6.3000 Final Exam Fall 2023

Name: Solutions

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This quiz is closed book, but you may use three 8.5×11 sheets of notes (six sides).

You may NOT use any electronic devices (such as calculators and phones).

If you have questions, please **come to us** at the front of the room to ask.

Please enter all solutions in the boxes provided.

Work on other pages with QR codes will be considered for partial credit.
Please provide a note if you continue work on worksheets at the end of the exam.

Please do not write on the QR codes at the bottom of each page.

We use those codes to identify which pages belong to each student.

Trigonometric Identities Reference

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \qquad \cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$

$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) \qquad \cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) \qquad \sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b) \qquad \cos(a+b)-\cos(a-b) = -2\sin(a)\sin(b)$$

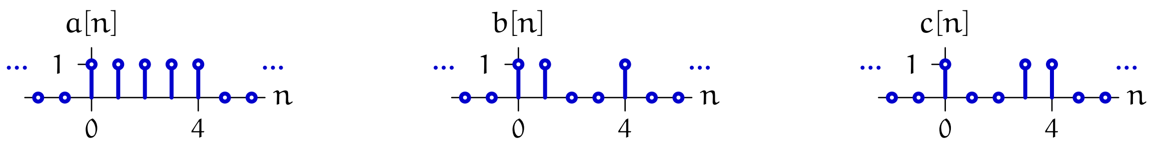
$$\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b) \qquad \sin(a+b)-\sin(a-b) = 2\cos(a)\sin(b)$$

$$2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b) \qquad 2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$$

$$2\sin(a)\cos(b) = \sin(a+b)+\sin(a-b) \qquad 2\cos(a)\sin(b) = \sin(a+b)-\sin(a-b)$$

1 Discrete-Time Convolutions

Values of the following discrete-time signals are zero outside the indicated ranges.



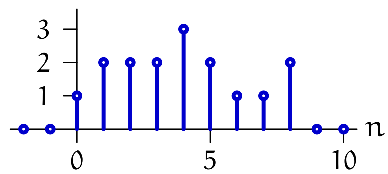
For each part below, determine if the target signal in the right column can be constructed by convolving (a or b or c) with (a or b or c). If it can, indicate which signals should be convolved. If it cannot, put an $\times$ in both boxes.

Notice that there are ten possible answers:

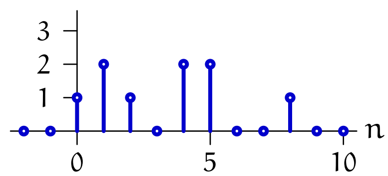
$(a * a)$, $(a * b)$, $(a * c)$, $(b * a)$, $(b * b)$, $(b * c)$, $(c * a)$, $(c * b)$, $(c * c)$, or $(\times, \times)$.

There may be more than one correct answer for each part, but you only need to find one correct answer for full credit.

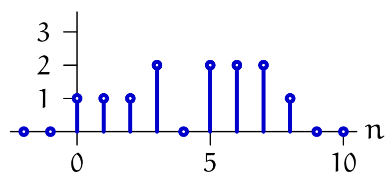
$\boxed{\times}$ (Enter a, b, c, or $\times$) $*$ $\boxed{\times}$ (Enter a, b, c, or $\times$) $=$



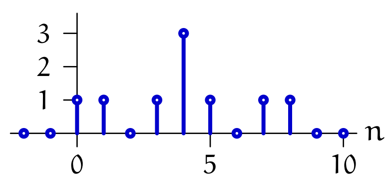
$\boxed{b}$ (Enter a, b, c, or $\times$) $*$ $\boxed{b}$ (Enter a, b, c, or $\times$) $=$



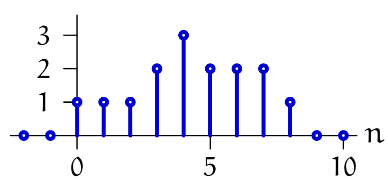
$\boxed{\times}$ (Enter a, b, c, or $\times$) $*$ $\boxed{\times}$ (Enter a, b, c, or $\times$) $=$



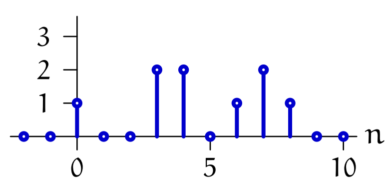
$\boxed{b \; c}$ (Enter a, b, c, or $\times$) $*$ $\boxed{c \; b}$ (Enter a, b, c, or $\times$) $=$



$\boxed{a \; c}$ (Enter a, b, c, or $\times$) $*$ $\boxed{c \; a}$ (Enter a, b, c, or $\times$) $=$



$\boxed{c}$ (Enter a, b, c, or $\times$) $*$ $\boxed{c}$ (Enter a, b, c, or $\times$) $=$



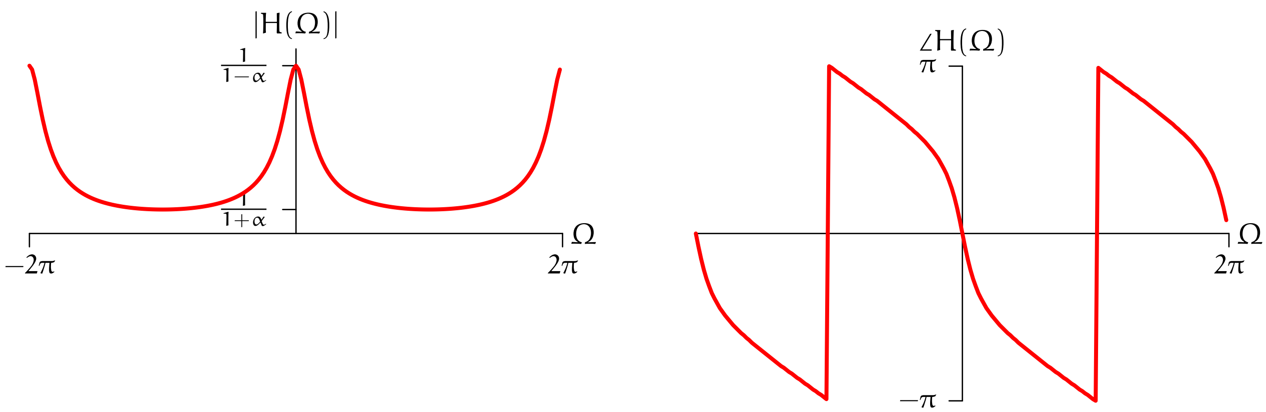
Worksheet (*intentionally blank*)

2 Frequency Response

Sketch the magnitude and angle of the frequency response of a linear, time-invariant system whose unit-sample response is given by

$$h[n] = \begin{cases} \alpha^{n-1} & \text{if } n \geqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha = 0.75$.



The unit-sample response $h[n]$ is a delayed version of a right-sided geometric sequence $g[n]$

$$g[n] = \alpha^n u[n]$$

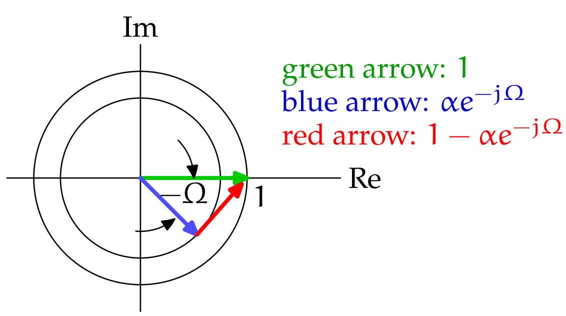
which has a Fourier transform

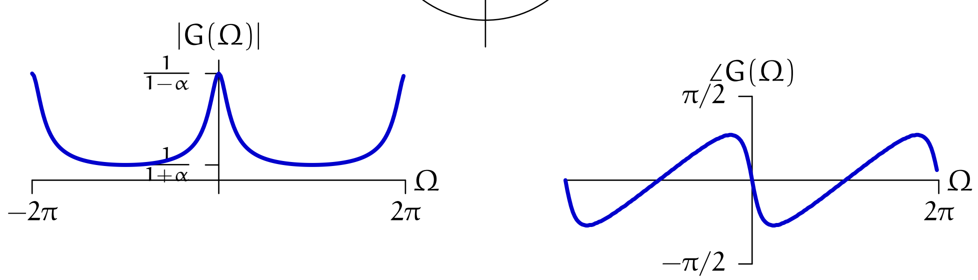
$$G(\Omega) = \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega n} = \frac{1}{1-\alpha e^{-j\Omega}}$$

The delay of one sample introduces a multiplicative phase factor $e^{-j\Omega}$ so the Fourier transform of $h[n]$ is

$$H(\Omega) = e^{-j\Omega} G(\Omega) = \frac{e^{-j\Omega}}{1-\alpha e^{-j\Omega}}$$

Note that denominator is the difference of 2 complex numbers.
If $0 < \alpha < 1$:





The multiplicative delay term $e^{-j\Omega}$ in $H(\Omega)$ has no effect on magnitude so

$$|H(\Omega)| = |G(\Omega)|$$

but subtracts Ω from each phase:

$$\angle H(\Omega) = \angle G(\Omega) - \Omega$$

3 Double DFTs

Let $f[n]$ represent a periodic discrete-time signal with period 4 as described by the following:

$f[n] = n \bmod 4$

Part a. Let $F_4[k]$ represent the discrete Fourier transform (DFT) of $f[n]$ when the analysis window $N = 4$. Determine the numerical values of $F_4[k]$ and enter these values in the boxes below.
Each entry should be simplified to a single complex-valued number of the form $a+jb$ (no sums of complex exponentials).

$F_4[0]$:	1.5
$F_4[1]$:	$-0.5+0.5j$
$F_4[2]$:	-0.5
$F_4[3]$:	$-0.5-0.5j$

Briefly explain your reasoning in the box below.

$$F_4[k] = \frac{1}{4}\sum_{n=0}^{3} f[n]e^{-j2\pi kn/4} = \frac{1}{4}\left(0 + e^{-j2\pi k/4} + 2e^{-j2\pi 2k/4} + 3e^{-j2\pi 3k/4}\right)$$

Evaluate this expression for $k = 0, 1, 2$ and 3.

Part b. Let $F_8[k]$ represent the DFT of $f[n]$ (same $f[n]$ as in part a) when the analysis window $N = 8$.
Determine the numerical values of $F_8[k]$ and enter these values in the boxes below.
Each entry should be simplified to a single complex-valued number of the form $a+jb$ (no sums of complex exponentials).

$F_8[0]$:	1.5
$F_8[1]$:	0
$F_8[2]$:	$-0.5 + 0.5j$
$F_8[3]$:	0
$F_8[4]$:	-0.5
$F_8[5]$:	0
$F_8[6]$:	$-0.5-0.5j$
$F_8[7]$:	0

Briefly explain your reasoning in the box below.

Write the expression for $G_8[k]$. Break that expression into two pieces: $n = 0$ to 3 and $n = 4$ to 7. Write each piece in terms of $G_4[k]$. See detailed argument below.

$$
\begin{aligned}
F_8[k] &= \frac{1}{8}\sum_{n=0}^{7} f[n]e^{-j2\pi kn/8} \\
&= \frac{1}{8}\left(0 + e^{-j2\pi k/8} + 2e^{-j2\pi 2k/8} + 3e^{-j2\pi 3k/8} + 0 + e^{-j2\pi 5k/8} + 2e^{-j2\pi 6k/8} + 3e^{-j2\pi 7k/8}\right) \\
&= \frac{1}{8}\left(0 + e^{-j2\pi k/8} + 2e^{-j2\pi 2k/8} + 3e^{-j2\pi 3k/8} + 0 + (-1)^k e^{-j2\pi 1k/8} + 2(-1)^k e^{-j2\pi 2k/8} + 3(-1)^k e^{-j2\pi 3k/8}\right) \\
&= \begin{cases} \frac{1}{4}\left(e^{-j2\pi k/8} + 2e^{-j2\pi 2k/8} + 3e^{-j2\pi 3k/8}\right) & \text{if k is even} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{4}\left(e^{-j2\pi (k/2)/4} + 2e^{-j2\pi 2(k/2)/4} + 3e^{-j2\pi 3(k/2)/4}\right) & \text{if k is even} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} F_4[k/2] & \text{if k is even} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Part c. Let $g[n]$ represent a periodic discrete-time signal with period 4.
Let $G_4[k]$ represent the DFT of $g[n]$ when the analysis window is $N = 4$.
Let $G_8[k]$ represent the DFT of $g[n]$ when the analysis window is $N = 8$.

Determine an expression for $G_8[k]$ in terms of $G_4[k]$.
Your relation should not include values of $g[n]$.
Enter your relation in the box below.

$$G_8[k] = \boxed{\begin{cases} G_4[k/2] & \text{if } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}}$$

Briefly explain your reasoning.

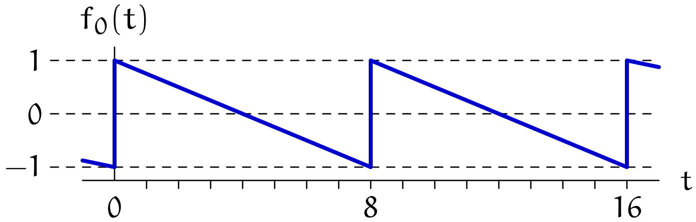
$$G_4[k] = \frac{1}{4}\sum_{n=0}^{3} g[n]e^{-j2\pi kn/4}$$

$$\begin{aligned}
G_8[k] &= \frac{1}{8}\sum_{n=0}^{7} g[n]e^{-j2\pi kn/8} \\
&= \frac{1}{8}\sum_{n=0}^{3} g[n]e^{-j2\pi kn/8} + \frac{1}{8}\sum_{n=4}^{7} g[n]e^{-j2\pi kn/8} \\
&= \frac{1}{8}\sum_{n=0}^{3} g[n]e^{-j2\pi kn/8} + \frac{1}{8}\sum_{n=0}^{3} g[n+4]e^{-j2\pi k(n+4)/8} \\
&= \frac{1}{8}\sum_{n=0}^{3} g[n]e^{-j2\pi kn/8} + \frac{1}{8}\sum_{n=0}^{3} g[n]e^{-j2\pi kn/8}e^{-j2\pi k4/8} \\
&= \frac{1}{8}\sum_{n=0}^{3} g[n]e^{-j2\pi kn/8} + \frac{1}{8}\sum_{n=0}^{3} g[n]e^{-j2\pi kn/8}(-1)^k \\
&= \frac{1+(-1)^k}{8}\sum_{n=0}^{3} g[n]e^{-j2\pi kn/8} \\
&= \begin{cases} G_4[k/2] & \text{if } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

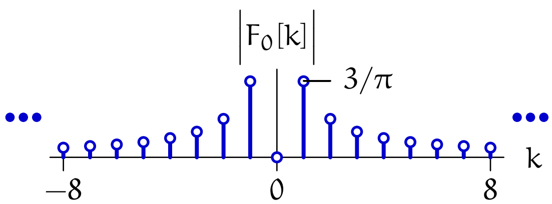
Worksheet (*intentionally blank*)

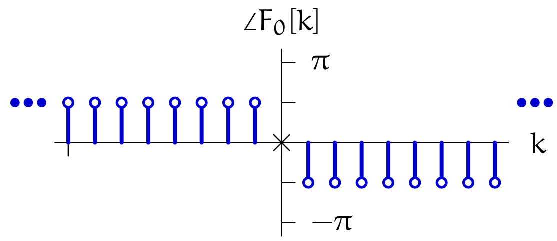
4 Sawtooth Waves

Let $f_0(t)$ represent the following continuous-time, periodic signal with period T=8

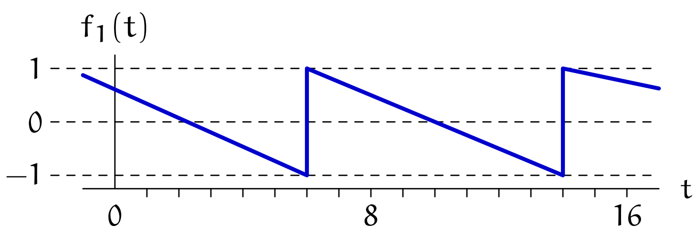


and let $F_0[k]$ represent the Fourier series coefficients of $f_0(t)$. The following plots show the magnitude and angle of $F_0[k]$.

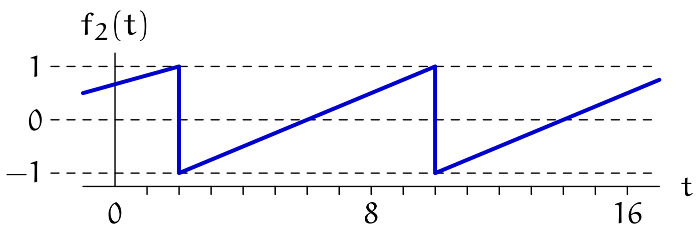




Each of the following parts defines a new signal $f_i(t)$ that is related to $f_0(t)$. Let $F_i[k]$ represent the Fourier series coefficients for $f_i(t)$. Determine which of panels A-F on the next page shows the magnitude of $F_i[k]$ and which of panels a-f shows the angle of $F_i[k]$. Enter your answers in the boxes provided.



$|F_1[k]| =$ **A** $\angle F_1[k] =$ **e**



$|F_2[k]| =$ **A** $\angle F_2[k] =$ **c**

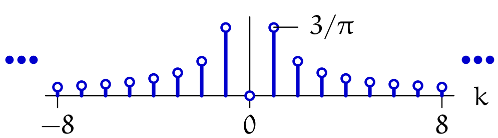
$f_3(t) = f_0(t) + f_1(t)$ $|F_3[k]| =$ **C** $\angle F_3[k] =$ **f**

$f_4(t) = f_0(t) + f_2(t)$ $|F_4[k]| =$ **D** $\angle F_4[k] =$ **d**

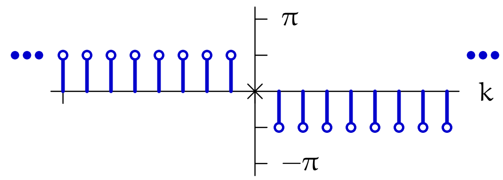
The following stem plots show six magnitude functions (A-F) in the left column and six angle functions (a-f) in the right column.

The $\times$ symbols in the angle plots represent values that are not defined because the magnitude at that k is zero.

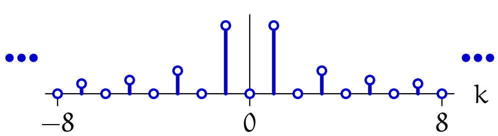
A



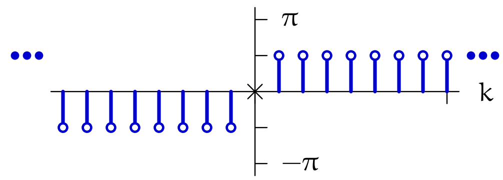
a



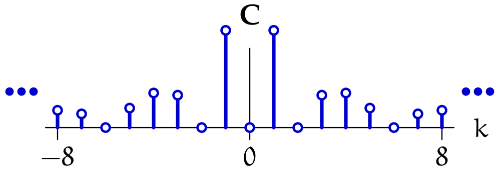
B



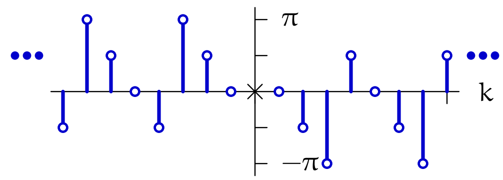
b



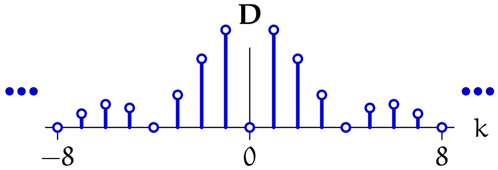
C



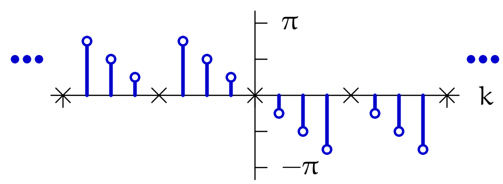
c



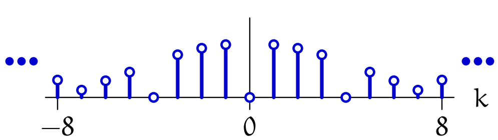
D



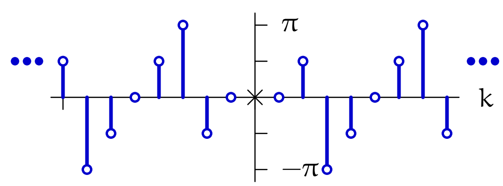
d



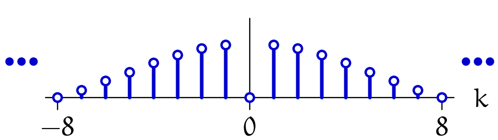
E



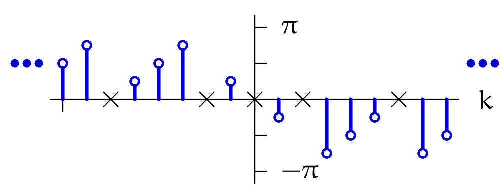
e



F



f



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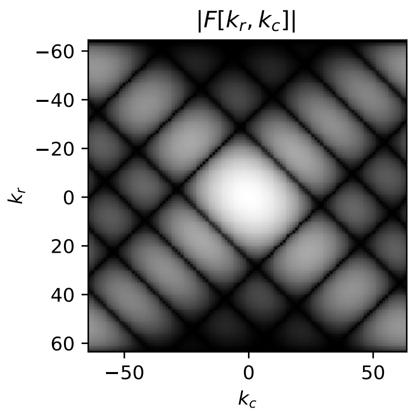
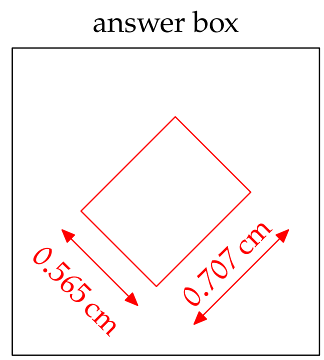
5 Mystery Photograph

A small white object is placed on a black background and photographed to produce an image $f[r, c]$, where r represents the row number and c represents the column number of pixels whose values range from 0 (black) to 1 (white). The image contains 128×128 pixels representing an area that measures 12.8 cm by 12.8 cm. The magnitude of the discrete Fourier transform (DFT) of this image is shown below.

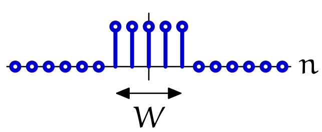


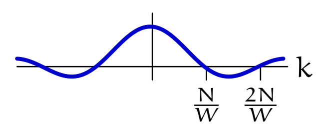


In the "answer" box above, sketch the approximate shape and orientation of the small white object as viewed in the spatial domain (r, c). Your sketch does not have to be to scale, but should include labels to indicate the dimensions of the object measured in centimeters (cm).

Briefly explain your reasoning.

The 2D DFT has the form of a sinc function in both of the diagonal directions. The inverse transform of a 2D sinc is a rectangle.





Except for $k = 0$, the zeros of the sinc function are at integer multiples of N/W. There are approximately 8 zero crossings for the diagonal with negative slope. Thus

$$\frac{8N}{W} \approx N\sqrt{2}$$

and $W \approx \frac{8}{\sqrt{2}} \approx 5.65$ pixels. There are approximately 10 zero crossings for the diagonal with positive slope. Thus

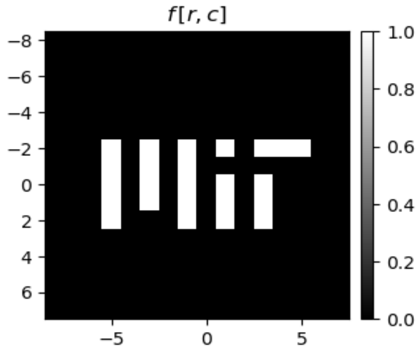
$$\frac{10N}{W} \approx N\sqrt{2}$$

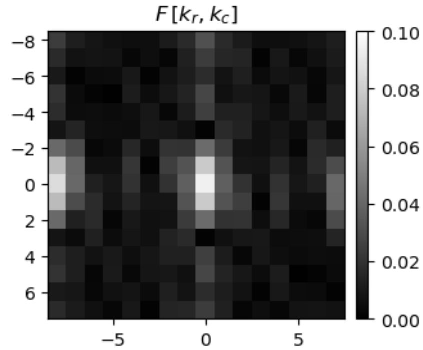
and $W \approx \frac{10}{\sqrt{2}} \approx 7.07$ pixels.

Pixels in the original image are separated by 12.8 cm divided by 128 pixels. Thus the sampling interval is 0.1 cm. Thus the rectangle is 0.707 cm by 0.565 cm.

6 Combining Images

Let $f[r, c]$ represent the MIT logo shown below where r represents the row number and c represents the column number of each pixel, and each pixel is either black (0) or white (1). Let $F[k_r, k_c]$ represent the two-dimensional (2D) Discrete Fourier Transform (DFT) of $f[r, c]$.





Let $h_1[r, c]$ represent a checkerboard image

$$h_1[r, c] = \frac{1}{2} + \frac{1}{2}(-1)^{(r+c)}$$

and let $H_1[k_r, k_c]$ represent the 2D DFT of $h_1[r, c]$.

Let $H_2[k_r, k_c]$ represent a two-dimensional DFT with a checkerboard pattern

$$H_2[k_r, k_c] = \frac{1}{2} + \frac{1}{2}(-1)^{(k_r+k_c)}$$

and let $h_2[r, c]$ represent the two-dimensional inverse DFT of $H_2[k_r, k_c]$.

Determine which of the images on the following page (A-I) is represented by each of the expressions below.

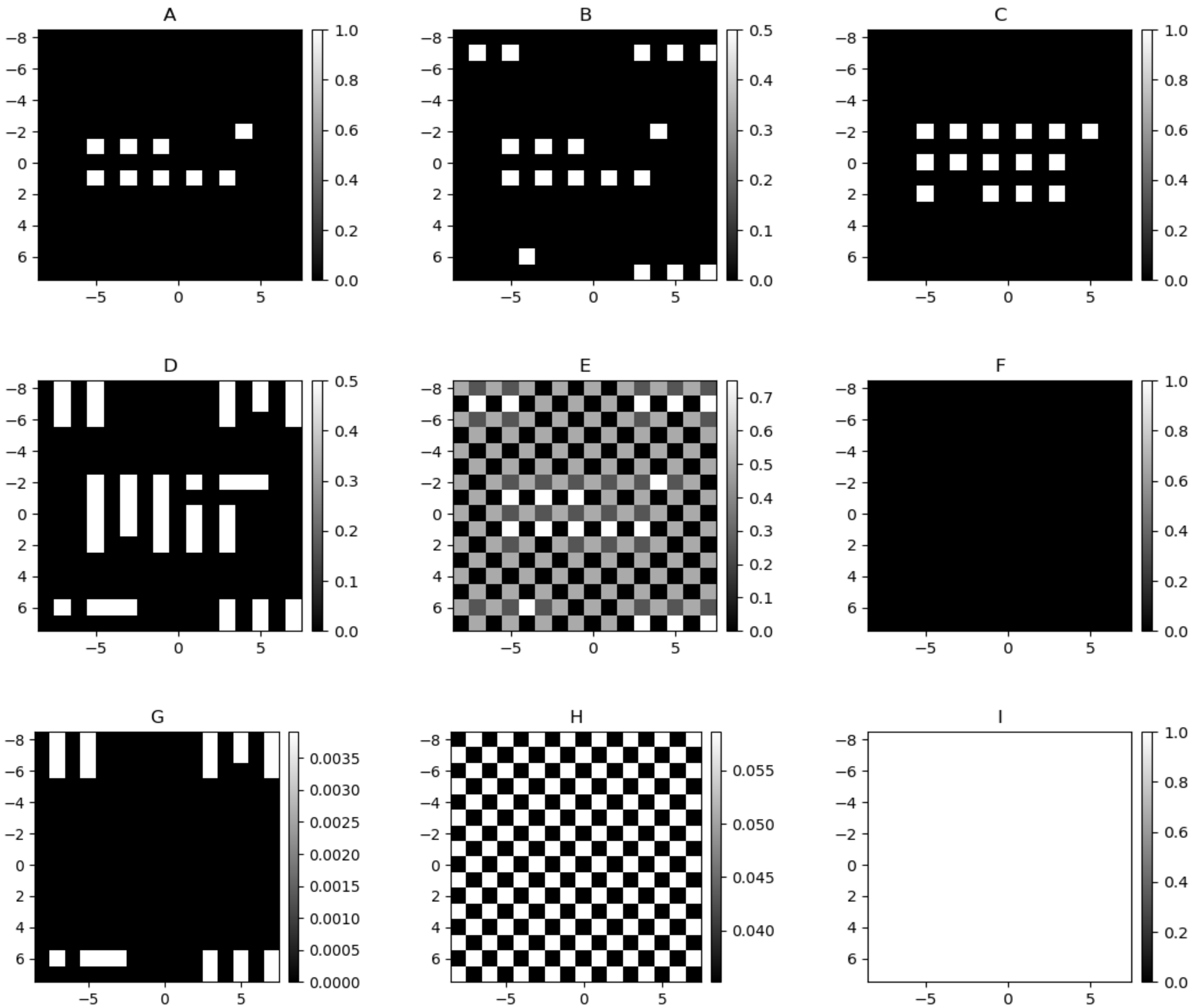
$(f \times h_1)[r, c]$: A

$(f \times h_2)[r, c]$: F

$(f \circledast h_1)[r, c]$: H

$(f \circledast h_2)[r, c]$: D

Pixel values in each of the following images are normalized so that black represents the smallest pixel value (not necessarily 0) and white represents the largest pixel value in that image (not necessarily 1).



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Worksheet (*intentionally blank*)

Worksheet (*intentionally blank*)