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Scaling time.

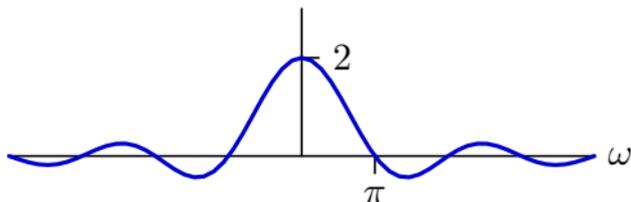
Consider the following signal and its Fourier transform.

Time representation:



Frequency representation:

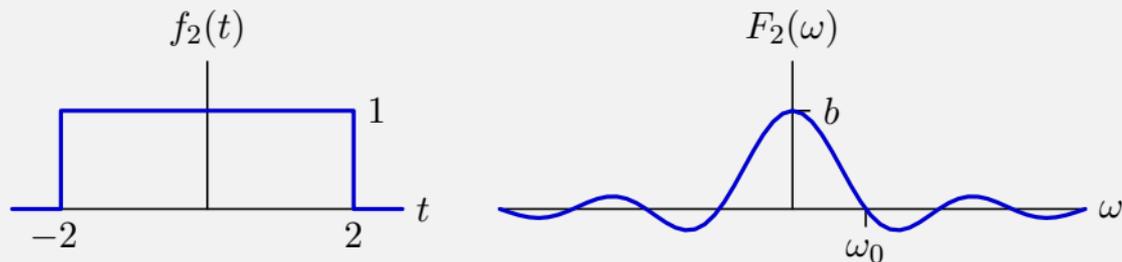
$$F_1(\omega) = \frac{2 \sin \omega}{\omega}$$



How would these functions scale if time were stretched?

## Check Yourself

Signal  $f_2(t)$  and its Fourier transform  $F_2(\omega)$  are shown below.



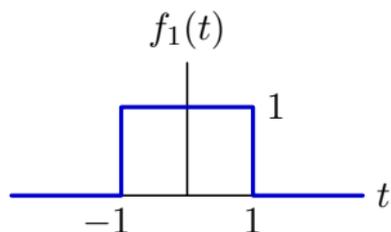
Which of the following is true?

1.  $b = 2$  and  $\omega_0 = \pi/2$
2.  $b = 2$  and  $\omega_0 = 2\pi$
3.  $b = 4$  and  $\omega_0 = \pi/2$
4.  $b = 4$  and  $\omega_0 = 2\pi$
5. none of the above

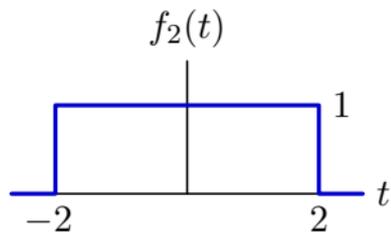
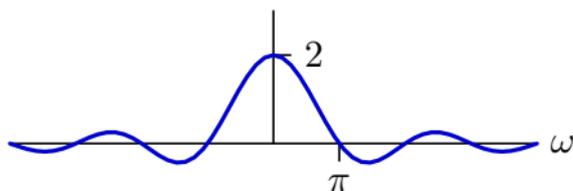
## Check Yourself

Find the Fourier transform.

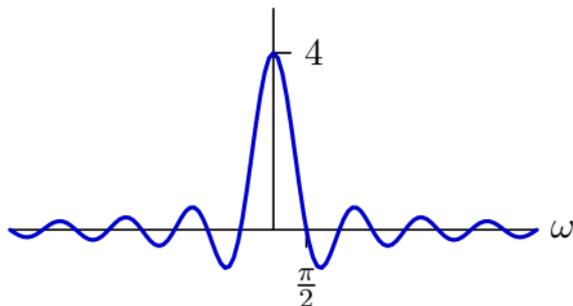
$$F_2(\omega) = \int_{-2}^2 e^{-j\omega t} dt = \frac{e^{-j\omega t}}{-j\omega} \Big|_{-2}^2 = \frac{2 \sin 2\omega}{\omega} = \frac{4 \sin 2\omega}{2\omega}$$



$$F_1(\omega) = \frac{2 \sin \omega}{\omega}$$



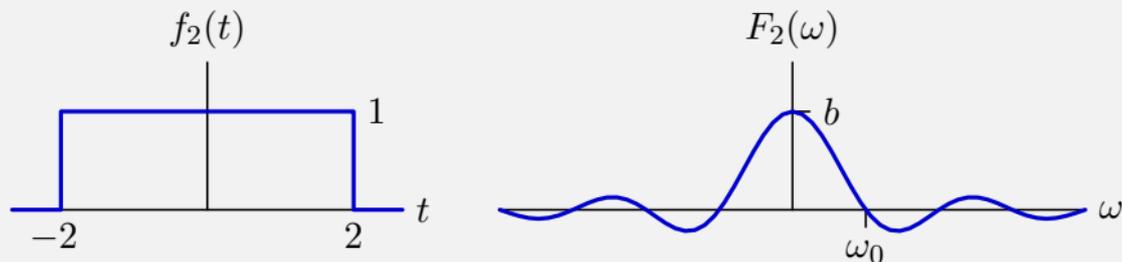
$$F_2(\omega) = \frac{4 \sin 2\omega}{2\omega}$$



Stretching time compresses frequency.

## Check Yourself

Signal  $f_2(t)$  and its Fourier transform  $F_2(\omega)$  are shown below.



Which of the following is true? 3

1.  $b = 2$  and  $\omega_0 = \pi/2$
2.  $b = 2$  and  $\omega_0 = 2\pi$
3.  $b = 4$  and  $\omega_0 = \pi/2$
4.  $b = 4$  and  $\omega_0 = 2\pi$
5. none of the above

## Properties of Fourier Transforms

---

Find a general scaling rule.

Let  $f_2(t) = f_1(at)$  where  $a > 0$ .

$$F_2(\omega) = \int_{-\infty}^{\infty} f_1(at) e^{-j\omega t} dt$$

Let  $\tau = at$ . Then  $d\tau = a dt$ .

$$F_2(\omega) = \int_{-\infty}^{\infty} f_1(\tau) e^{-j\omega\tau/a} \frac{1}{a} d\tau = \frac{1}{a} F_1\left(\frac{\omega}{a}\right)$$

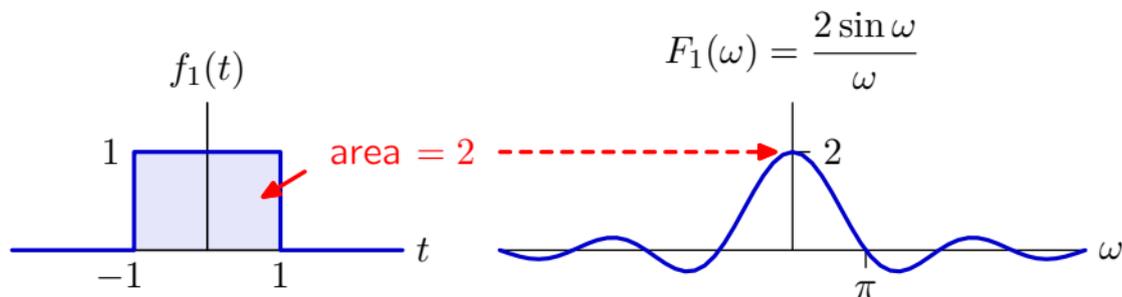
Stretching time compresses frequency and increases amplitude (preserving area).

## Area Properties

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The value of  $F(\omega)$  at  $\omega = 0$  is the integral of  $f(t)$  over time  $t$ .

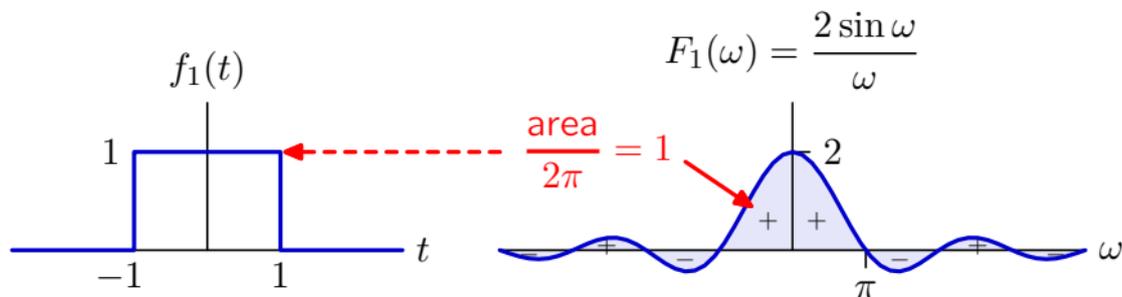
$$F(\omega)|_{\omega=0} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt = \int_{-\infty}^{\infty} f(t)e^{-j0t} dt = \int_{-\infty}^{\infty} f(t) dt$$



## Areas

The value of  $f(0)$  is the integral of  $F(\omega)$  divided by  $2\pi$ .

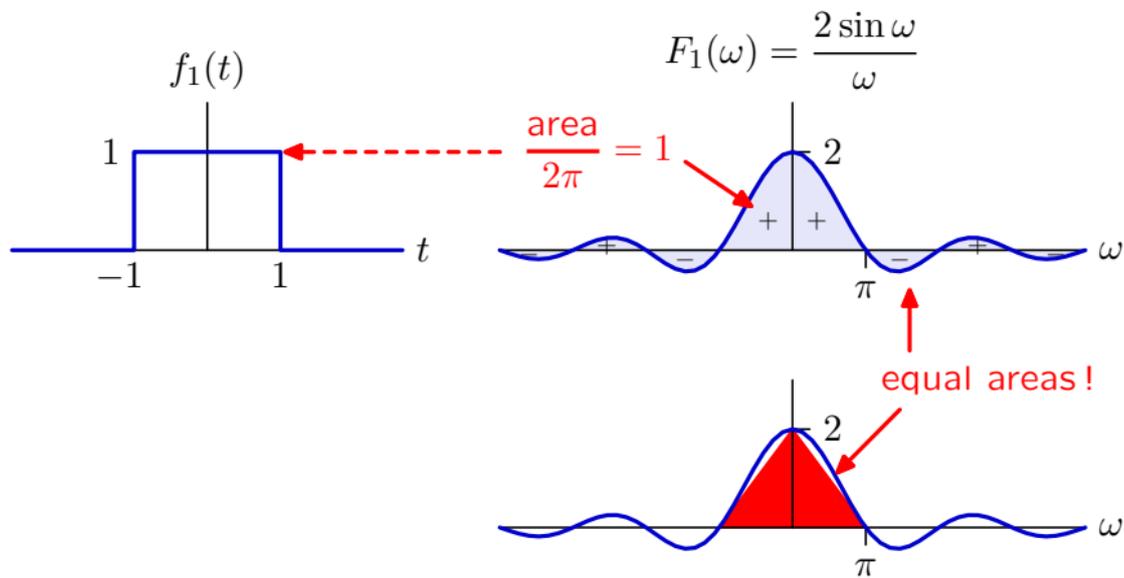
$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) d\omega$$



## Areas

The value of  $f(0)$  is the integral of  $F(\omega)$  divided by  $2\pi$ .

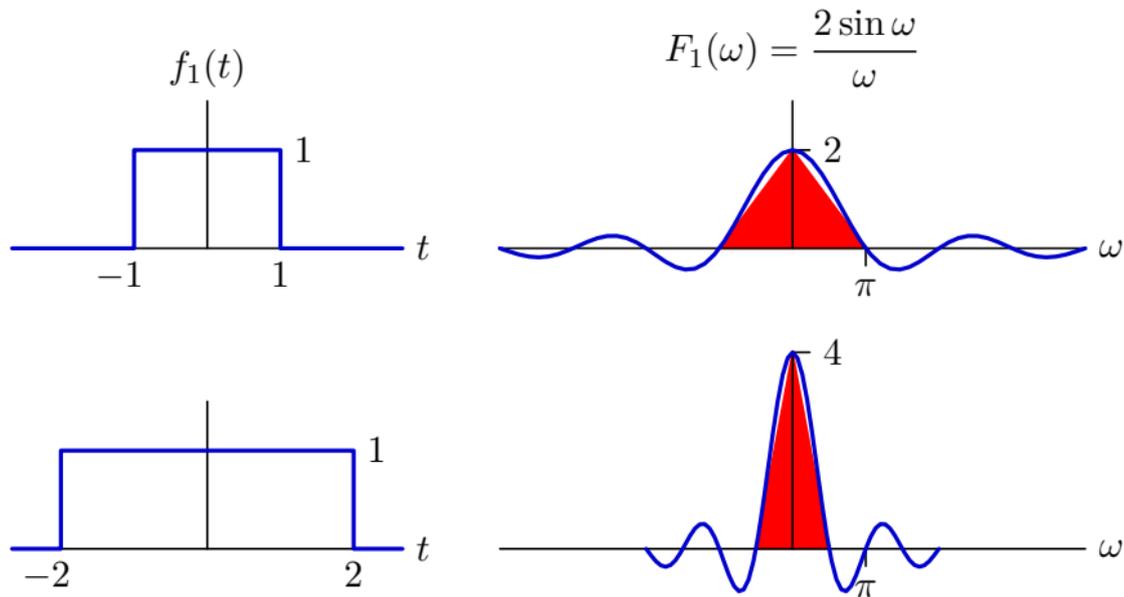
$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) d\omega$$



## Stretching Time

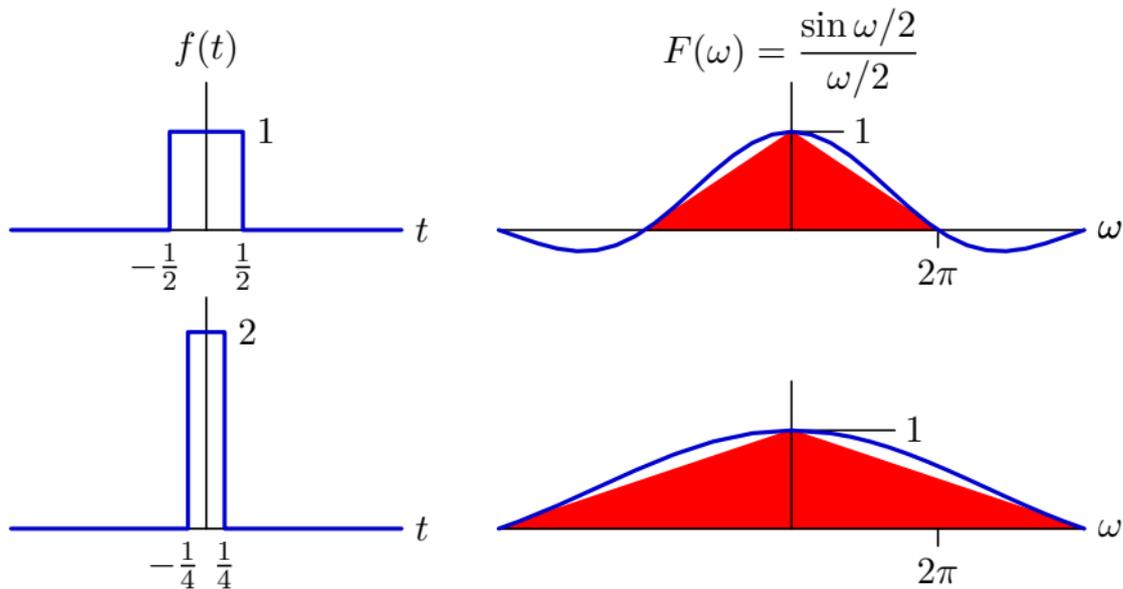
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Stretching time compresses frequency and increases amplitude (preserving area).



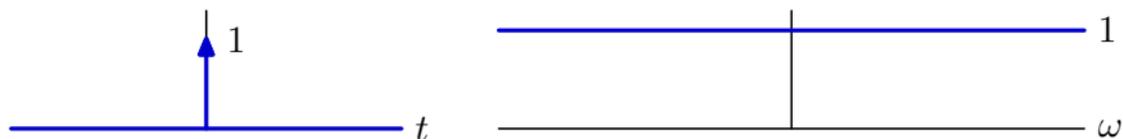
## Compressing Time to the Limit

Alternatively, we could compress time while keeping area = 1.



In the limit, the pulse has zero width but area 1!

We represent this limit with the delta (or impulse) function:  $\delta(t)$ .



## Math With Impulses

---

Although physically unrealizable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Example 1: Find the Fourier transform of a unit impulse function.

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t} dt$$

Since  $\delta(t)$  is zero except near  $t=0$ , only values of  $e^{-j\omega t}$  near  $t=0$  are important. Because  $e^{-j\omega t}$  is a smooth function of  $t$ ,  $e^{-j\omega t}$  can be replaced by  $e^{-j\omega 0}$ :

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega 0} dt = \int_{-\infty}^{\infty} \delta(t) dt = 1$$

This matches our previous result which was based explicitly on a limit. Here the limit is implicit.

## Math With Impulses

---

Although physically unrealizable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Example 2: Find the function whose Fourier transform is an impulse.

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j0t} d\omega = \frac{1}{2\pi}$$

$$1 \xrightarrow{\text{CTFT}} 2\pi\delta(\omega)$$

Notice the similarity to the previous result:

$$\delta(t) \xrightarrow{\text{CTFT}} 1$$

These relations are **duals** of each other.

- A constant in time consists of a single frequency at  $\omega = 0$ .
- An impulse in time contains components at all frequencies.

## Math With Impulses

---

Although physically unrealizable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Example 3: Find the function whose Fourier transform is a shifted impulse.

$$\begin{aligned}f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_o) e^{j\omega t} d\omega \\&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_o) e^{j\omega_o t} d\omega \\&= \frac{1}{2\pi} e^{j\omega_o t} \int_{-\infty}^{\infty} \delta(\omega - \omega_o) d\omega \\&= \frac{1}{2\pi} e^{j\omega_o t}\end{aligned}$$

$$e^{j\omega_o t} \stackrel{\text{CTFT}}{\implies} 2\pi \delta(\omega - \omega_o)$$

Use this result to relate Fourier series to Fourier transforms.

## Relation Between Fourier Series and Fourier Transforms

---

If a periodic signal  $f(t) = f(t + T)$  has a Fourier series representation, then it can also be represented by an equivalent Fourier transform.

$$e^{j\omega_0 t} \xrightarrow{\text{FT}} 2\pi\delta(\omega - \omega_0)$$

$$f(t) = f(t + T) = \sum_{k=-\infty}^{\infty} F[k] e^{j\frac{2\pi}{T}kt} \quad \begin{array}{c} \text{CTFS} \\ \longleftrightarrow \end{array} \quad F[k]$$

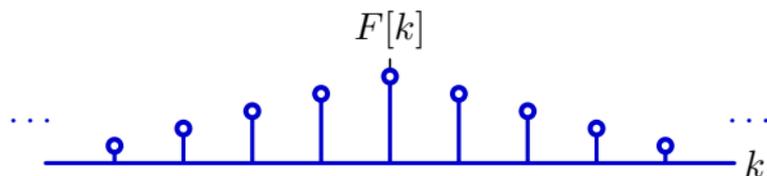
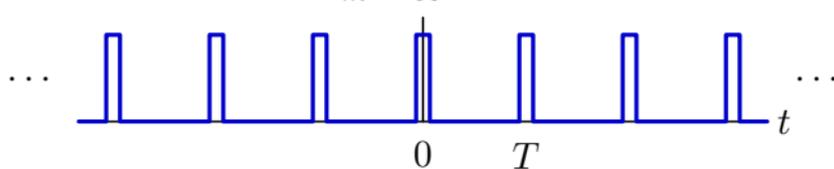
$$f(t) = f(t + T) = \sum_{k=-\infty}^{\infty} F[k] e^{j\frac{2\pi}{T}kt} \quad \begin{array}{c} \text{CTFT} \\ \longleftrightarrow \end{array} \quad \sum_{k=-\infty}^{\infty} 2\pi F[k] \delta\left(\omega - \frac{2\pi}{T}k\right)$$

Each term in the Fourier series is replaced by an impulse in the Fourier transform.

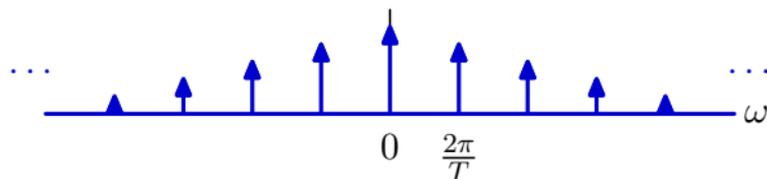
## Relation between Fourier Transform and Fourier Series

Each Fourier series term is replaced by an impulse in the Fourier transform.

$$f(t) = \sum_{m=-\infty}^{\infty} f(t - mT)$$



$$F(\omega) = \sum_{k=-\infty}^{\infty} 2\pi F[k] \delta\left(\omega - k\frac{2\pi}{T}\right)$$



## Summary

---

Fourier series and transforms are similar:  
both represent signals by their frequency content.

### Continuous-Time Fourier Series

$$F[k] = \frac{1}{T} \int_T f(t) e^{-jk\omega_0 t} dt$$

analysis equation

$$f(t) = f(t + T) = \sum_{k=-\infty}^{\infty} F[k] e^{jk\omega_0 t}$$

synthesis equation

$$\text{where } \omega_0 = \frac{2\pi}{T}$$

### Continuous-Time Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

analysis equation

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

synthesis equation

**Next time:** Fourier Transform for discrete-time signals.

## Question of the Day

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Determine a closed-form expression for  $\int_{-\infty}^{\infty} \delta(t-1)dt$ .

---

Please enter your response at the following url:



<http://bit.ly/4qehFmF>