6.3000: Signal Processing

Continuous-Time Fourier Transform

Synthesis Equation

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Analysis Equation

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

February 25, 2025

Continuous-Time Fourier Transform

Find the Fourier transforms of the following continuous-time signals.

- $x_1(t) = \begin{cases} e^{-t} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$
- $x_2(t) = x_1(t - t_0)$
- $x_3(t) = \text{Symmetric}\{x_1(t)\}$
- $x_4(t) = \text{Antisymmetric}\{x_1(t)\}$
- $x_5(t) = \frac{d}{dt}\text{Symmetric}\{x_1(t)\}$

Continuous-Time Fourier Transform

Find the Fourier transform of the following signal

$$x_1(t) = \begin{cases} e^{-t} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

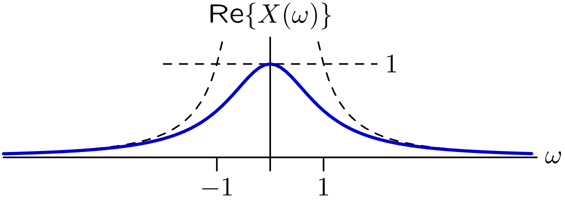
$$\begin{aligned} X_1(\omega) &= \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt \\ &= \int_{0}^{\infty} e^{-t} e^{-j\omega t} dt \\ &= \int_{0}^{\infty} e^{-(1+j\omega)t} dt \\ &= -\frac{1}{1+j\omega} e^{-(1+j\omega)t} \Big|_0^{\infty} \\ &= \frac{1}{1+j\omega} \end{aligned}$$

Continuous-Time Fourier Transform

Find the real and imaginary parts of $X(\omega)$.

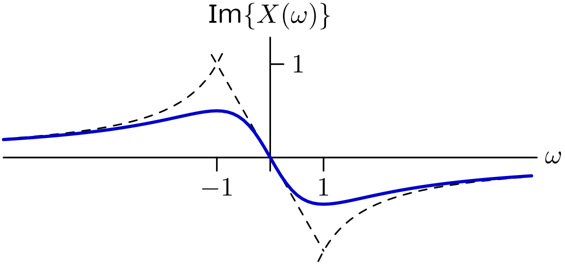
$$X(\omega) = \frac{1}{1+j\omega} \times \frac{1-j\omega}{1-j\omega} = \frac{1-j\omega}{1+\omega^2}$$

Plot these functions.



$|\omega| << 1: \quad \frac{1}{1+\omega^2} \to 1$

$|\omega| >> 1: \quad \frac{1}{1+\omega^2} \to \frac{1}{\omega^2}$



$|\omega| << 1: \quad -\frac{\omega}{1+\omega^2} \to -\omega$

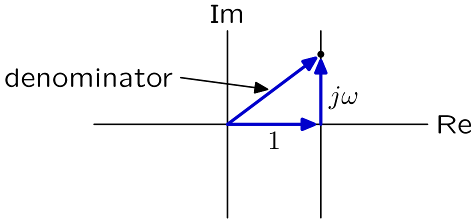
$|\omega| >> 1: \quad -\frac{\omega}{1+\omega^2} \to -\frac{1}{\omega}$

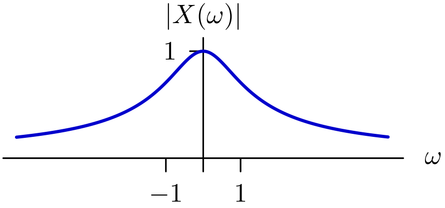
Continuous-Time Fourier Transform

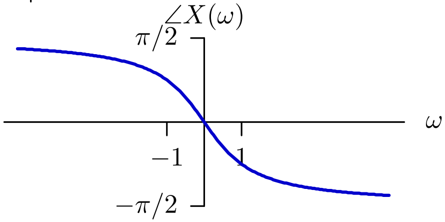
Alternative graphical method.

$$X(\omega) = \frac{1}{1 + j\omega}$$

The denominator is the sum of two vectors.





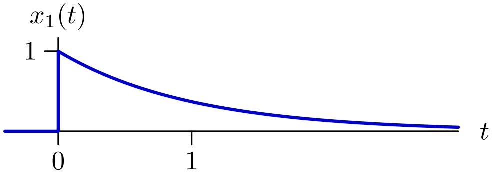


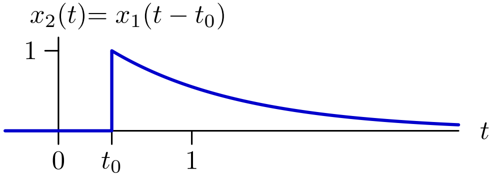
Magnitude is symmetric, angle is antisymmetric in Ω.

Continuous-Time Fourier Transform

Find the Fourier transform of the following signal.

$$x_2(t) = x_1(t - t_0)$$





Continuous-Time Fourier Transform

Find the Fourier transform of the following signal.

$$x_2(t) = x_1(t - t_0)$$

$$\begin{aligned} X_2(\omega) &= \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} \, dt \\ &= \int_{-\infty}^{\infty} x_1(t - t_0) e^{-j\omega t} \, dt \end{aligned}$$

Let $\tau = t - t_0$.

$$\begin{aligned} X_2(\omega) &= \int_{-\infty}^{\infty} x_1(\tau) e^{-j\omega(\tau + t_0)} \, d\tau \\ &= e^{-j\omega t_0} \int_{-\infty}^{\infty} x_1(\tau) e^{-j\omega \tau} \, d\tau \\ &= e^{-j\omega t_0} X_1(\omega) \end{aligned}$$

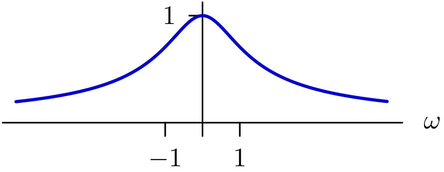
Continuous-Time Fourier Transform

Find the Fourier transform of the following signal.

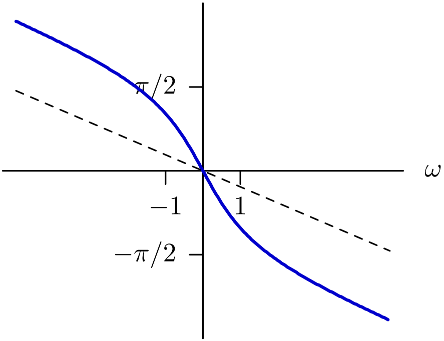
$$x_2(t) = x_1(t - t_0)$$

$$X_2(\omega) = e^{-j\omega t_0} X_1(\omega)$$

$$|X_2(\omega)| = |X_1(\omega)|$$



$$\angle X_2(\omega) = \angle X_1(\omega) - \omega t_0$$



Magnitude is unchanged.

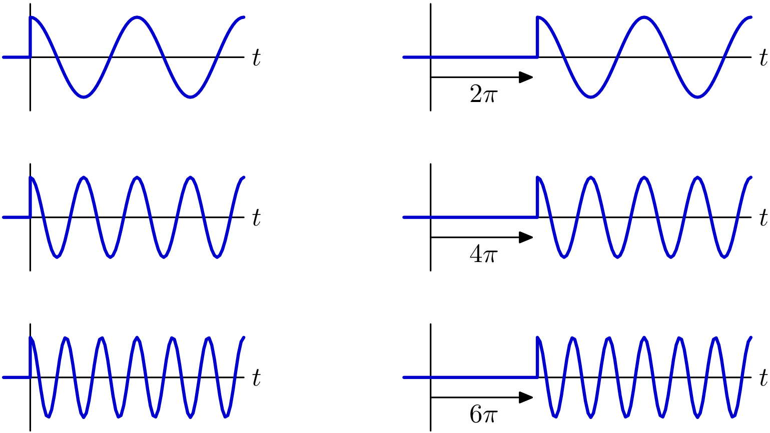
Angle offset by straight line through zero (still antisymmetric).

Why does time delay shift phase by angle proportional to frequency?

Continuous-Time Fourier Transform

Why does time delay shift phase by angle proportional to frequency?

Think about Fourier components of a signal that are each delayed by same time t_0.



The same amount of time corresponds to different amounts of phase.

Continuous-Time Fourier Transform

Find the Fourier transform of the following signal.

$$x_3(t) = \text{Symmetric}\{x_1(t)\}$$

$$\begin{aligned}
X_3(\omega) &= \int_{-\infty}^{\infty} \text{Symmetric}\{x_1(t)\} e^{-j\omega t} dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} (x_1(t) + x_1(-t)) e^{-j\omega t} dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + \frac{1}{2}\int_{-\infty}^{\infty} x_1(-t) e^{-j\omega t} dt \\
&= \frac{1}{2} X_1(\omega) + \frac{1}{2}\int_{-\infty}^{\infty} x_1(\tau) e^{j\omega\tau} d\tau \\
&= \frac{1}{2} X_1(\omega) + \frac{1}{2} X_1(-\omega) \\
&= \frac{1}{2}\left(\frac{1}{1+j\omega} + \frac{1}{1-j\omega}\right) = \frac{1}{1+\omega^2}
\end{aligned}$$

real and symmetric $\overset{\text{CTFT}}{\Longrightarrow}$ real and symmetric

Continuous-Time Fourier Transform

Find the Fourier transform of the following signal.

$$x_4(t) = \text{Antisymmetric}\{x_1(t)\}$$

$$\begin{aligned}
X_4(\omega) &= \int_{-\infty}^{\infty} \text{Antisymmetric}\{x_1(t)\} e^{-j\omega t} dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} (x_1(t) - x_1(-t)) e^{-j\omega t} dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt - \frac{1}{2}\int_{-\infty}^{\infty} x_1(-t) e^{-j\omega t} dt \\
&= \frac{1}{2} X_1(\omega) - \frac{1}{2}\int_{-\infty}^{\infty} x_1(\tau) e^{j\omega\tau} d\tau \\
&= \frac{1}{2} X_1(\omega) - \frac{1}{2} X_1(-\omega) \\
&= \frac{1}{2}\Big(\frac{1}{1+j\omega} - \frac{1}{1-j\omega}\Big) = \frac{-j\omega}{1+\omega^2}
\end{aligned}$$

real and antisymmetric $\overset{\text{CTFT}}{\Longrightarrow}$ imaginary and antisymmetric

Continuous-Time Fourier Transform

Find the Fourier transform of the following signal.

$$x_5(t) = \frac{d}{dt}\text{Symmetric}\{x_1(t)\}$$

$$X_5(\omega) = \int_{-\infty}^{\infty} \underbrace{\frac{dx_3(t)}{dt}}_{dv} \underbrace{e^{-j\omega t}}_{u} \underbrace{dt}_{dv}$$

$$= \underbrace{e^{-j\omega t}}_{u} \underbrace{x_3(t)}_{v} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \underbrace{x_3(t)}_{v} \underbrace{(-j\omega)e^{-j\omega t}dt}_{du}$$

The first term is zero since the CTFT of $x_3(t)$ converged.

$$X_5(\omega) = j\omega \int_{-\infty}^{\infty} x_3(t)\, e^{-j\omega t}dt$$

$$= j\omega X_3(\omega) = \frac{j\omega}{1+\omega^2}$$

real and antisymmetric $\overset{\text{CTFT}}{\Longrightarrow}$ imaginary and antisymmetric

Continuous-Time Fourier Transform

Alternative method – start with synthesis equation:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

$$y(t) = \frac{dx(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{d}{dt}e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)(j\omega)e^{j\omega t}\,d\omega$$

$$Y(\omega) = j\omega X(\omega)$$

Straightforward approach (start with analysis) not always simplest.

This is a great example of how Fourier transforms simplify calculus.

Differentiation in time corresponds to multiplication by $j\omega$ in frequency.

Inverse Continuous-Time Fourier Transform

Find the signal whose Fourier transform is

$$X(\omega) = e^{-|\omega|}$$

Inverse Continuous-Time Fourier Transform

Find the signal whose Fourier transform is

$$X(\omega) = e^{-|\omega|}$$

$$\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} e^{\omega}e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{0}^{\infty} e^{-\omega}e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\left.\frac{e^{(jt+1)\omega}}{jt+1}\right|_{-\infty}^{0} + \frac{1}{2\pi}\left.\frac{e^{(jt-1)\omega}}{jt-1}\right|_{0}^{\infty} \\
&= \frac{1}{2\pi}\frac{1}{jt+1} - \frac{1}{2\pi}\frac{1}{jt-1} \\
&= \frac{1/\pi}{1+t^2}
\end{aligned}$$