6.3000: Signal Processing

Discrete-Time Fourier Transform

Synthesis Equation

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

Analysis Equation

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

February 27, 2025

Discrete-Time Fourier Transform

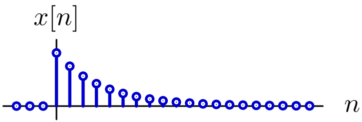
Find the Fourier transforms of the following discrete-time signals.

- $x_1[n] = \begin{cases} a^n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$
- $x_2[n] = x_1[n-n_0]$
- $x_3[n] = \text{Symmetric}\{x_1[n]\}$
- $x_4[n] = \text{Antisymmetric}\{x_1[n]\}$
- $x_5[n] = nx_1[n]$

Discrete-Time Fourier Transform

Find the Fourier transform of

$$x_1[n] = \begin{cases} a^n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$\begin{aligned} X_1(\Omega) &= \sum_{n=-\infty}^{\infty} x_1[n] e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} a^n e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} \left(a e^{-j\Omega}\right)^n \\ &= \frac{1}{1 - a e^{-j\Omega}} \quad \text{provided } |a| < 1 \end{aligned}$$

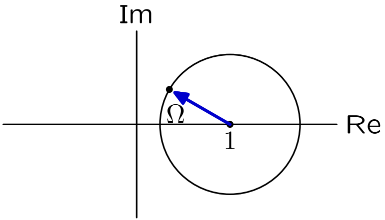
Discrete-Time Fourier Transform

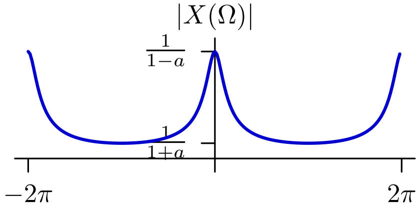
Plot the transform.

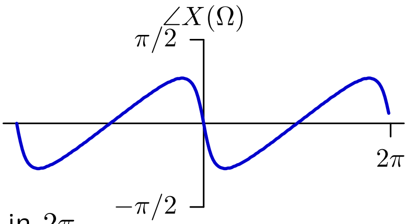
$$x_1[n] = \begin{cases} a^n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \overset{\text{DTFT}}{\Longrightarrow} \quad X_1(\Omega) = \frac{1}{1 - ae^{-j\Omega}}$$

Note that denominator is sum of 2 complex numbers.

$0 < a < 1$:







Magnitude is symmetric in Ω and periodic in 2π.

Angle is antisymmetric and periodic in 2π.

How would these change if $-1 < a < 0$?

Discrete-Time Fourier Transform

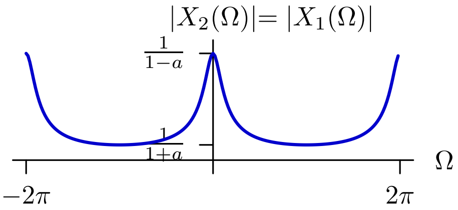
Find the Fourier transform of $x_2[n] = x_1[n - n_0]$.

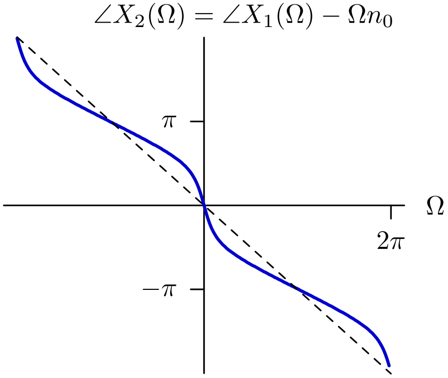
$$\begin{aligned}
X_2(\Omega) &= \sum_{n=-\infty}^{\infty} x_2[n] e^{-j\Omega n} \\
&= \sum_{n=-\infty}^{\infty} x_1[n - n_0] e^{-j\Omega n} \\
&= \sum_{m=-\infty}^{\infty} x_1[m] e^{-j\Omega(m+n_0)} \\
&= e^{-j\Omega n_0} \sum_{m=-\infty}^{\infty} x_1[m] e^{-j\Omega m} \\
&= e^{-j\Omega n_0}\, X_1(\Omega)
\end{aligned}$$

Discrete-Time Fourier Transform

Find the Fourier transform of $x_2[n] = x_1[n - n_0]$.

$$X_2(\Omega) = e^{-j\Omega n_0}\, X_1(\Omega)$$





Magnitude is unchanged.

Phase offset by $-\Omega n_0$.

- still antisymmetric?
- still periodic in 2π?

Discrete-Time Fourier Transform

Find the Fourier transform of $x_3[n] = \text{Symmetric}\{x_1[n]\}$.

$$
\begin{aligned}
X_3(\Omega) &= \sum_{n=-\infty}^{\infty} \frac{1}{2}(x_1[n] + x_1[-n])e^{-j\Omega n} \\
&= \frac{1}{2}\sum_{n=-\infty}^{\infty} x_1[n]\, e^{-j\Omega n} + \frac{1}{2}\sum_{n=-\infty}^{\infty} x_1[-n]\, e^{-j\Omega n} \\
&= \frac{1}{2}X_1(\Omega) + \frac{1}{2}\sum_{m=-\infty}^{\infty} x_1[m]\, e^{j\Omega m} \\
&= \frac{1}{2}X_1(\Omega) + \frac{1}{2}X_1(-\Omega) \\
&= \frac{1}{2}\left(\frac{1}{1-ae^{-j\Omega}} + \frac{1}{1-ae^{j\Omega}}\right) = \frac{1}{2}\left(\frac{1-ae^{-j\Omega}+1-ae^{j\Omega}}{1-ae^{-j\Omega}-ae^{j\Omega}+a^2}\right) \\
&= \frac{1-a\cos\Omega}{1-2a\cos\Omega+a^2}
\end{aligned}
$$

real and symmetric $\overset{\text{DTFT}}{\Longrightarrow}$ real and symmetric

Discrete-Time Fourier Transform

Find the Fourier transform of $x_4[n] = \text{Antisymmetric}\{x_1[n]\}$.

$$
\begin{aligned}
X_4\Omega &= \sum_{n=-\infty}^{\infty} \frac{1}{2}(x_1[n] - x_1[-n])e^{-j\Omega n} \\
&= \frac{1}{2}\sum_{n=-\infty}^{\infty} x_1[n]\, e^{-j\Omega n} - \frac{1}{2}\sum_{n=-\infty}^{\infty} x_1[-n]\, e^{-j\Omega n} \\
&= \frac{1}{2}X_1(\Omega) - \frac{1}{2}\sum_{m=-\infty}^{\infty} x_1[m]\, e^{j\Omega m} \\
&= \frac{1}{2}X_1(\Omega) - \frac{1}{2}X_1(-\Omega) \\
&= \frac{1}{2}\left(\frac{1}{1-ae^{-j\Omega}} - \frac{1}{1-ae^{j\Omega}}\right) = \frac{1}{2}\left(\frac{1-ae^{j\Omega}-1+ae^{-j\Omega}}{1-ae^{-j\Omega}-ae^{j\Omega}+a^2}\right) \\
&= \frac{-ja\sin\Omega}{1-2a\cos\Omega+a^2}
\end{aligned}
$$

real and antisymmetric $\overset{\text{DTFT}}{\Longrightarrow}$ imaginary and antisymmetric

Discrete-Time Fourier Transform

Find the Fourier transform of $x_5[n] = nx_1[n]$.

$$X_1(\Omega) = \sum_{n=-\infty}^{\infty} x_1[n]\, e^{-j\Omega n}$$

$$\frac{d}{d\Omega} X_1(\Omega) = \sum_{n=-\infty}^{\infty} x_1[n]\, (-jn) e^{-j\Omega n}$$

$$j\frac{d}{d\Omega} X_1(\Omega) = \sum_{n=-\infty}^{\infty} nx_1[n]\, e^{-j\Omega n}$$

$$x_1[n] \quad \overset{\text{DTFT}}{\Longrightarrow} \quad \frac{1}{1 - ae^{-j\Omega}}$$

$$x_5[n] \quad \overset{\text{DTFT}}{\Longrightarrow} \quad j\frac{d}{d\Omega}\left(\frac{1}{1 - ae^{-j\Omega}}\right)$$

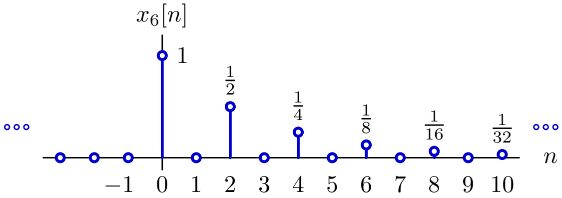
$$= -j\left(\frac{1}{1 - ae^{-j\Omega}}\right)^2 (-ae^{-j\Omega})(-j)$$

$$= \frac{ae^{-j\Omega}}{(1 - ae^{-j\Omega})^2}$$

Discrete-Time Fourier Transform

Find the Fourier transform of $x_6[n]$:

$$x_6[n] = \begin{cases} \left(\frac{1}{2}\right)^{n/2} & n = 0, 2, 4, 6, 8, ..., \infty \\ 0 & \text{otherwise} \end{cases}$$



Discrete-Time Fourier Transform

Find the Fourier transform of $x_6[n]$:

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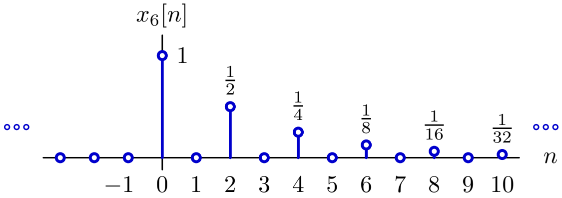

$$X_6(\Omega) = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \left(\frac{1}{2}\right)^{n/2} e^{-j\Omega n}$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m} e^{-j\Omega 2m} = \sum_{m=0}^{\infty} \left(\frac{1}{2} e^{-j2\Omega}\right)^{m} = \frac{1}{1 - \frac{1}{2} e^{-j2\Omega}}$$

Stretching in time $\rightarrow$ compressing in frequency

Inverse Discrete-Time Fourier Transform

Find the signal whose Fourier transform is

$$X(\Omega) = e^{-j3\Omega}$$

Inverse Discrete-Time Fourier Transform

Find the signal whose Fourier transform is

$$X(\Omega) = e^{-j3\Omega}$$

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\int_{2\pi} e^{-j3\Omega}e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\int_{2\pi} e^{j\Omega(n-3)}d\Omega \\
&= \begin{cases} 1 & n = 3 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$