6.3000: Signal Processing

Short-Time Fourier Transform

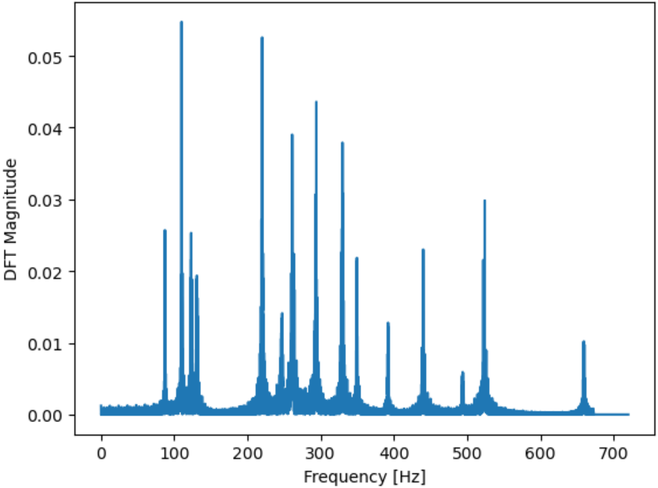
- Spectrograms
- Window Functions

April 01, 2025

Music Clip

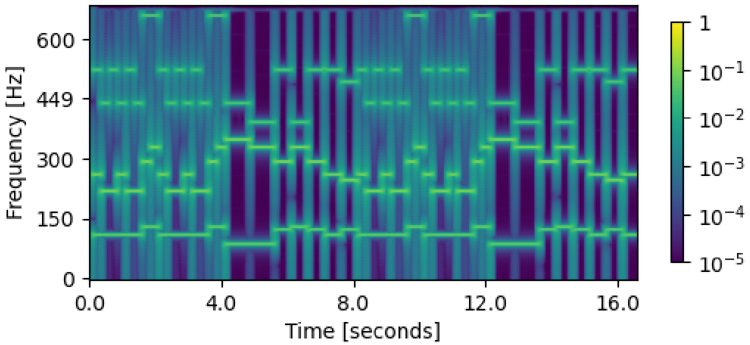
In lecture, we saw three representations for the same music clip.

The first was the magnitude of the DFT (shown below).



Music Clip

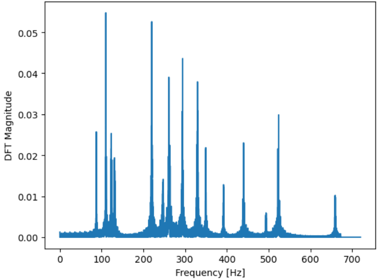
The second was the spectrogram.

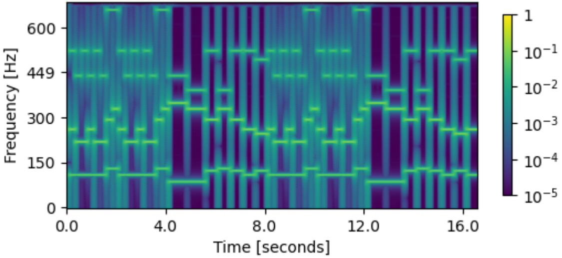


The third was the musical score.

Music Clip

Compare features of these representations.

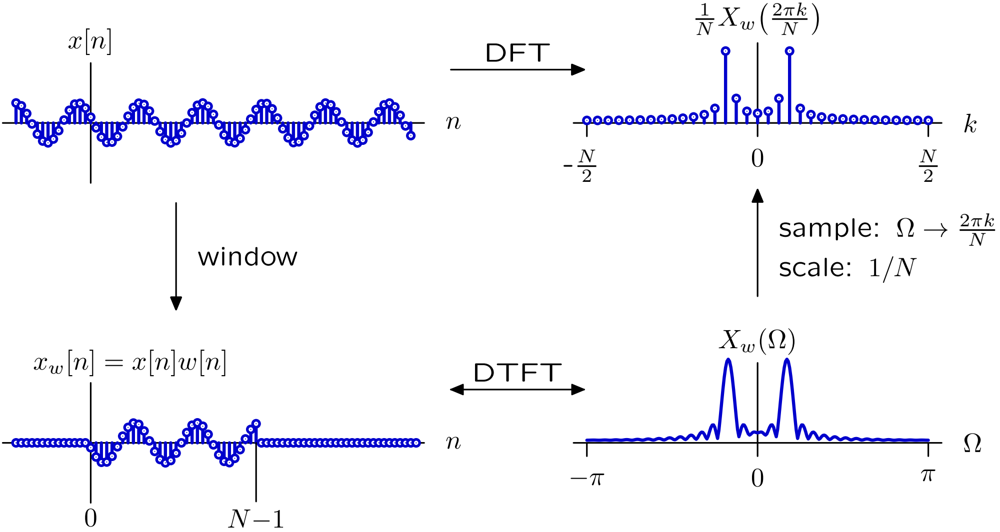




- What do these representations have in common?
  How are they different?
- Which representation has the best frequency resolution? Why?
- Which representation has the best time resolution? Why?
- The left panel has peaks with different heights. How do the different heights in the left panel correspond to features of the spectrogram?

Window Functions

A defining feature of the DFT is its finite length N, which plays a critical role in determining both time and frequency resolution.



The finite length constraint is equivalent to multiplication by a rectangular window. What would happen if we used a different type of window?

Window Functions

Dozens of different window functions are in common use. We will look at three of them:

- rectangular window
- triangular window
- Hann window

These and other window functions have a variety of different properties. We would like to understand which properties are important in which applications.

Rectangular Window

Definition:

$$w_r[n] = \begin{cases} \frac{1}{2M-1} & 0 \le n < 2M-1 \\ 0 & \text{otherwise} \end{cases}$$

- Make a plot of $w_r[n]$ versus n.
- Determine the DT Fourier Transform $W_r(\Omega)$.
- Make a plot of $W_r(\Omega)$ versus Ω.

Rectangular Window

Definition:

$$w_r[n] = \begin{cases} \frac{1}{2M-1} & 0 \leq n < 2M-1 \\ 0 & \text{otherwise} \end{cases}$$

One approach would be to close the following sum analytically:

$$W_r(\Omega) = \sum_{n=-\infty}^{\infty} w_r[n] e^{-j\Omega n}$$

Alternatively, we could evaluate the above sum for $\Omega = \frac{2\pi k}{N}$ using a DFT:

$$W_r\left(\frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{\infty} w_r[n] e^{-j\frac{2\pi k}{N} n}$$

Since $w_r[n] = 0$ outside the range $0 \leq n \leq 2M-2$ we can reduce the infinite sum to a finite sum, which can then be evaluated with a DFT.

$$W_r\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{2M-2} w_r[n] e^{-j\frac{2\pi k}{N} n} = N \times \text{DFT}\{w_r\}$$

We can choose the analysis length N of the DFT based on our desired frequency resolution.

Rectangular Window

Definition:

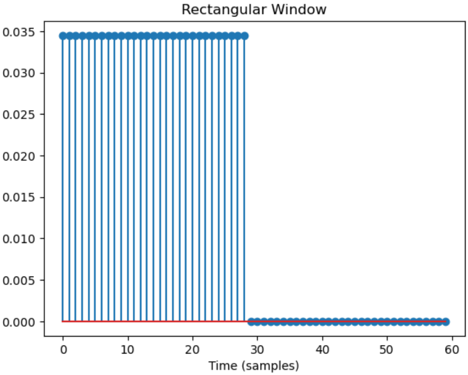
$$w_r[n] = \begin{cases} \frac{1}{2M-1} & 0 \le n < 2M-1 \\ 0 & \text{otherwise} \end{cases}$$

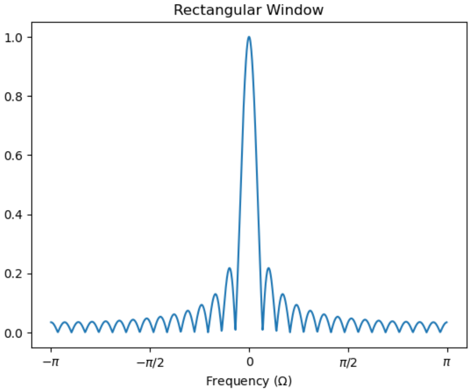
```
from matplotlib.pyplot import plot,stem,xticks,xlabel,title,legend,show
from lib6003.fft import fft
from math import pi,cos,sin

M = 15
N = 1024
w = [1/(2*M-1) for n in range(2*M-1)]
stem(w+(60-len(w))*[0])
xlabel('Time (samples)')
title('Rectangular Window')
show()
W = fft(w+(N-len(w))*[0])
plot([2*pi*k/N for k in range(-N//2,N//2)],[abs(W[k]/W[0]) for k in range(-N//2,N//2)])
xticks([-pi,-pi/2,0,pi/2,pi],['$-\pi$','$-\pi/2$','0','$\pi/2$','$\pi$'])
xlabel('Frequency ($\Omega$)')
title('Rectangular Window')
show()
```

Rectangular Window

Definition:

$$w_r[n] = \begin{cases} \frac{1}{2M-1} & 0 \leq n < 2M-1 \\ 0 & \text{otherwise} \end{cases}$$





Triangular Window

Definition:

$$w_t[n] = \begin{cases} \frac{n+1}{M^2} & \text{if } 0 \leq n < M \\ \frac{2M-n-1}{M^2} & \text{if } M \leq n < 2M-1 \\ 0 & \text{otherwise} \end{cases}$$

- Make a plot of $w_t[n]$ versus n.
- Determine the DT Fourier Transform $W_t(\Omega)$.
- Make a plot of $W_t(\Omega)$ versus Ω.

Triangular Window

Definition:

$$w_t[n] = \begin{cases} \frac{n+1}{M^2} & \text{if } 0 \leq n < M \\ \frac{2M-n-1}{M^2} & \text{if } M \leq n < 2M-1 \\ 0 & \text{otherwise} \end{cases}$$

```
from matplotlib.pyplot import ion,plot,stem,xticks,xlabel,title,legend,show
from lib6003.fft import fft
from math import pi,cos,sin

w = [(n+1)/M/M for n in range(M)]+[(2*M-n-1)/M/M for n in range(M,2*M-1)]
stem(w+100*[0])
xlabel('Time (samples)')
title('Triangular Window')
show()
W = fft(w+(N-len(w))*[0])
plot([2*pi*k/N for k in range(-N//2,N//2)],[abs(W[k]/W[0]) for k in range(-N//2,N//2)])
xticks([-pi,-pi/2,0,pi/2,pi],['$-\pi$','$-\pi/2$','0','$\pi/2$','$\pi$'])
xlabel('Frequency ($\Omega$)')
title('Triangular Window')
show()
```

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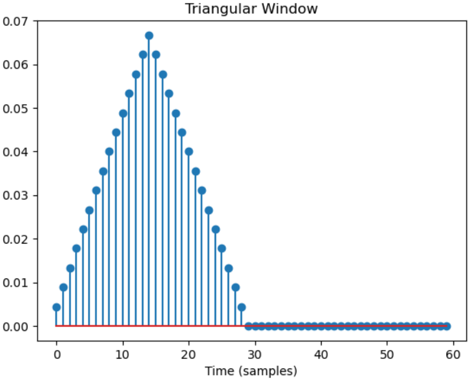
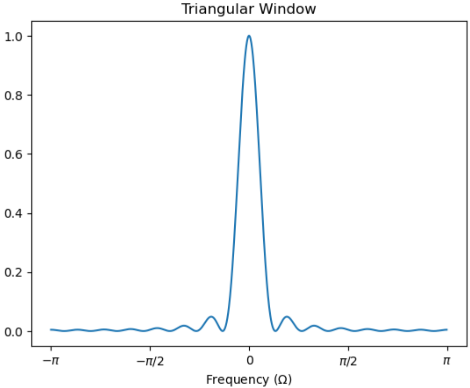
$$w_t[n] = \begin{cases} \frac{n+1}{M^2} & \text{if } 0 \le n < M \\ \frac{2M-n-1}{M^2} & \text{if } M \le n < 2M-1 \\ 0 & \text{otherwise} \end{cases}$$





Hann Window

Definition:

$$w_h[n] = \begin{cases} \frac{1}{5}\sin^2\left(\frac{\pi*(n+1)}{2M-1}\right) & 0 \le n < 2M-1 \\ 0 & \text{otherwise} \end{cases}$$

- Make a plot of $w_h[n]$ versus n.
- Determine the DT Fourier Transform $W_h(\Omega)$.
- Make a plot of $W_h(\Omega)$ versus Ω.

Hann Window

Definition:

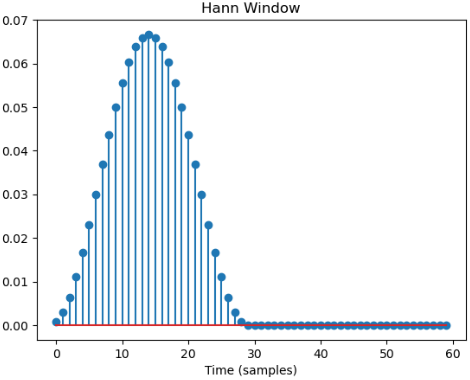
$$w_h[n] = \begin{cases} \frac{1}{5}\sin^2\left(\frac{\pi*(n+1)}{2M-1}\right) & 0 \leq n < 2M-1 \\ 0 & \text{otherwise} \end{cases}$$

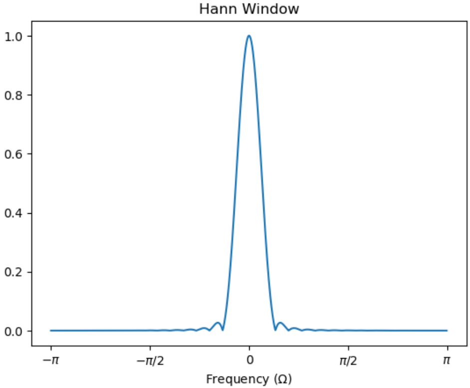
```
from matplotlib.pyplot import ion,plot,stem,xticks,xlabel,title,legend,show
from lib6003.fft import fft
from math import pi,cos,sin

M = 15
N = 1024
w = [sin(pi*(n+1)/(2*M))**2/M for n in range(2*M-1)]
stem(w+100*[0])
xlabel('Time (samples)')
title('Hann Window')
show()
W = fft(w+(N-len(w))*[0])
plot([2*pi*k/N for k in range(-N//2,N//2)],[abs(W[k]/W[0]) for k in range(-N//2,N//2)])
xticks([-pi,-pi/2,0,pi/2,pi],['$-\pi$','$-\pi/2$','0','$\pi/2$','$\pi$'])
xlabel('Frequency ($\Omega$)')
title('Hann Window')
show()
```

Hann Window

Definition:

$$w_h[n] = \begin{cases} \frac{1}{5}\sin^2\left(\frac{\pi*(n+1)}{2M-1}\right) & 0 \le n < 2M-1 \\ 0 & \text{otherwise} \end{cases}$$





Compare

Superpose the plots of $W_r(\Omega)$, $W_t(\Omega)$, and $W_h(\Omega)$.

What are the important differences?

Compare

Superpose the plots of $W_r(\Omega)$, $W_t(\Omega)$, and $W_h(\Omega)$.

What are the important differences?