6.3000 Quiz 1 Fall 2023

Name: Solutions

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This quiz is closed book, but you may use one 8.5×11 sheet of notes (both sides).

You may NOT use any electronic devices (such as calculators and phones).

If you have questions, please **come to us** at the front of the room to ask.

Please enter all solutions in the boxes provided.

Work on other pages with QR codes will be considered for partial credit.
Please provide a note if you continue work on worksheets at the end of the exam.

Please do not write on the QR codes at the bottom of each page.

We use those codes to identify which pages belong to each student.

Trigonometric Identities Reference

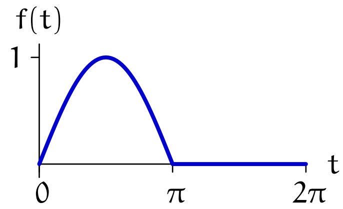
$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$
$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b)$$
$$\cos(a+b)-\cos(a-b) = -2\sin(a)\sin(b)$$
$$\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$$
$$\sin(a+b)-\sin(a-b) = 2\cos(a)\sin(b)$$
$$2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b)$$
$$2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$$
$$2\sin(a)\cos(b) = \sin(a+b)+\sin(a-b)$$
$$2\cos(a)\sin(b) = \sin(a+b)-\sin(a-b)$$

1 Top of the Sine (15/70 points)

Let $f(t)$ represent a signal that is periodic in time t with period $T = 2\pi$. One period of $f(t)$ is described by the equation

$$f(t) = \begin{cases} \sin(t) & \text{if } 0 \leqslant t \leqslant \pi \\ 0 & \text{if } \pi \leqslant t \leqslant 2\pi \end{cases}$$

as shown in the following plot.



Determine the coefficients a_k to represent $f(t)$ as a Fourier series of the following form:

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$$

Enter numerical expressions for a_k for k in the range $k = -5$ to 5 in the following table. Your expressions should not include integrals or infinite sums.

k	a_k	a_{-k}
0	$\frac{1}{\pi}$	XXXXXXXXXXXXXXXXXXXX
1	$\frac{1}{4j}$	$-\frac{1}{4j}$
2	$\frac{-1}{2\pi} + \frac{1}{6\pi}$	$\frac{-1}{2\pi} + \frac{1}{6\pi}$
3	0	0
4	$\frac{-1}{6\pi} + \frac{1}{10\pi}$	$\frac{-1}{6\pi} + \frac{1}{10\pi}$
5	0	0

$$\sin(t) = \frac{1}{j2}\left(e^{jt} - e^{-jt}\right)$$

$$\begin{aligned} a_k &= \frac{1}{2\pi}\int_0^\pi f(t)e^{-jkt}\,dt \\ &= \frac{1}{2\pi}\int_0^\pi \frac{1}{j2}\left(e^{jt} - e^{-jt}\right)e^{-jkt}\,dt \\ &= \underbrace{-\frac{1}{4\pi}\left[\frac{e^{j(1-k)t}}{1-k}\right]_0^\pi}_{\begin{cases} \frac{1}{2\pi(1-k)} & k \text{ even} \\ \frac{1}{4j} & k = 1 \\ 0 & \text{otherwise} \end{cases}} \quad \underbrace{-\frac{1}{4\pi}\left[\frac{e^{-j(1+k)t}}{1+k}\right]_0^\pi}_{\begin{cases} \frac{1}{2\pi(1+k)} & k \text{ even} \\ -\frac{1}{4j} & k = -1 \\ 0 & \text{otherwise} \end{cases}} \end{aligned}$$

Worksheet (intentionally blank)

2 Discrete-Time Fourier Series (15/70 points)

Is it possible to represent the following discrete-time signal

$$f[n] = \cos\left(\frac{\pi n}{3} + \frac{\pi}{4}\right) + \cos\left(\frac{\pi n}{2}\right)$$

as a Fourier series of the following form?

$$f[n] = \sum_{k=<N>} a_k e^{j2\pi kn/N}$$

Enter **YES** or **NO**: **YES**

If yes, determine the smallest value of N for which the Fourier series exists, and enter the Fourier series components a_{-6} to a_6 in the table below.

Smallest value of N: **12**

k	a_k	a_{-k}
0	0	XXXXXXXXXXXXXXXXXXXX
1	0	0
2	$\frac{1+j}{2\sqrt{2}}$	$\frac{1-j}{2\sqrt{2}}$
3	$\frac{1}{2}$	$\frac{1}{2}$
4	0	0
5	0	0
6	0	0

If no, briefly explain why.

Not applicable.

$$\begin{aligned}
f[n] &= \frac{1}{2}e^{j\pi n/3}e^{j\pi/4} + \frac{1}{2}e^{-j\pi n/3}e^{-j\pi/4} + \frac{1}{2}e^{j\pi n/2} + \frac{1}{2}e^{-j\pi n/2} \\
&= \frac{1+j}{2\sqrt{2}}e^{j\pi n/3} + \frac{1-j}{2\sqrt{2}}e^{-j\pi n/3} + \frac{1}{2}e^{j\pi n/2} + \frac{1}{2}e^{-j\pi n/2} \\
&= \frac{1+j}{2\sqrt{2}}e^{j4\pi n/12} + \frac{1-j}{2\sqrt{2}}e^{-j4\pi n/12} + \frac{1}{2}e^{j6\pi n/12} + \frac{1}{2}e^{-6j\pi n/12}
\end{aligned}$$

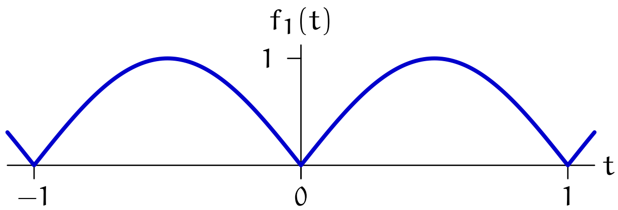
Worksheet (*intentionally blank*)

3 Transformations (20/70 points)

Part a. Let $f_1(t)$ represent a continuous-time function given by the following expression:

$f_1(t) = |\sin(\pi t)|$

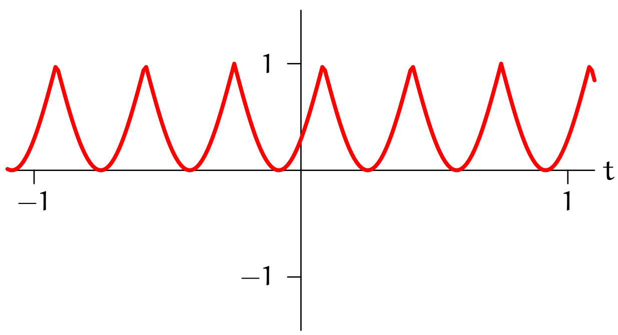
as shown in the following plot.



Now define a new function $g_1(t)$ in terms of $f_1(t)$ as follows:

$g_1(t) = 1 - f_1(3t-1/4)$

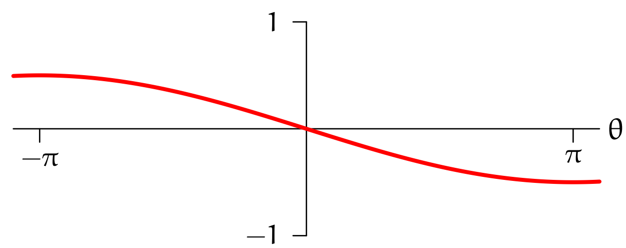
Sketch $g_1(t)$ as a function of t on the axes below. Label the important parameters of your sketch.



Part b. Let $f_2(\theta)$ represent the following function:

$$f_2(\theta) = \mathrm{Im}\left(\frac{d}{d\theta}\left(je^{-j\theta/2}\right)\right)$$

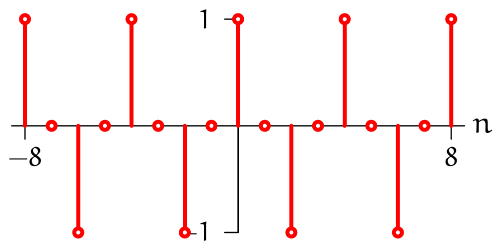
Sketch $f_2(\theta)$ as a function of θ on the axes below. Label the important parameters of your sketch.



Part c. Let $f_3[n]$ represent the following function:

$$f_3[n] = \text{Re}\left(\left(\frac{1}{(\cos(\theta) + j\sin(\theta))^n}\right)^2\right)$$

Plot $f_3[n]$ as a function of discrete-time n for the case when $\theta = \pi/4$. Label the important parameters of your plot.

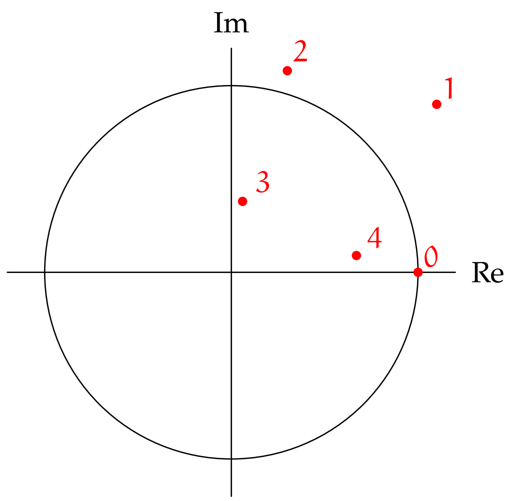


Part d. Define a discrete function $f_4[n]$ as follows

$$f_4[n] = \sum_{k=0}^{n} a^k$$

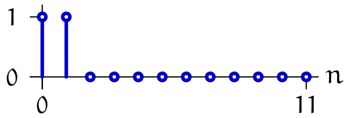
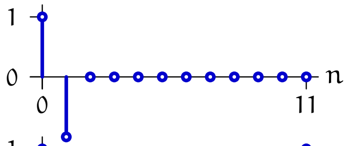
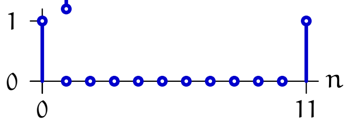
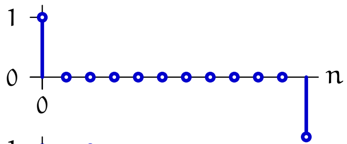
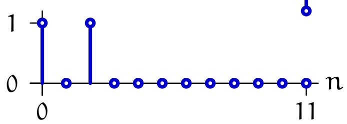
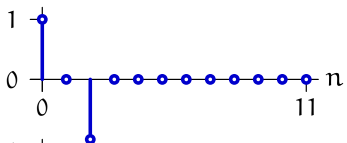
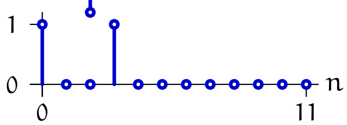
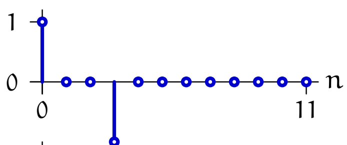
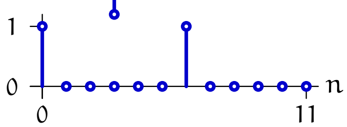
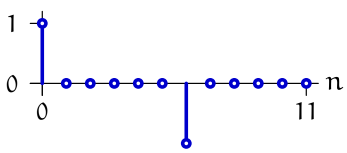
where $a = 0.1 + 0.9j$.

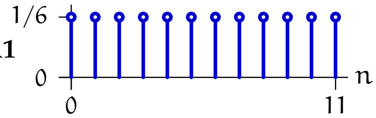
The following figure shows the complex plane where the circle has unit radius. Draw dots on that figure to indicate the values of $f_4[0]$, $f_4[1]$, $f_4[2]$, $f_4[3]$, and $f_4[4]$. Label the dots as 0, 1, 2, 3, or 4, to indicate which dot corresponds to each time index n.



4 Siren Series (20/70 points)

Each of the following plots shows one period (N=12) of a periodic, discrete-time signal $f_i[n]$ that can be represented by its Fourier series coefficients $F_i[k]$. Determine which plot on the next page shows the real and imaginary parts of $F_i[k]$ as functions of k and enter their identifiers in the boxes below. Some plots on the next page may be used more than once.

		$\mathrm{Re}(F_i[k])$	$\mathrm{Im}(F_i[k])$
$f_1[n]$		A16	A6
$f_2[n]$		A18	A4
$f_3[n]$		A16	A4
$f_4[n]$		A18	A6
$f_5[n]$		A20	A10
$f_6[n]$		A22	A8
$f_7[n]$		A32	A28
$f_8[n]$		A34	A26
$f_9[n]$		A3	A12
$f_{10}[n]$		A14	A12

A1


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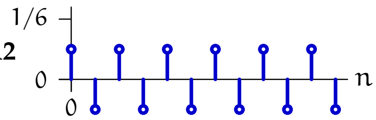

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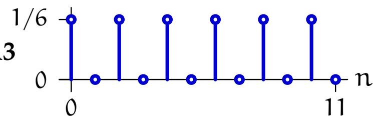

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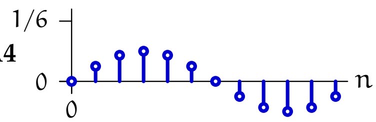

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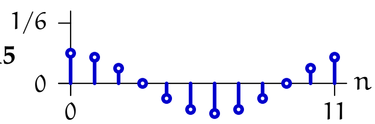

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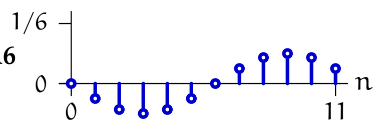

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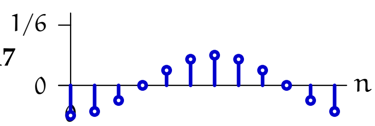

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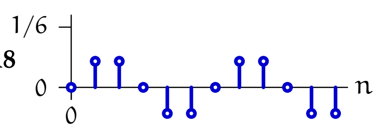

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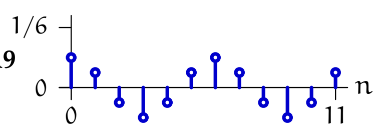

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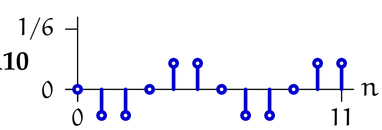

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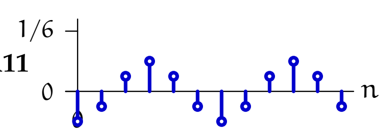

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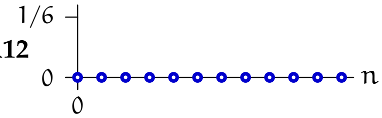

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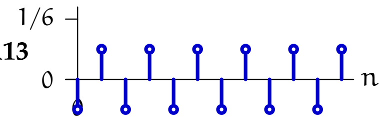

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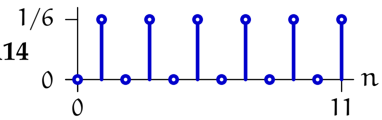

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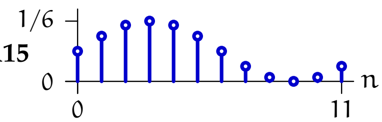

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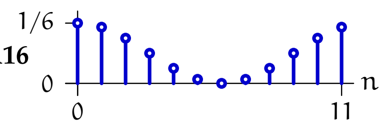

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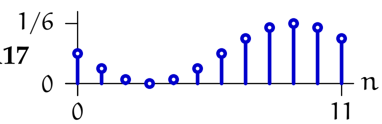

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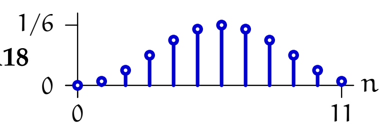

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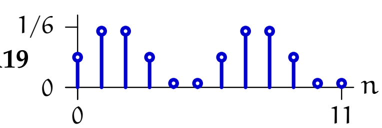

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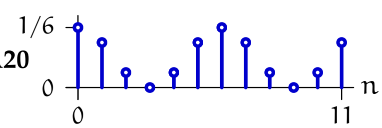

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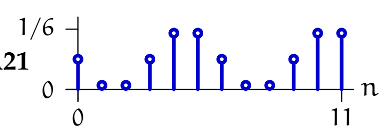

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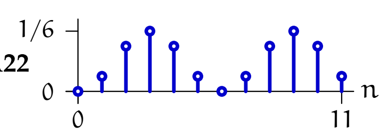

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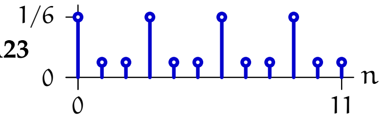

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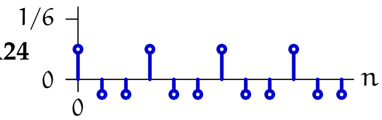

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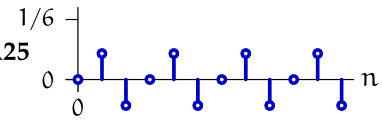

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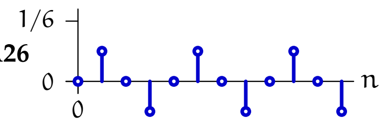

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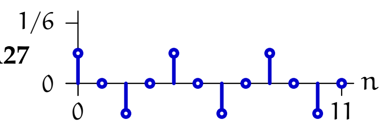

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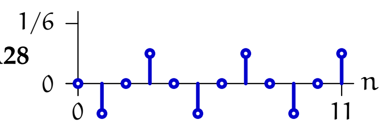

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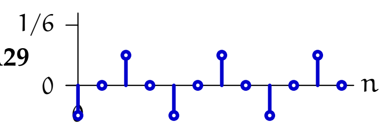

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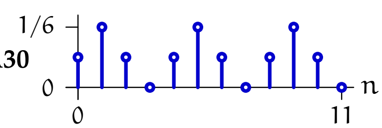

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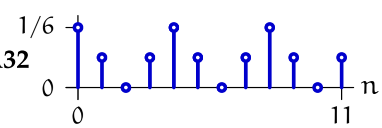

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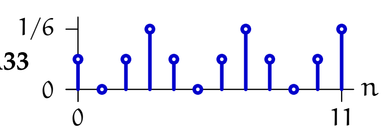

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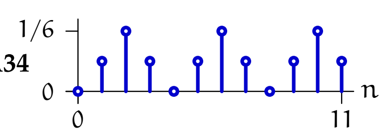

Part a. $F_1[k]$:

$$F_1[k] = \frac{1}{N}\sum_{n=<N>} f_1[n]e^{-j2\pi kn/N} = \frac{1}{12}(1+e^{-j2\pi k/12}) = \frac{1}{12}\Big(1+\cos(2\pi k/12) - j\sin(2\pi k/12)\Big)$$

$$\text{Re}\,(F_1[k]) = \frac{1}{12}\Big(1+\cos(2\pi k/12)\Big)$$

$$\text{Im}\,(F_1[k]) = -\frac{1}{12}\sin(2\pi k/12)$$

Notice that the real and imaginary parts of $F_1[k]$ are periodic with period 12.
The real part is an offset cosine with a period of 12: **A16**.
The imaginary part is a negative sine with a period of 12: **A6**.

Part b. $F_2[k]$:

$$F_2[k] = \frac{1}{N}\sum_{n=<N>} f_2[n]e^{-j2\pi kn/N} = \frac{1}{12}(1-e^{-j2\pi k/12}) = \frac{1}{12}\Big(1-\cos(2\pi k/12) + j\sin(2\pi k/12)\Big)$$

$$\text{Re}\,(F_2[k]) = \frac{1}{12}\Big(1-\cos(2\pi k/12)\Big)$$

$$\text{Im}\,(F_2[k]) = \frac{1}{12}\sin(2\pi k/12)$$

Notice that the real and imaginary parts of $F_2[k]$ are periodic with period 12.
The real part is an offset negative cosine with a period of 12: **A18**.
The imaginary part is a sine with a period of 12: **A4**.

Part c. $F_3[k]$:

$$F_3[k] = \frac{1}{12}(1+e^{-j2\pi 11k/12}) = \frac{1}{12}(1+e^{j2\pi k/12}) = \frac{1}{12}\Big(1+\cos(2\pi 11k/12) - j\sin(2\pi 11k/12)\Big)$$

$$\text{Re}\,(F_3[k]) = \frac{1}{12}\Big(1+\cos(2\pi 11k/12)\Big)$$

$$\text{Im}\,(F_3[k]) = \frac{1}{12}\sin(2\pi 11k/12)$$

Notice that the real and imaginary parts of $F_3[k]$ are periodic with period 12.
The real part is an offset cosine with a period of 12: **A16**.
The imaginary part is a sine with a period of 12: **A4**.

Part d. $F_4[k]$:

$$F_4[k] = \frac{1}{12}(1-e^{-j2\pi 11k/12}) = \frac{1}{12}(1-e^{j2\pi k/12}) = \frac{1}{12}\Big(1-\cos(2\pi 11k/12) - j\sin(2\pi 11k/12)\Big)$$

$$\text{Re}\,(F_4[k]) = \frac{1}{12}\Big(1-\cos(2\pi 11k/12)\Big)$$

$$\text{Im}\,(F_4[k]) = \frac{1}{12} - \sin(2\pi 11k/12)$$

Notice that the real and imaginary parts of $F_4[k]$ are periodic with period 12.
The real part is an offset negative cosine with a period of 12: **A18**.
The imaginary part is a negative sine with a period of 12: **A6**.

Part e. $F_5[k]$:

$$F_5[k] = \frac{1}{N}\sum_{n=<N>} f_5[n]e^{-j2\pi kn/N} = \frac{1}{12}(1+e^{-j2\pi 2k/12}) = \frac{1}{12}\Big(1+\cos(2\pi 2k/12) - j\sin(2\pi 2k/12)\Big)$$

$$\text{Re}\,(F_5[k]) = \frac{1}{12}\Big(1+\cos(2\pi 2k/12)\Big)$$

$$\text{Im}\,(F_5[k]) = -\frac{1}{12}\sin(2\pi 2k/12)$$

Notice that the real and imaginary parts of $F_5[k]$ are periodic with period 6.
The real part is an offset cosine with a period of 6: **A20**.
The imaginary part is a negative sine with a period of 6: **A10**.

Part f. $F_6[k]$:

$$F_6[k] = \frac{1}{N}\sum_{n=<N>} f_6[n]e^{-j2\pi kn/N} = \frac{1}{12}(1-e^{-j2\pi 2k/12}) = \frac{1}{12}\Big(1-\cos(2\pi 2k/12) + j\sin(2\pi 2k/12)\Big)$$

$$\text{Re}\,(F_6[k]) = \frac{1}{12}\Big(1-\cos(2\pi 2k/12)\Big)$$

$$\text{Im}\,(F_6[k]) = \frac{1}{12}\sin(2\pi 2k/12)$$

Notice that the real and imaginary parts of $F_6[k]$ are periodic with period 6.
The real part is an offset negative cosine with a period of 6: **A22**.
The imaginary part is a sine with a period of 6: **A8**.

Part g. $F_7[k]$:

$$F_7[k] = \frac{1}{N}\sum_{n=<N>} f_7[n]e^{-j2\pi kn/N} = \frac{1}{12}(1+e^{-j2\pi 3k/12}) = \frac{1}{12}\Big(1+\cos(2\pi 3k/12) - j\sin(2\pi 3k/12)\Big)$$

$$\text{Re}\,(F_7[k]) = \frac{1}{12}\Big(1+\cos(2\pi 3k/12)\Big)$$

$$\text{Im}\,(F_7[k]) = -\frac{1}{12}\sin(2\pi 3k/12)$$

Notice that the real and imaginary parts of $F_7[k]$ are periodic with period 4.
The real part is an offset cosine with a period of 4: **A32**.
The imaginary part is a negative sine with a period of 4: **A28**.

Part h. $F_8[k]$:

$$F_8[k] = \frac{1}{N}\sum_{n=<N>} f_8[n]e^{-j2\pi kn/N} = \frac{1}{12}(1-e^{-j2\pi 2k/12}) = \frac{1}{12}\Big(1-\cos(2\pi 2k/12) + j\sin(2\pi 2k/12)\Big)$$

$$\text{Re}\,(F_8[k]) = \frac{1}{12}\Big(1-\cos(2\pi 2k/12)\Big)$$

$$\text{Im}\,(F_8[k]) = \frac{1}{12}\sin(2\pi 2k/12)$$

Notice that the real and imaginary parts of $F_8[k]$ are periodic with period 4.
The real part is an offset negative cosine with a period of 4: **A34**.
The imaginary part is a sine with a period of 4: **A26**.

Part i. $F_9[k]$:

$$F_9[k] = \frac{1}{N}\sum_{n=<N>} f_9[n]e^{-j2\pi kn/N} = \frac{1}{12}(1+e^{-j2\pi 6k/12}) = \frac{1}{12}\Big(1+\cos(2\pi 6k/12) - j\sin(2\pi 6k/12)\Big)$$

$$\text{Re}\left(F_9[k]\right) = \frac{1}{12}\Big(1+\cos(2\pi 6k/12)\Big)$$

$$\text{Im}\left(F_9[k]\right) = -\frac{1}{12}\sin(2\pi 6k/12)$$

Notice that the real of $F_9[k]$ is periodic with period 2.
The real part is an offset cosine with a period of 3: **A3**.
The imaginary part is zero for all k: **A12**.

Part j. $F_{10}[k]$:

$$F_{10}[k] = \frac{1}{N}\sum_{n=<N>} f_{10}[n]e^{-j2\pi kn/N} = \frac{1}{12}(1-e^{-j2\pi 6k/12}) = \frac{1}{12}\Big(1-\cos(2\pi 6k/12) + j\sin(2\pi 6k/12)\Big)$$

$$\text{Re}\left(F_{10}[k]\right) = \frac{1}{12}\Big(1-\cos(2\pi 6k/12)\Big)$$

$$\text{Im}\left(F_{10}[k]\right) = \frac{1}{12}\sin(2\pi 6k/12)$$

Notice that the real part of $F_8[k]$ is periodic with period 2.
The real part is an offset negative cosine with a period of 2: **A14**.
The imaginary part is zero for all k: **A12**.

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