6.3000: Signal Processing

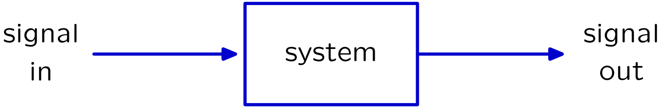
Unit-Sample Response and Convolution

- Unit Sample Signal and Unit Sample Response
- Discrete-Time Convolution
- Impulse Function and Impulse Response
- Continuous-Time Convolution

March 11, 2025

Last Time: The System Abstraction

Represent a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



Properties of Systems

We will focus primarily on systems that have two important properties:

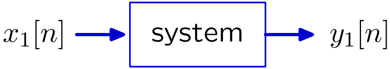
- **linearity**
- **time invariance**

Such systems are both prevalent and mathematically tractable.

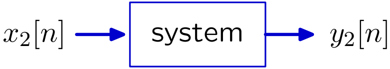
Additivity

A system is additive if its **response to a sum** of signals is equal to the **sum of the responses** to each signal taken one at a time.

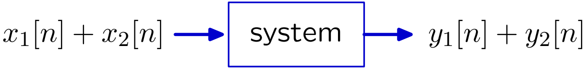
Given



and



the **system is additive** if

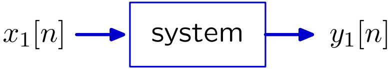


for all possible inputs and all times n.

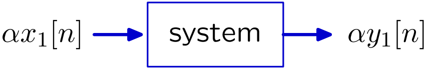
Homogeneity

A system is homogeneous if multiplying its input signal by a constant multiplies the output signal by the same constant.

Given



the **system is homogeneous** if

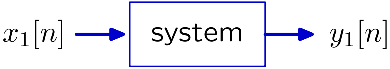


for all α and all possible inputs and all times n.

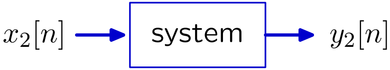
Linearity

A system is linear if its **response to a weighted sum** of input signals is equal to the **weighted sum of its responses** to each of the input signals.

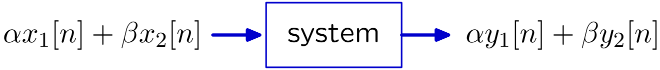
Given



and



the **system is linear** if



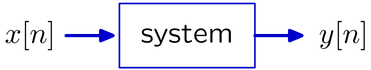
for all α and β and all possible inputs and all times n.

A system is linear if it is both additive and homogeneous.

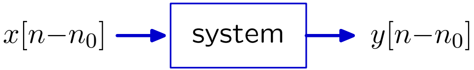
Time-Invariance

A system is time-invariant if delaying the input signal simply delays the output signal by the same amount of time.

Given



the **system is time invariant** if



for all n_0 and for all possible inputs and all times n.

Linear Difference Equations with Constant Coefficients

If a discrete-time system can be described by a linear difference equation with constant coefficients, then the system is linear and time-invariant.

General form:

$$\sum_l c_l y[n-l] = \sum_m d_m x[n-m]$$

Such systems are easily shown to be linear and time-invariant.

Additivity: output of sum is sum of outputs

$$\sum_l c_l (y_1[n-l] + y_2[n-l]) = \sum_m d_m (x_1[n-m] + x_2[n-m])$$

Homogeneity: scaling an input scales its output

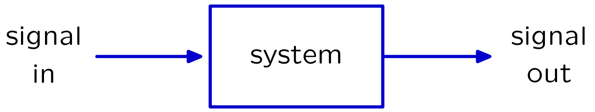
$$\sum_l c_l (\alpha y[n-l]) = \sum_m d_m (\alpha x[n-m])$$

Time invariance: delaying an input delays its output

$$\sum_l c_l y[(n-n_0)-l] = \sum_m d_m x[(n-n_0)-m]$$

Today: Representing a System by its Unit-Sample Response

Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



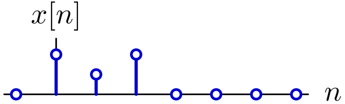
This abstraction is particularly powerful for **linear and time-invariant** systems, which are both **prevalent** and **mathematically tractable**.

Three important representations for LTI systems:

- **Difference Equation:** algebraic **constraint** on samples $\surd$
- **Convolution:** represent a system by its **unit-sample response**
- **Filter:** represent a system by its **frequency response**

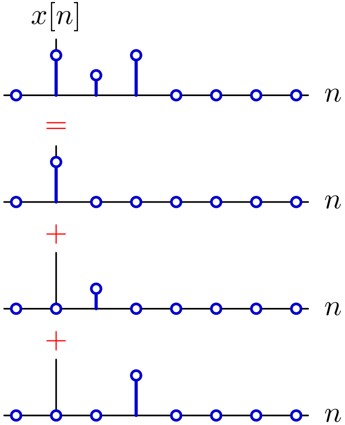
Superposition

Break the input signal into additive parts and sum responses to the parts.



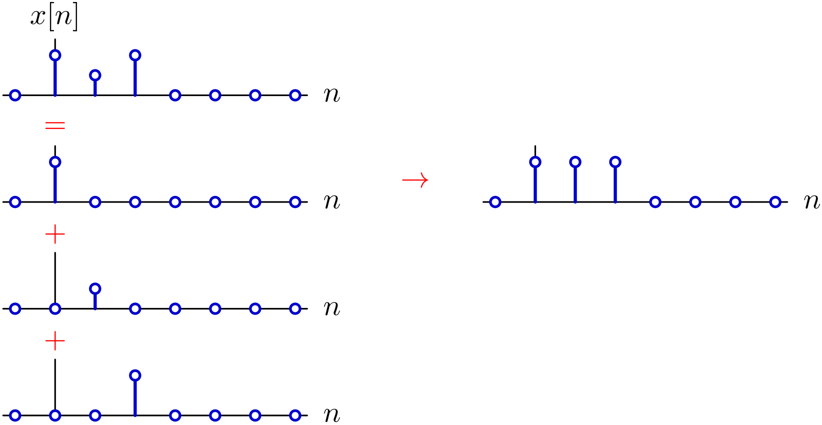
Superposition

Break the input signal into additive parts.



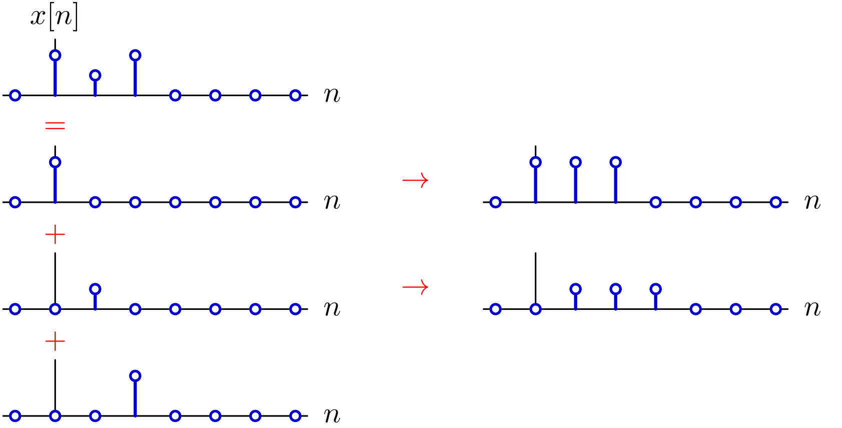
Superposition

Find the response to each part.



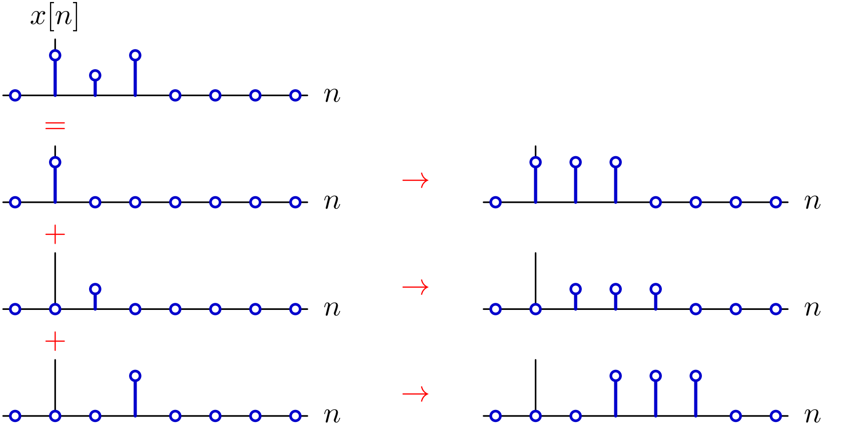
Superposition

Find the response to each part.



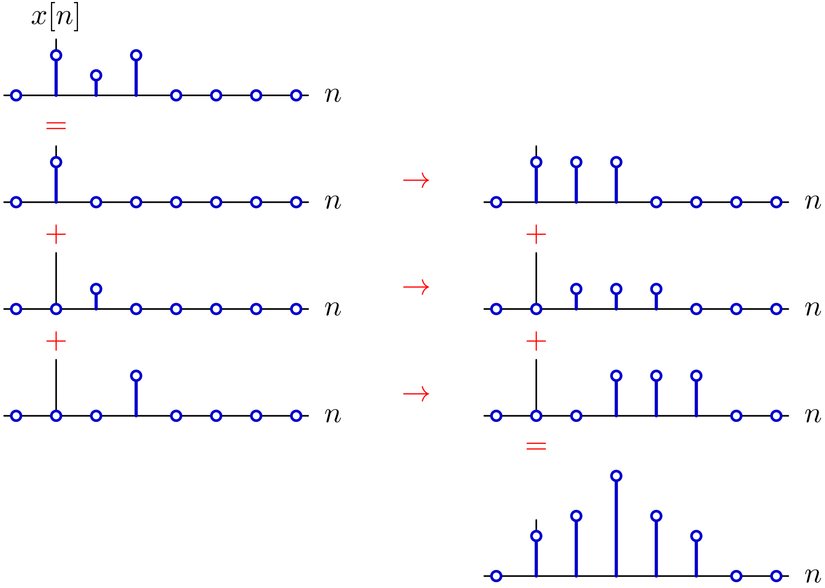
Superposition

Find the response to each part.



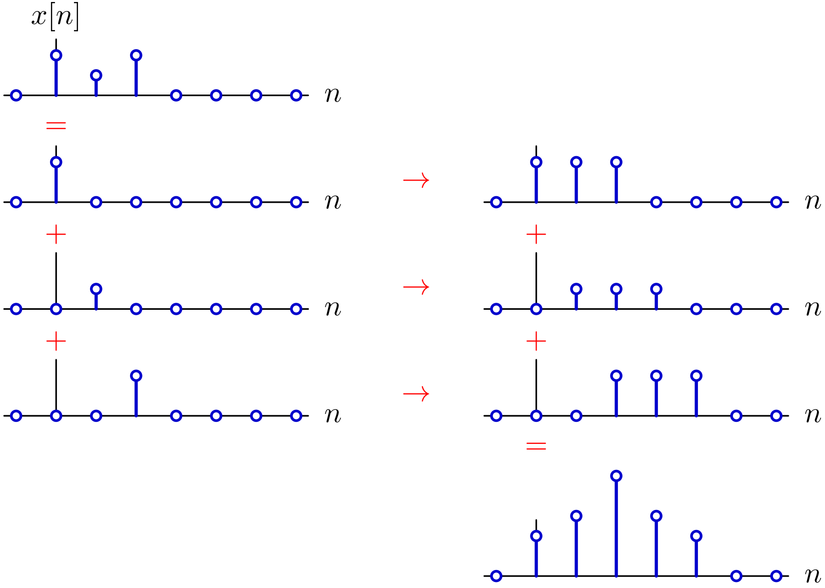
Superposition

Add the responses to the parts.



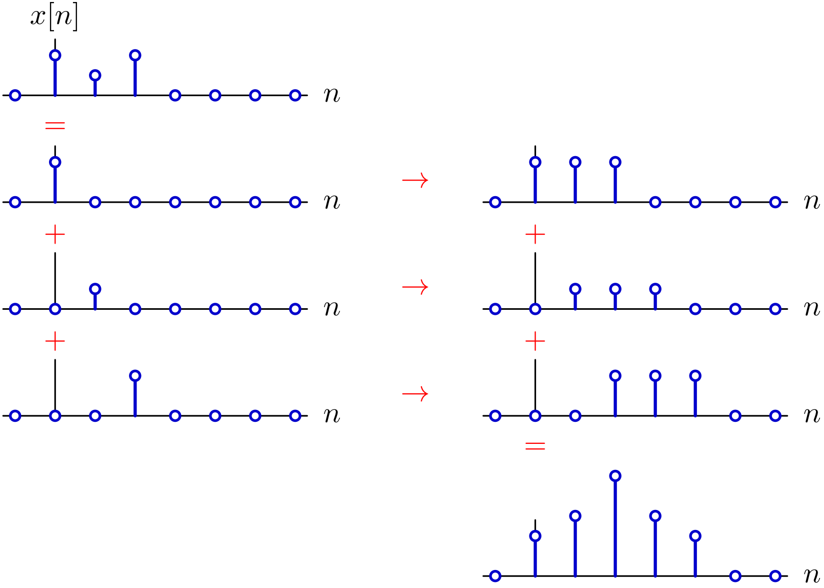
Superposition

Superposition only works if the system is **additive**.



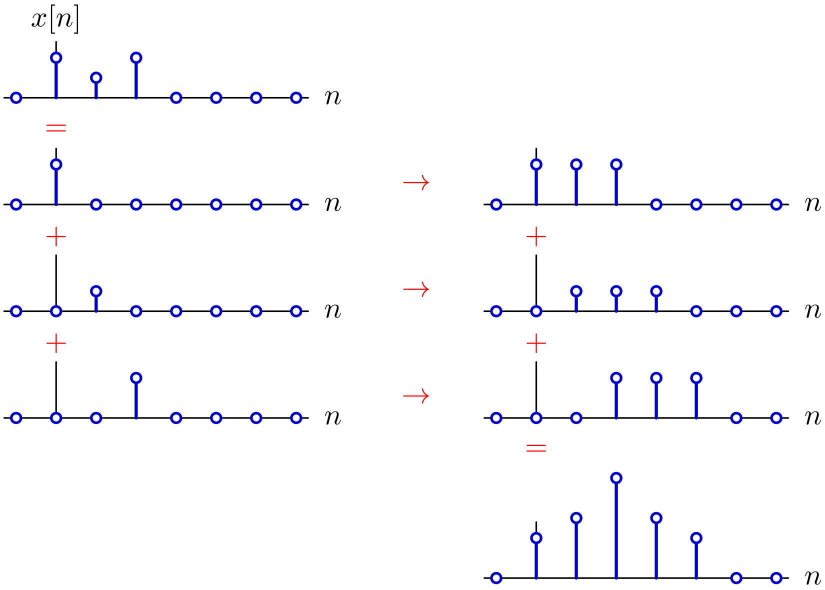
Superposition

Superposition is easy if system is also **homogeneous** and **time-invariant**.



Superposition

Response of a **linear** system is determined by its response to $\delta[n]$.



Unit-Sample Response and Convolution

If a system is linear and time-invariant (LTI), its input-output relation is **completely specified** by the system's unit-sample response $h[n]$.

1. One can always find the unit-sample response of a system.

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

2. Time invariance implies that shifting the input simply shifts the output.

$$\delta[n-m] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n-m]$$

3. Homogeneity implies that scaling the input simply scales the output.

$$x[m]\delta[n-m] \longrightarrow \boxed{\text{LTI}} \longrightarrow x[m]h[n-m]$$

4. Additivity implies that the response to a sum is the sum of responses.

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

This rule for combining the input $x[n]$ with the unit-sample response $h[n]$ is called **convolution**.

Convolution

The response of an LTI system to an arbitrary input $x[n]$ can be found by **convolving** that input with the **unit-sample response** $h[n]$ of the system.

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \equiv (x * h)[n]$$

This is an amazing result.

We can represent the operation of an LTI system by a **single signal**!

Notation

Convolution is represented with an asterisk.

$$\sum_{m=-\infty}^{\infty} x[m]h[n-m] \equiv (x * h)[n]$$

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Convolution is represented with an asterisk.

$$\sum_{m=-\infty}^{\infty} x[m]h[n-m] \equiv (x * h)[n]$$

It is customary (but confusing) to abbreviate this notation:

$$(x * h)[n] = x[n] * h[n]$$

$x[n] * h[n]$ looks like an operation of samples; but it is not!

$$x[1] * h[1] \neq (x * h)[1]$$

Convolution operates on signals not samples.

Unambiguous notation:

$$y = x * h$$

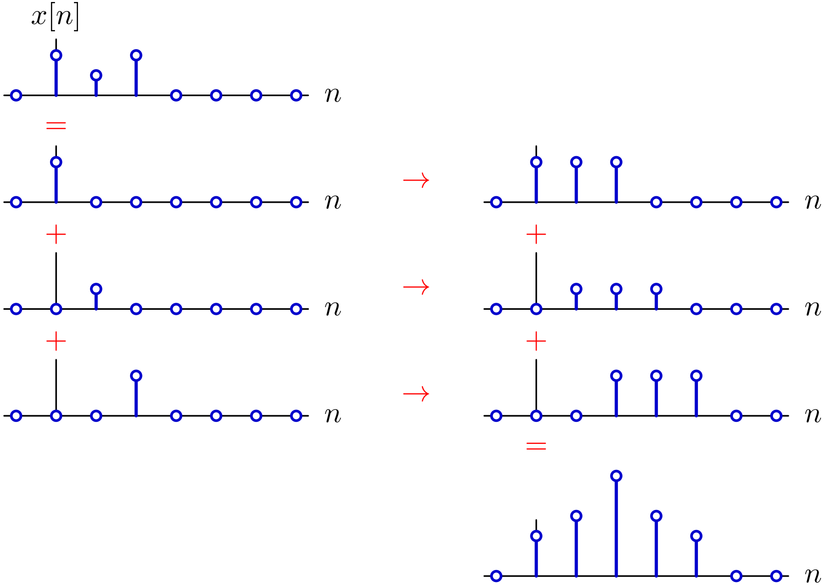
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \equiv (x * h)[n]$$

The symbols x and h represent DT signals.

Convolving x with h generates a new DT signal $y = x * h$.

Structure of Convolution

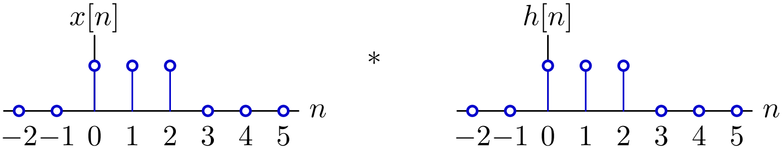
Convolution as an application of superposition.



Structure of Convolution

Focus on computing the n^{th} output sample.

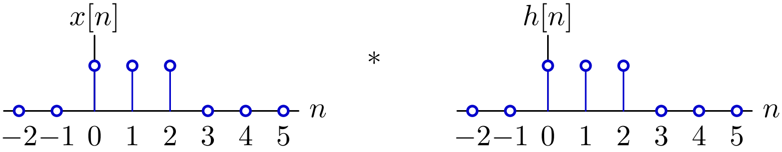
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$



Structure of Convolution

Focus on computing the n^{th} output sample: start with $n = 0$.

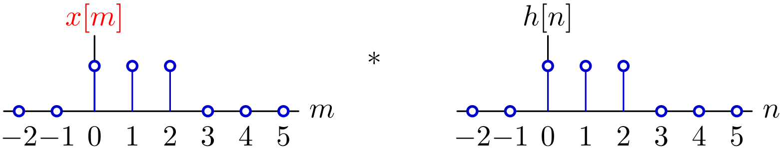
$$y[0] = \sum_{m=-\infty}^{\infty} x[m]h[0-m]$$



Structure of Convolution

The sum is over $x[m]$ (not $x[n]$).

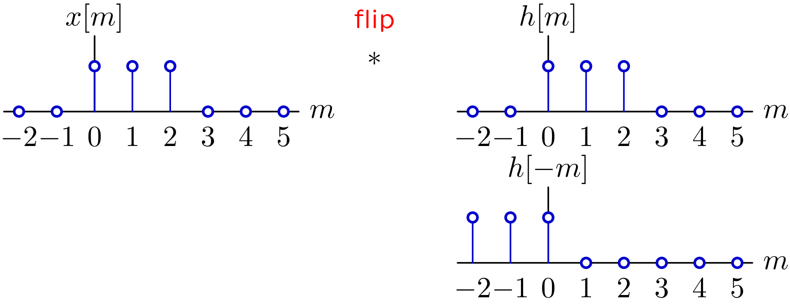
$$y[0] = \sum_{m=-\infty}^{\infty} x[m]h[0-m]$$



Structure of Convolution

The sum is over $x[m]$ and $h[-m]$ ($h[m]$ is **flipped**).

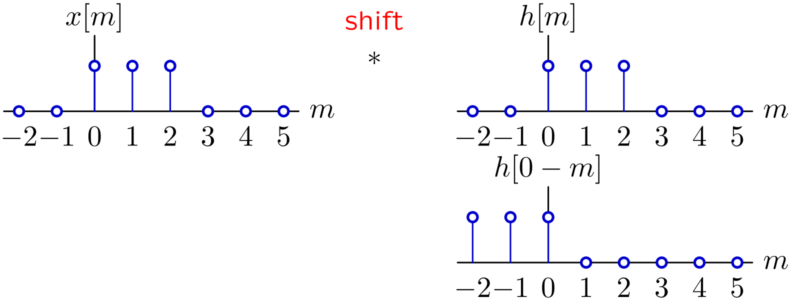
$$y[0] = \sum_{m=-\infty}^{\infty} x[m]h[0 - m]$$



Structure of Convolution

Focus on computing the n^{th} output sample.

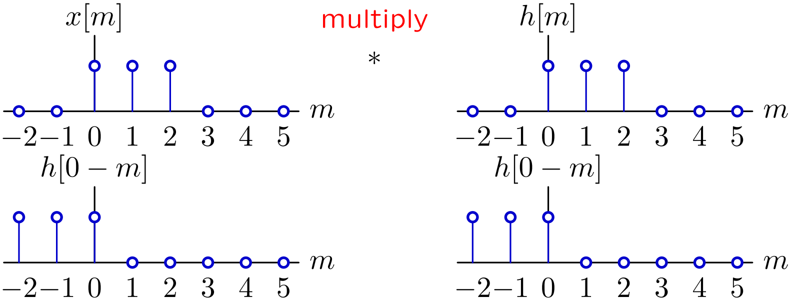
$$y[0] = \sum_{m=-\infty}^{\infty} x[m]h[{\color{red}0} - m]$$



Structure of Convolution

Focus on computing the n^{th} output sample.

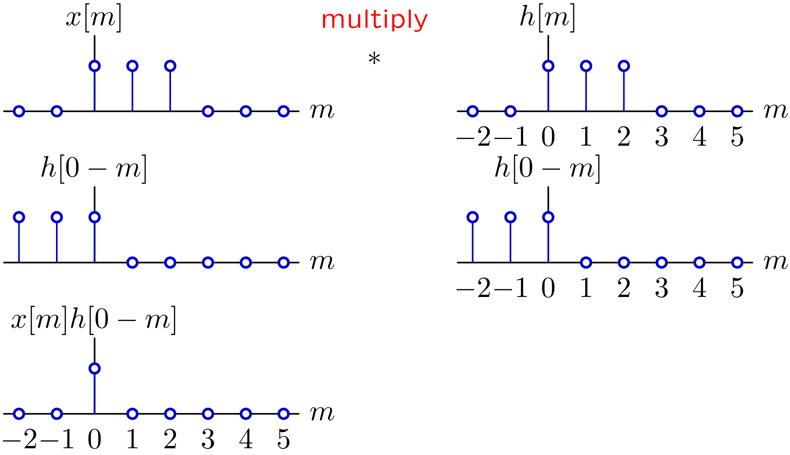
$$y[0] = \sum_{m=-\infty}^{\infty} x[m]h[0-m]$$



Structure of Convolution

Focus on computing the n^{th} output sample.

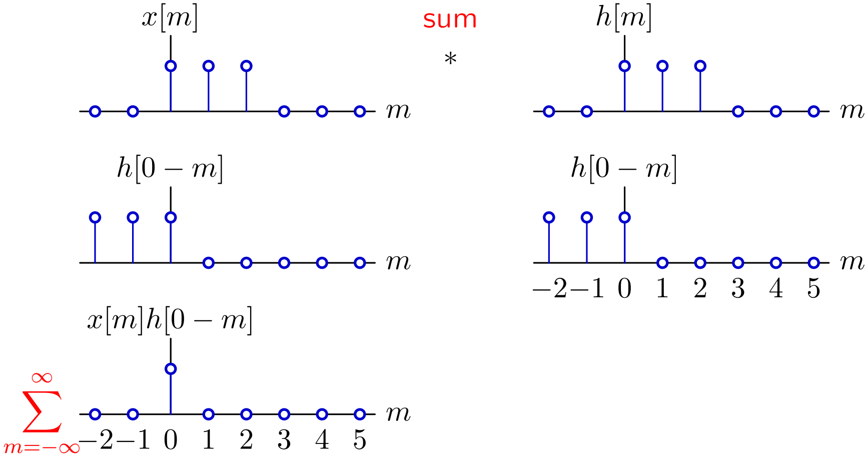
$$y[0] = \sum_{m=-\infty}^{\infty} x[m]h[0-m]$$



Structure of Convolution

Focus on computing the n^{th} output sample.

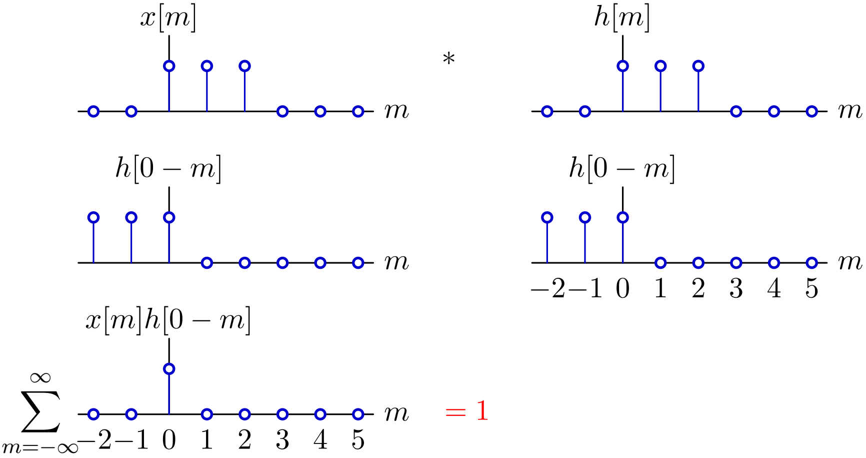
$$y[0] = \sum_{m=-\infty}^{\infty} x[m]h[0-m]$$



Structure of Convolution

Focus on computing the n^{th} output sample.

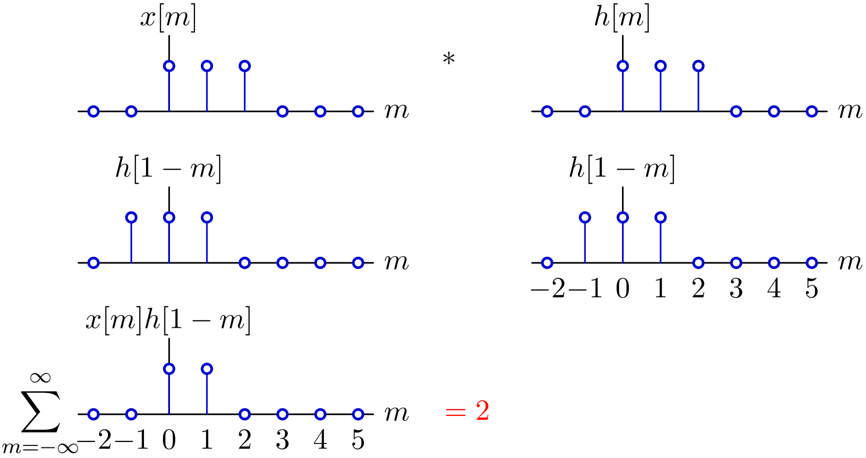
$$y[0] = \sum_{m=-\infty}^{\infty} x[m]h[0-m]$$



Structure of Convolution

Focus on computing the n^{th} output sample.

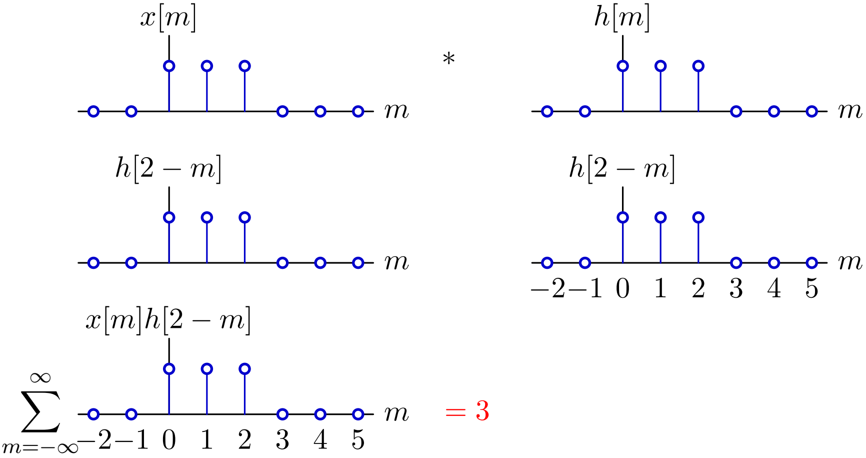
$$y[1] = \sum_{m=-\infty}^{\infty} x[m]h[1-m]$$



Structure of Convolution

Focus on computing the n^{th} output sample.

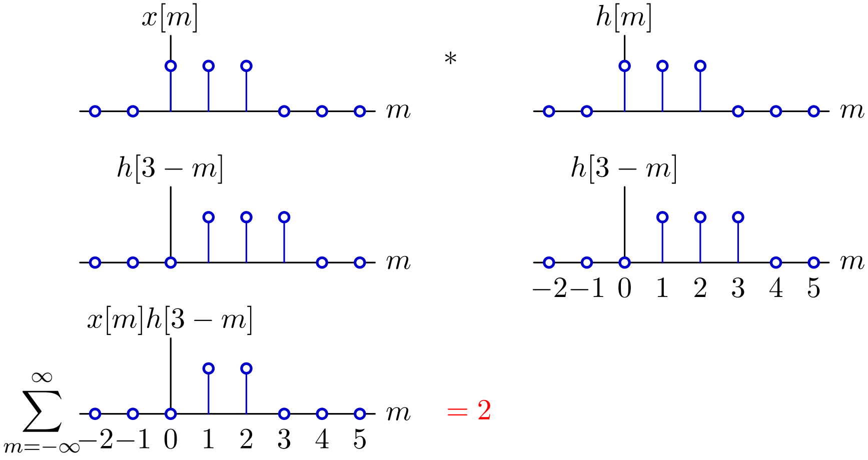
$$y[{\color{red}2}] = \sum_{m=-\infty}^{\infty} x[m]h[2-m]$$



Structure of Convolution

Focus on computing the n^{th} output sample.

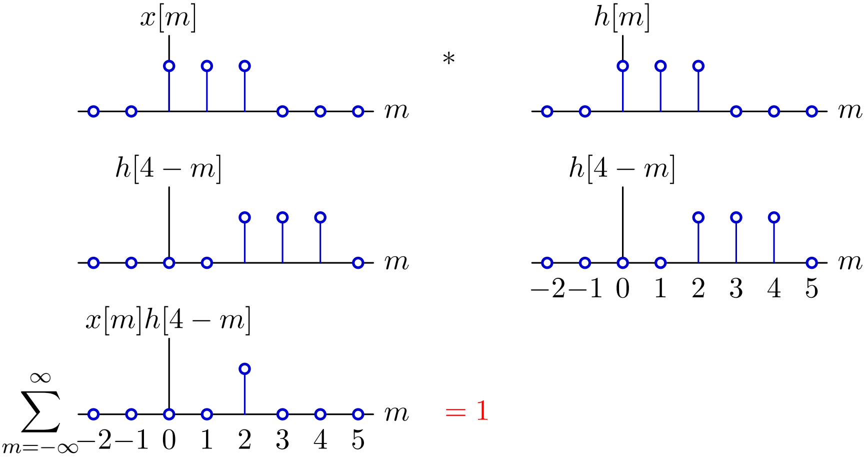
$$y[{\color{red}3}] = \sum_{m=-\infty}^{\infty} x[m]h[3-m]$$



Structure of Convolution

Focus on computing the n^{th} output sample.

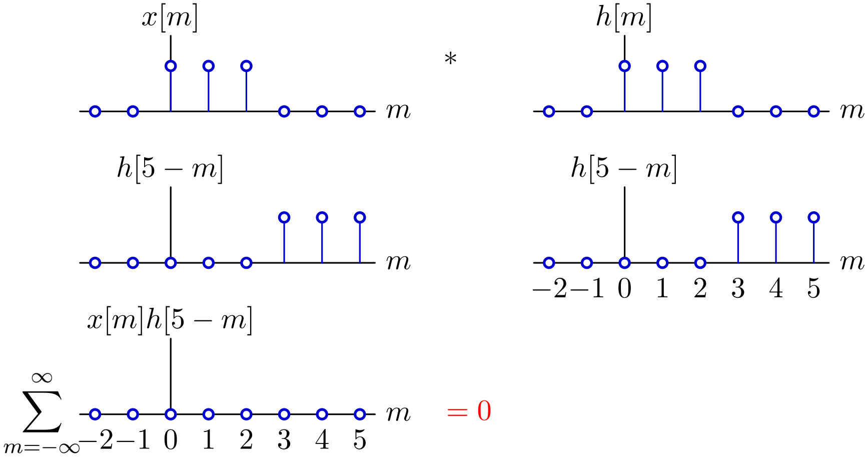
$$y[4] = \sum_{m=-\infty}^{\infty} x[m]h[4-m]$$



Structure of Convolution

Focus on computing the n^{th} output sample.

$$y[5] = \sum_{m=-\infty}^{\infty} x[m]h[5-m]$$

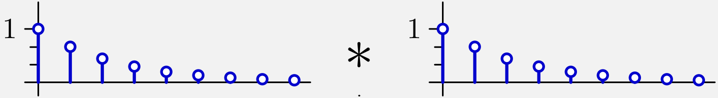


$$\sum_{m=-\infty}^{\infty} = 0$$

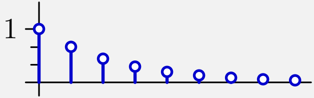
Check Yourself

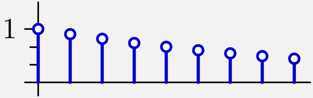
Consider the convolution of two geometric sequences:

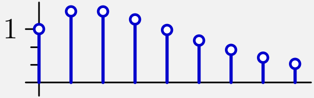
$$\left(\frac{2}{3}\right)^n u[n] \quad * \quad \left(\frac{2}{3}\right)^n u[n]$$

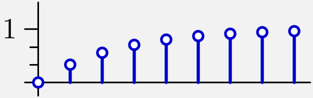


Which plot below shows the result of the convolution above?

1. 

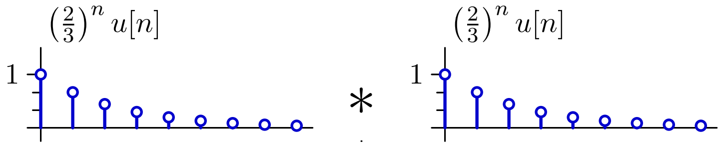
2. 

3. 

4. 

5. none of the above

Check Yourself



Express mathematically:

$$\left(\left(\frac{2}{3}\right)^n u[n]\right) * \left(\left(\frac{2}{3}\right)^n u[n]\right) = \sum_{m=-\infty}^{\infty} \left(\left(\frac{2}{3}\right)^m u[m]\right) \times \left(\left(\frac{2}{3}\right)^{n-m} u[n-m]\right)$$

$$= \sum_{m=0}^{n} \left(\frac{2}{3}\right)^m \times \left(\frac{2}{3}\right)^{n-m}$$

$$= \sum_{m=0}^{n} \left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^n \sum_{m=0}^{n} 1$$

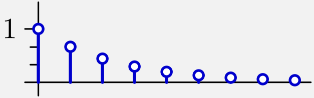
$$= (n+1)\left(\frac{2}{3}\right)^n u[n]$$

$$= 1,\ \frac{4}{3},\ \frac{4}{3},\ \frac{32}{27},\ \frac{80}{81},\ \ldots$$

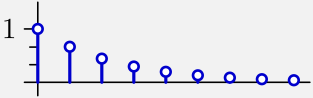
Check Yourself

Consider the convolution of two geometric sequences:

$$\left(\frac{2}{3}\right)^n u[n] \quad * \quad \left(\frac{2}{3}\right)^n u[n]$$

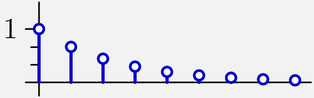


$*$

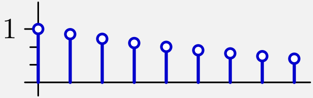


Which plot shows the result of the convolution above? 3

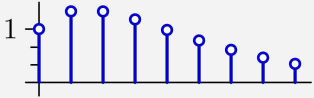
1.



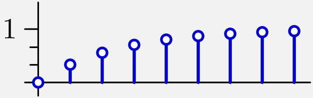
2.



3.



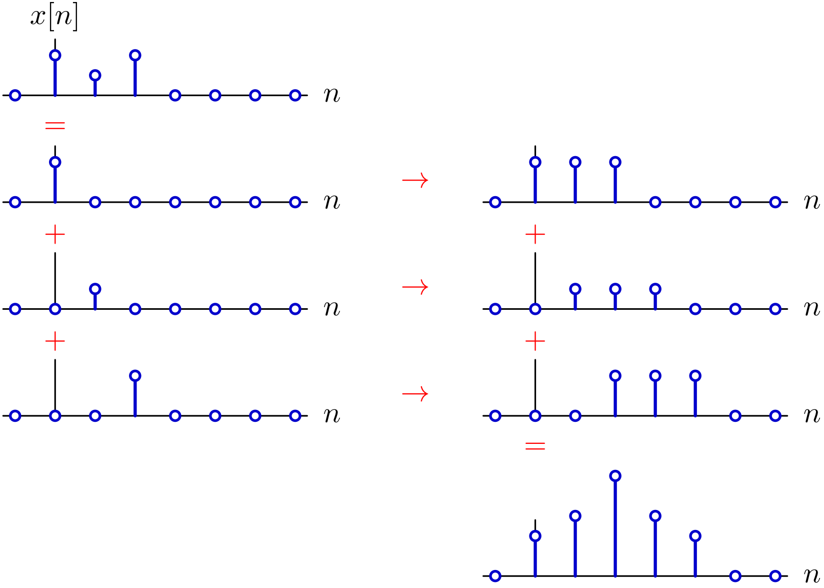
4.



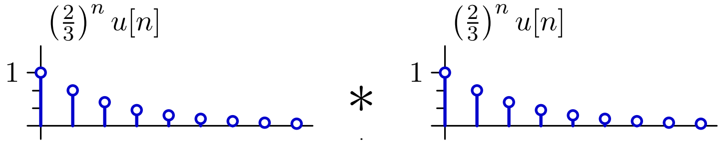
5. none of the above

Superposition

Superposition is often an easy way to implement convolution.



Check Yourself

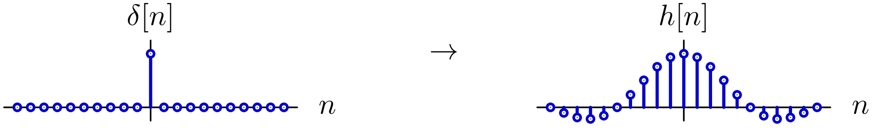


n:	0	1	2	3	4	5	
$x[0] \times h[n]$:	1	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{27}$	$\frac{16}{81}$	$\frac{32}{243}$	$\ldots$
$x[1] \times h[n-1]$:		$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{27}$	$\frac{16}{81}$	$\frac{32}{243}$	$\ldots$
$x[2] \times h[n-2]$:			$\frac{4}{9}$	$\frac{8}{27}$	$\frac{16}{81}$	$\frac{32}{243}$	$\ldots$
$x[3] \times h[n-3]$:				$\frac{8}{27}$	$\frac{16}{81}$	$\frac{32}{243}$	$\ldots$
$x[4] \times h[n-4]$:					$\frac{16}{81}$	$\frac{32}{243}$	$\ldots$
$x[5] \times h[n-5]$:						$\frac{32}{243}$	$\ldots$
$\sum x[m]h[n-m]$:	1	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{32}{27}$	$\frac{80}{81}$	$\frac{192}{243}$	$\ldots$

Unit-Sample Response

The unit-sample response is a **complete** description of an LTI system.

The response of a linear system to a unit sample signal



can be used to compute the response to any arbitrary input signal.

$$y[n] = (x * h)[n] \equiv \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Continuous-Time Systems

Superposition and convolution are also important for CT systems.

Linear Differential Equations with Constant Coefficients

If a continuous-time system can be described by a linear differential equation with constant coefficients, then the system is linear and time-invariant.

General form:

$$\sum_l c_l \frac{d^l y(t)}{dt^l} = \sum_m d_m \frac{d^m x(t)}{dt^m}$$

Such systems are easily shown to be linear and time-invariant.

Additivity: output of sum is sum of outputs

$$\sum_l c_l \left(\frac{d^l y_1(t)}{dt^l} + \frac{d^l y_2(t)}{dt^l} \right) = \sum_m d_m \left(\frac{d^m x_1(t)}{dt^m} + \frac{d^m x_2(t)}{dt^m} \right)$$

Homogeneity: scaling an input scales its output

$$\sum_l c_l \left(\alpha \frac{d^l y(t)}{dt^l} \right) = \sum_m d_m \left(\alpha \frac{d^m x(t)}{dt^m} \right)$$

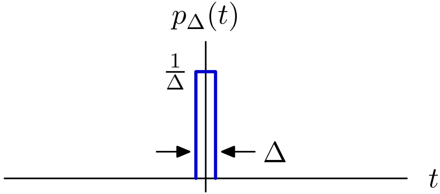
Time invariance: delaying an input delays its output

$$\sum_l c_l \frac{d^l y(t-\tau)}{dt^l} = \sum_m d_m \frac{d^m x(t-\tau)}{dt^m}$$

Impulse Response

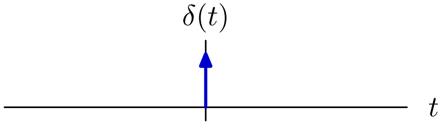
A CT system is completely characterized by its **impulse response**, much as a DT system is completely characterized by its unit-sample response.

We have worked with the impulse (Dirac delta) function $\delta(t)$ previously.
It's defined in a limit as follows.
Let $p_\Delta(t)$ represent a pulse of width Δ and height $\frac{1}{\Delta}$ so that its area is 1.



Then

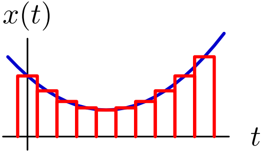
$$\delta(t) = \lim_{\Delta \to 0} p_\Delta(t)$$



Impulse Response

An arbitrary CT signal can be represented by an infinite sum of infinitesimal impulses (which define an integral).

Approximate an arbitrary signal $x(t)$ (blue) as a sum of pulses $p_\Delta(t)$ (red).



$$x_\Delta(t) = \sum_{m=-\infty}^{\infty} x(m\Delta)p_\Delta(t-m\Delta)\Delta$$

and the limit of $x_\Delta(t)$ as $\Delta \to 0$ will approximate $x(t)$.

$$\lim_{\Delta\to 0} x_\Delta(t) = \lim_{\Delta\to 0} \sum_{m=-\infty}^{\infty} x(m\Delta)p_\Delta(t-m\Delta)\Delta \to \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

The result in CT is much like the result for DT:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \qquad\qquad x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta(n-m)$$

Impulse Response

If a system is linear and time-invariant (LTI), its input-output relation is **completely specified** by the system's impulse response $h(t)$.

1. One can always find the impulse response of a system.

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

2. Time invariance implies that shifting the input simply shifts the output.

$$\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow h(t-\tau)$$

3. Homogeneity implies that scaling the input simply scales the output.

$$x(\tau)\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow x(\tau)h(t-\tau)$$

4. Additivity implies that the response to a sum is the sum of responses.

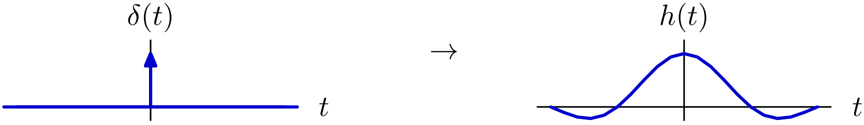
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \boxed{\text{system}} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

This rule for combining the input $x(t)$ with the impulse response $h(t)$ is called **convolution**.

Impulse Response

The impulse response is a **complete** description of an LTI system.

The response of a linear system to an impulse function



can be used to compute the response to any arbitrary input signal.

$$y(t) = (x * h)(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Comparison of CT and DT Convolution

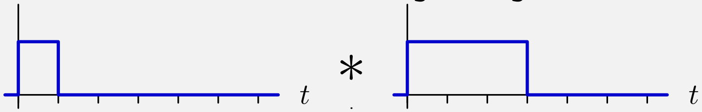
Convolution of CT signals is analogous to convolution of DT signals.

DT: $$y[n] = (x * h)[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

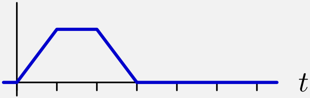
CT: $$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

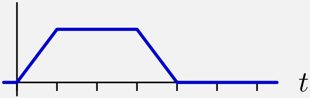
Check Yourself

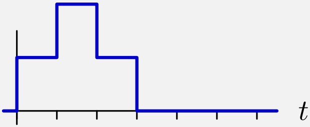
Consider the convolution of two rectangular signals:

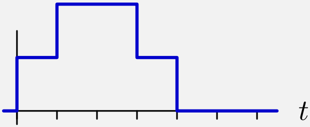


Which plot shows the result of the convolution above?

1. 

2. 

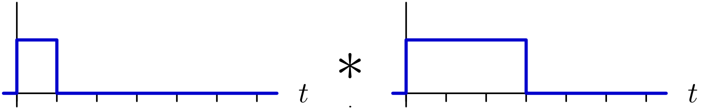
3. 

4. 

5. none of the above

Check Yourself

Which plot shows the result of the following convolution?



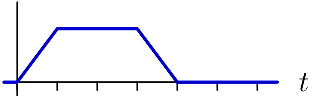
Start by flipping the first function about $t = 0$.

Multiply by the second function. No overlap $\rightarrow 0$.

As the flipped function slides right, the overlap increases linearly for first unit.

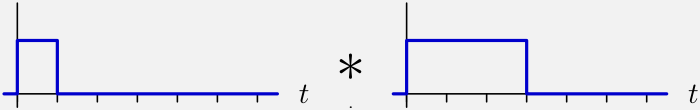
Then there is constant overlap for 2 units.

Then the overlap linearly decreases for the next unit till it is back to zero.

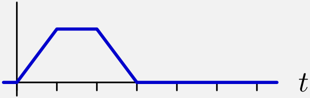


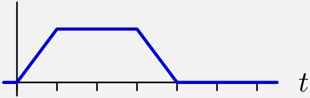
Check Yourself

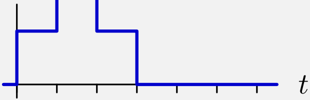
Consider the convolution of two rectangular signals:

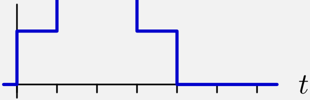


Which plot shows the result of the convolution above? 2

1. 

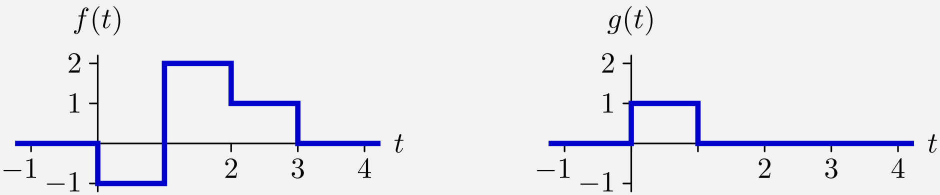
2. 

3. 

4. 

5. none of the above

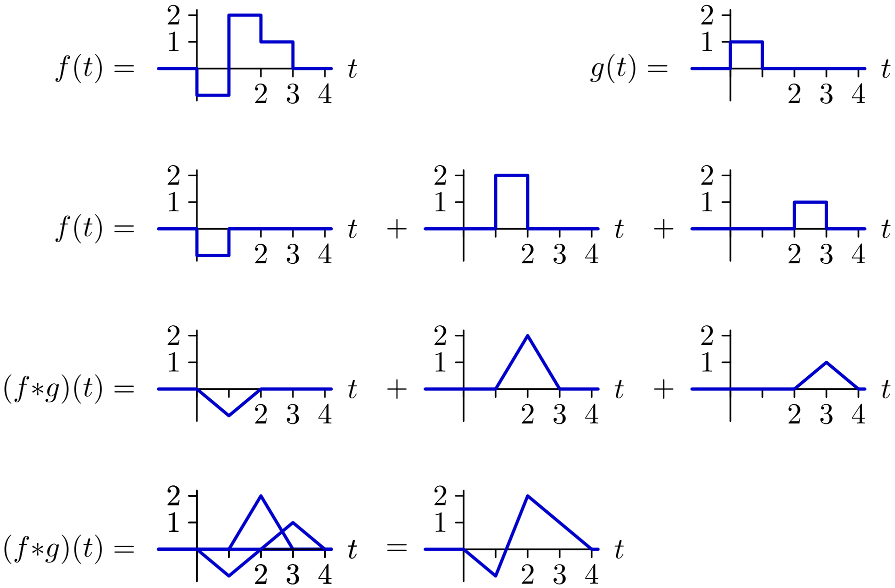
Check Yourself



For what value of t is $(f*g)(t)$ greatest?

What is the maximum value of $(f*g)(t)$?

Check Yourself



The peak value is 2 and it occurs at $t = 2$.

Properties of Convolution

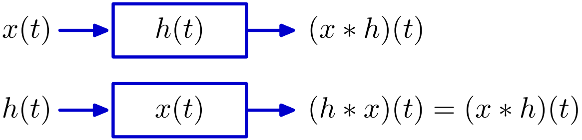
Commutivity:

$$(x * y)(t) = (y * x)(t)$$

$$(x * y)(t) \equiv \int_{-\infty}^{\infty} x(\tau) y(t-\tau) \, d\tau$$

Evaluate the integral with $\lambda = t-\tau$:

$$\begin{aligned} (x * y)(t) &= \int_{\infty}^{-\infty} x(t-\lambda) y(\lambda)(-d\lambda) \\ &= \int_{-\infty}^{\infty} y(\lambda) x(t-\lambda) \, d\lambda \\ &= (y * x)(t) \end{aligned}$$



Properties of Convolution

Associativity.

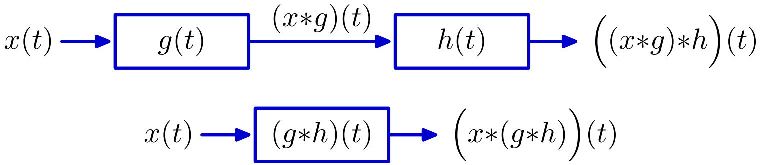
$$\Big((x*y)*z\Big)(t) = \Big(x*(y*z)\Big)(t)$$

$$\Big((x*y)*z\Big)(t) \equiv \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} x(\tau)y(\lambda-\tau)\,d\tau\right)}_{(x*y)(\lambda)} z(t-\lambda)\,d\lambda$$

Replace λ with $\lambda+\tau$ and swap the order of integration:

$$\Big((x*y)*z\Big)(t) = \int_{-\infty}^{\infty} x(\tau) \underbrace{\left(\int_{-\infty}^{\infty} y(\lambda)z(t-\lambda-\tau)\,d\lambda\right)}_{(y*z)(t-\lambda)} d\tau$$

$$= \Big(x*(y*z)\Big)$$

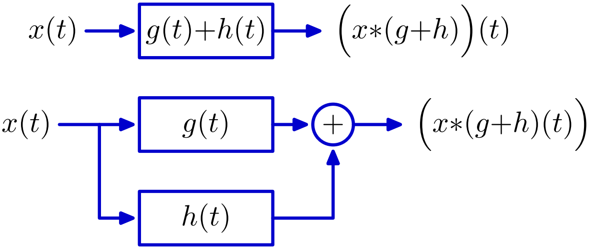


Properties of Convolution

Distributivity over addition.

$$\Big(x * (g+h)\Big)(t) = (x*g)(t) + (x*h)(t)$$

$$\begin{aligned}\Big(x * (g+h)\Big) &= \int_{-\infty}^{\infty} x(\tau)\Big(g(t-\tau)+h(t-\tau)\Big)\, d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)g(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= (x*g)(t) + (x*h)(t)\end{aligned}$$



Check Yourself

Match expressions on the left with functions on the right where

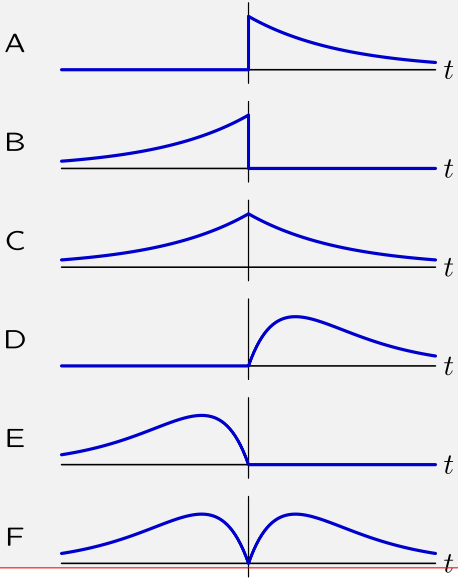
$$f(t) = e^{-t}\,u(t)$$
$$g(t) = e^{t}\,u(-t)$$

$(f * f)(t)$ ☐

$(g * g)(t)$ ☐

$(f * g)(t)$ ☐

$(g * f)(t)$ ☐



CT Convolution

Let

$$f(t) = e^{-t}\,u(t)$$

$$g(t) = e^{t}\,u(-t)$$

Find $(f * f)(t)$, $(g * g)(t)$, $(f * g)(t)$, and $(g * f)(t)$.

$$(f * f)(t) = \int_{-\infty}^{\infty} f(\tau)f(t-\tau)\,d\tau = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-(t-\tau)}u(t-\tau)\,d\tau$$

The integrand is zero unless $\tau > 0$ and $t - \tau > 0$.

Therefore $0 < \tau < t$ and $t > 0$.

$$(f * f)(t) = e^{-t}\int_0^t d\tau\,u(t) = te^{-t}\,u(t)$$

Answer: D

CT Convolution

Let

$$f(t) = e^{-t}\,u(t)$$

$$g(t) = e^{t}\,u(-t)$$

Find $(f * f)(t)$, $(g * g)(t)$, $(f * g)(t)$, and $(g * f)(t)$.

Since $g(t) = f(-t)$,

$$(g * g)(t) = \int_{-\infty}^{\infty} g(\tau)g(t-\tau)\,d\tau = \int_{-\infty}^{\infty} f(-\tau)f(-t+\tau)\,d\tau$$

Now let $\lambda = -\tau$:

$$\begin{aligned}(g * g)(t) &= \int_{\infty}^{-\infty} f(\lambda)f(-t-\lambda)\,(-d\lambda) \\ &= \int_{-\infty}^{\infty} f(\lambda)f(-t-\lambda)\,d\lambda \\ &= (f * f)(-t)\end{aligned}$$

Answer: E

CT Convolution

Let

$$f(t) = e^{-t}\,u(t)$$
$$g(t) = e^{t}\,u(-t)$$

Find $(f*f)(t)$, $(g*g)(t)$, $(f*g)(t)$, and $(g*f)(t)$.

$$(f*g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{t-\tau}u(\tau-t)\,d\tau$$

The integrand is zero unless $\tau > 0$ and $\tau > t$. Therefore $\tau > \max(0,t)$.

$$(f*g)(t) = \begin{cases} e^{t}\displaystyle\int_{0}^{\infty} e^{-2\tau}\,d\tau = \frac{1}{2}e^{t} & \text{if } t<0 \\ e^{t}\displaystyle\int_{t}^{\infty} e^{-2\tau}\,d\tau = \frac{1}{2}e^{-t} & \text{if } t>0 \end{cases}$$

$$= \frac{1}{2}e^{-|t|}$$

$$= (g*f)(t) \quad \text{since convolution is commutative}$$

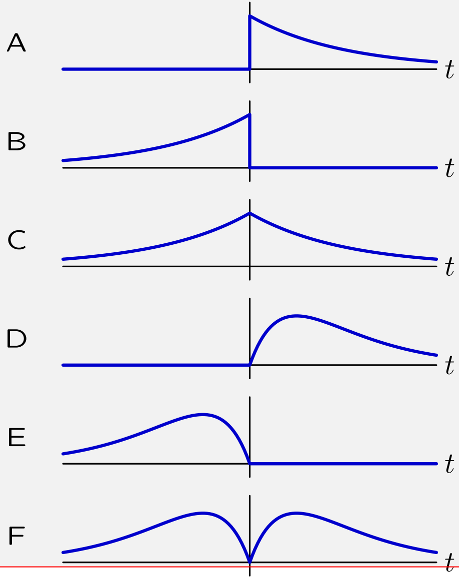
Answer: C

Check Yourself

Match expressions on the left with functions on the right where

$$f(t) = e^{-t}\,u(t)$$
$$g(t) = e^{t}\,u(-t)$$

$(f*f)(t)$	D
$(g*g)(t)$	E
$(f*g)(t)$	C
$(g*f)(t)$	C



Summary

The response of a discrete-time, LTI system to an input $x[n]$ can be computed by convolving the input with the system's **unit sample response**.

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \equiv (x * h)[n]$$

The response of a continuous-time, LTI system to an input $x(t)$ can be computed by convolving the input with the system's **impulse response**.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \equiv (x * h)(t)$$

Convolution allows us to represent a system by a **single signal!**

Question of the Day

Let

$$f[n] = \begin{cases} n+1 & \text{if } 0 \leq n \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$g[n] = (f * f)[n]$$

Determine $g[3]$.