6.3000: Signal Processing

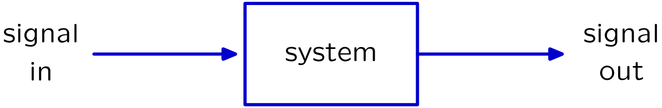
Unit-Sample Response and Convolution

- Unit Sample Signal and Unit Sample Response
- Discrete-Time Convolution
- Impulse Function and Impulse Response
- Continuous-Time Convolution

March 11, 2025

Last Time: The System Abstraction

Represent a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



Properties of Systems

We will focus primarily on systems that have two important properties:

- **linearity**
- **time invariance**

Such systems are both prevalent and mathematically tractable.

Additivity

A system is additive if its **response to a sum** of signals is equal to the **sum of the responses** to each signal taken one at a time.

Given

$$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

and

$$x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_2[n]$$

the **system is additive** if

$$x_1[n] + x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n] + y_2[n]$$

for all possible inputs and all times n.

Homogeneity

A system is homogeneous if multiplying its input signal by a constant multiplies the output signal by the same constant.

Given

$$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

the **system is homogeneous** if

$$\alpha x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow \alpha y_1[n]$$

for all α and all possible inputs and all times n.

Linearity

A system is linear if its **response to a weighted sum** of input signals is equal to the **weighted sum of its responses** to each of the input signals.

Given

$$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

and

$$x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_2[n]$$

the **system is linear** if

$$\alpha x_1[n] + \beta x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow \alpha y_1[n] + \beta y_2[n]$$

for all α and β and all possible inputs and all times n.

A system is linear if it is both additive and homogeneous.

Time-Invariance

A system is time-invariant if delaying the input signal simply delays the output signal by the same amount of time.

Given

$$x[n] \longrightarrow \boxed{\text{system}} \longrightarrow y[n]$$

the **system is time invariant** if

$$x[n-n_0] \longrightarrow \boxed{\text{system}} \longrightarrow y[n-n_0]$$

for all n_0 and for all possible inputs and all times n.

Linear Difference Equations with Constant Coefficients

If a discrete-time system can be described by a linear difference equation with constant coefficients, then the system is linear and time-invariant.

General form:

$$\sum_l c_l y[n-l] = \sum_m d_m x[n-m]$$

Such systems are easily shown to be linear and time-invariant.

Additivity: output of sum is sum of outputs

$$\sum_l c_l (y_1[n-l] + y_2[n-l]) = \sum_m d_m (x_1[n-m] + x_2[n-m])$$

Homogeneity: scaling an input scales its output

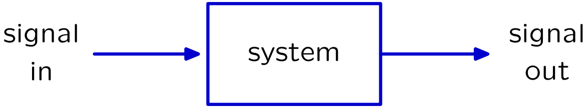
$$\sum_l c_l (\alpha y[n-l]) = \sum_m d_m (\alpha x[n-m])$$

Time invariance: delaying an input delays its output

$$\sum_l c_l y[(n-n_0)-l] = \sum_m d_m x[(n-n_0)-m]$$

Today: Representing a System by its Unit-Sample Response

Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



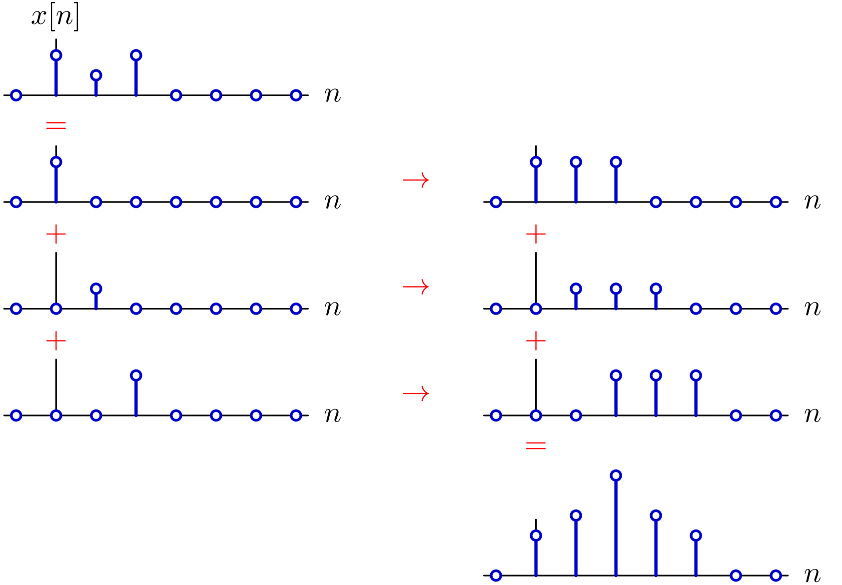
This abstraction is particularly powerful for **linear and time-invariant** systems, which are both **prevalent** and **mathematically tractable**.

Three important representations for LTI systems:

- **Difference Equation:** algebraic **constraint** on samples $\surd$
- **Convolution:** represent a system by its **unit-sample response**
- **Filter:** represent a system by its **frequency response**

Superposition

Response of an LTI system is determined by the system's response to $\delta[n]$.



Unit-Sample Response and Convolution

If a system is linear and time-invariant (LTI), its input-output relation is **completely specified** by the system's unit-sample response $h[n]$.

1. One can always find the unit-sample response of a system.

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

2. Time invariance implies that shifting the input simply shifts the output.

$$\delta[n-m] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n-m]$$

3. Homogeneity implies that scaling the input simply scales the output.

$$x[m]\delta[n-m] \longrightarrow \boxed{\text{LTI}} \longrightarrow x[m]h[n-m]$$

4. Additivity implies that the response to a sum is the sum of responses.

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

This rule for combining the input $x[n]$ with the unit-sample response $h[n]$ is called **convolution**.

Convolution

The response of an LTI system to an arbitrary input $x[n]$ can be found by **convolving** that input with the **unit-sample response** $h[n]$ of the system.

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]$$

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \equiv (x * h)[n]$$

This is an amazing result.

We can represent the operation of an LTI system by a **single signal**!

Notation

Convolution is represented with an asterisk.

$$\sum_{m=-\infty}^{\infty} x[m]h[n-m] \equiv (x * h)[n]$$

It is customary (but confusing) to abbreviate this notation:

$$(x * h)[n] = x[n] * h[n]$$

$x[n] * h[n]$ looks like an operation of samples; but it is not!

$$x[1] * h[1] \neq (x * h)[1]$$

Convolution operates on signals not samples.

Unambiguous notation:

$$y = x * h$$

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \equiv (x * h)[n]$$

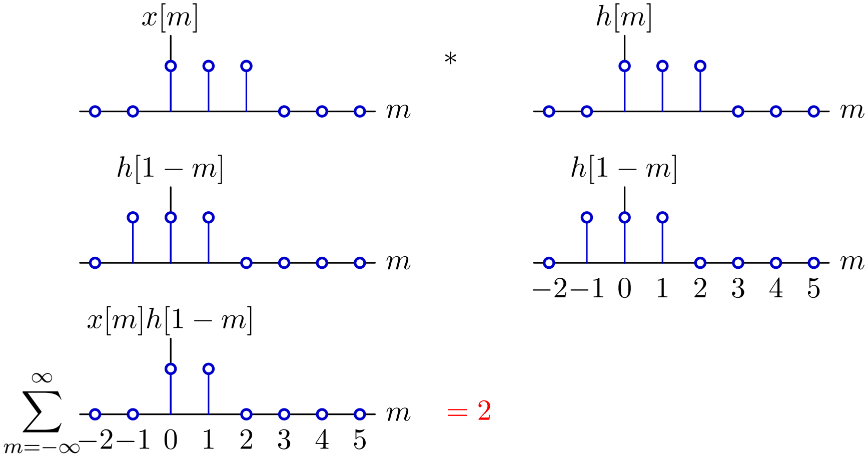
The symbols x and h represent DT signals.

Convolving x with h generates a new DT signal $y = x * h$.

Structure of Convolution

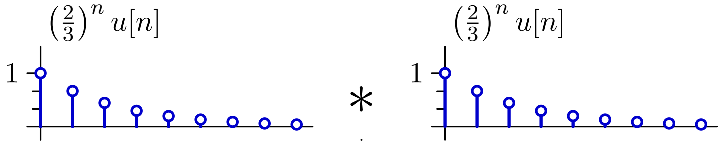
Focus on computing the n^{th} output sample.

$$y[1] = \sum_{m=-\infty}^{\infty} x[m]h[1-m]$$



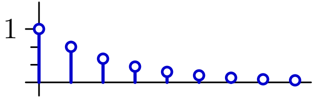
Check Yourself

Consider the convolution of two geometric sequences:



Which plot below shows the result of the convolution above?

1.



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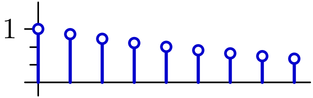

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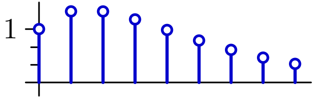

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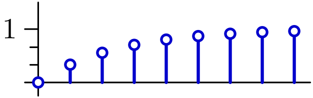


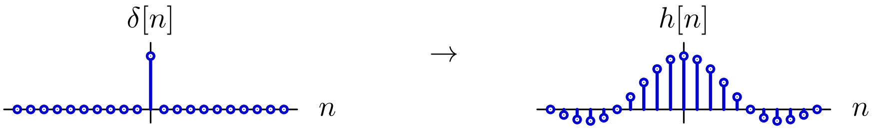
5. none of the above

Unit-Sample Response

The unit-sample response is a **complete** description of an LTI system.

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

The response of a linear system to a unit sample signal



can be used to compute the response to any arbitrary input signal.

$$y[n] = (x * h)[n] \equiv \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Continuous-Time Systems

Superposition and convolution are also important for CT systems.

Linear Differential Equations with Constant Coefficients

If a continuous-time system can be described by a linear differential equation with constant coefficients, then the system is linear and time-invariant.

General form:

$$\sum_l c_l \frac{d^l y(t)}{dt^l} = \sum_m d_m \frac{d^m x(t)}{dt^m}$$

Such systems are easily shown to be linear and time-invariant.

Additivity: output of sum is sum of outputs

$$\sum_l c_l \left(\frac{d^l y_1(t)}{dt^l} + \frac{d^l y_2(t)}{dt^l} \right) = \sum_m d_m \left(\frac{d^m x_1(t)}{dt^m} + \frac{d^m x_2(t)}{dt^m} \right)$$

Homogeneity: scaling an input scales its output

$$\sum_l c_l \left(\alpha \frac{d^l y(t)}{dt^l} \right) = \sum_m d_m \left(\alpha \frac{d^m x(t)}{dt^m} \right)$$

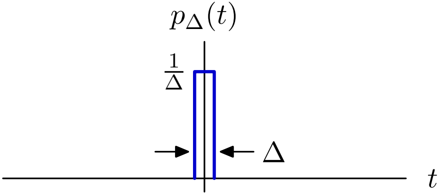
Time invariance: delaying an input delays its output

$$\sum_l c_l \frac{d^l y(t-\tau)}{dt^l} = \sum_m d_m \frac{d^m x(t-\tau)}{dt^m}$$

Impulse Response

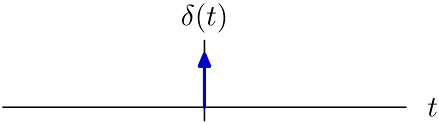
A CT system is completely characterized by its **impulse response**, much as a DT system is completely characterized by its unit-sample response.

We have worked with the impulse (Dirac delta) function $\delta(t)$ previously. It's defined in a limit as follows.
Let $p_\Delta(t)$ represent a pulse of width Δ and height $\frac{1}{\Delta}$ so that its area is 1.



Then

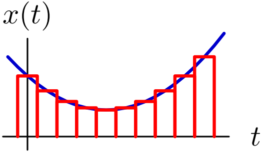
$$\delta(t) = \lim_{\Delta \to 0} p_\Delta(t)$$



Impulse Response

An arbitrary CT signal can be represented by an infinite sum of infinitesimal impulses (which define an integral).

Approximate an arbitrary signal $x(t)$ (blue) as a sum of pulses $p_\Delta(t)$ (red).



$$x_\Delta(t) = \sum_{m=-\infty}^{\infty} x(m\Delta)p_\Delta(t-m\Delta)\Delta$$

and the limit of $x_\Delta(t)$ as $\Delta \to 0$ will approximate $x(t)$.

$$\lim_{\Delta\to 0} x_\Delta(t) = \lim_{\Delta\to 0} \sum_{m=-\infty}^{\infty} x(m\Delta)p_\Delta(t-m\Delta)\Delta \to \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

The result in CT is much like the result for DT:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \qquad\qquad x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta(n-m)$$

Impulse Response

If a system is linear and time-invariant (LTI), its input-output relation is **completely specified** by the system's impulse response $h(t)$.

1. One can always find the impulse response of a system.

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

2. Time invariance implies that shifting the input simply shifts the output.

$$\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow h(t-\tau)$$

3. Homogeneity implies that scaling the input simply scales the output.

$$x(\tau)\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow x(\tau)h(t-\tau)$$

4. Additivity implies that the response to a sum is the sum of responses.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \boxed{\text{system}} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

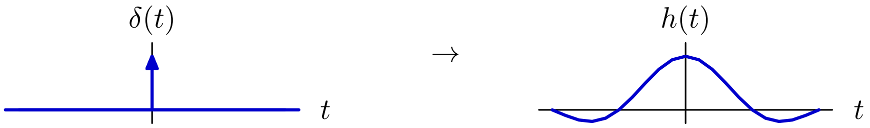
This rule for combining the input $x(t)$ with the impulse response $h(t)$ is called **convolution**.

Impulse Response

The impulse response is a **complete** description of an LTI system.

$$\delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t)$$

The response of a linear system to an impulse function



can be used to compute the response to any arbitrary input signal.

$$y(t) = (x * h)(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Comparison of CT and DT Convolution

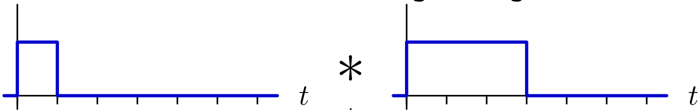
Convolution of CT signals is analogous to convolution of DT signals.

DT: $$y[n]=(x*h)[n]=\sum_{m=-\infty}^{\infty}x[m]h[n-m]$$

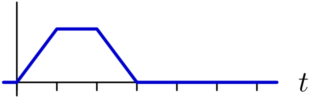
CT: $$y(t)=(x*h)(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$$

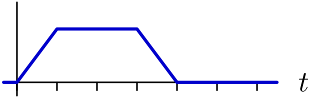
Check Yourself

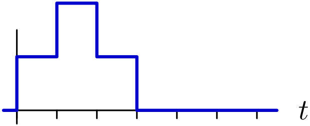
Consider the convolution of two rectangular signals:

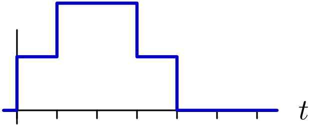


Which plot shows the result of the convolution above?

1. 

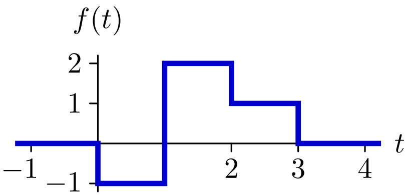
2. 

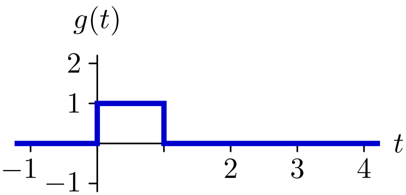
3. 

4. 

5. none of the above

Check Yourself





For what value of t is $(f*g)(t)$ greatest?

What is the maximum value of $(f*g)(t)$?

Properties of Convolution

Commutivity:

$$(x * y)(t) = (y * x)(t)$$

$$(x * y)(t) \equiv \int_{-\infty}^{\infty} x(\tau) y(t-\tau)\, d\tau$$

Evaluate the integral with $\lambda = t-\tau$:

$$\begin{aligned}(x * y)(t) &= \int_{\infty}^{-\infty} x(t-\lambda) y(\lambda)(-d\lambda) \\ &= \int_{-\infty}^{\infty} y(\lambda) x(t-\lambda)\, d\lambda \\ &= (y * x)(t)\end{aligned}$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow (x * h)(t)$$

$$h(t) \longrightarrow \boxed{x(t)} \longrightarrow (h * x)(t) = (x * h)(t)$$

Properties of Convolution

Associativity.

$$\Big((x*y)*z\Big)(t) = \Big(x*(y*z)\Big)(t)$$

$$\Big((x*y)*z\Big)(t) \equiv \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} x(\tau)y(\lambda-\tau)\,d\tau\right)}_{(x*y)(\lambda)} z(t-\lambda)\,d\lambda$$

Replace λ with $\lambda+\tau$ and swap the order of integration:

$$\Big((x*y)*z\Big)(t) = \int_{-\infty}^{\infty} x(\tau) \underbrace{\left(\int_{-\infty}^{\infty} y(\lambda)z(t-\lambda-\tau)\,d\lambda\right)}_{(y*z)(t-\lambda)} d\tau$$

$$= \Big(x*(y*z)\Big)$$

$$x(t) \longrightarrow \boxed{g(t)} \xrightarrow{(x*g)(t)} \boxed{h(t)} \longrightarrow \Big((x*g)*h\Big)(t)$$

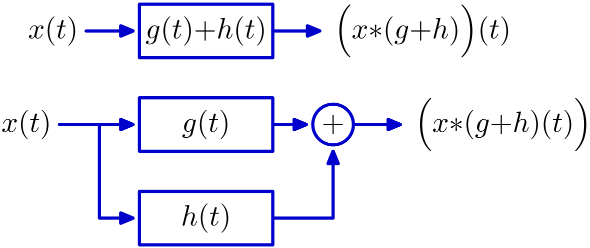
$$x(t) \longrightarrow \boxed{(g*h)(t)} \longrightarrow \Big(x*(g*h)\Big)(t)$$

Properties of Convolution

Distributivity over addition.

$$\Big(x * (g+h)\Big)(t) = (x*g)(t) + (x*h)(t)$$

$$\begin{aligned}\Big(x * (g+h)\Big) &= \int_{-\infty}^{\infty} x(\tau)\Big(g(t-\tau)+h(t-\tau)\Big)\, d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)g(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= (x*g)(t) + (x*h)(t)\end{aligned}$$



Check Yourself

Match expressions on the left with functions on the right where

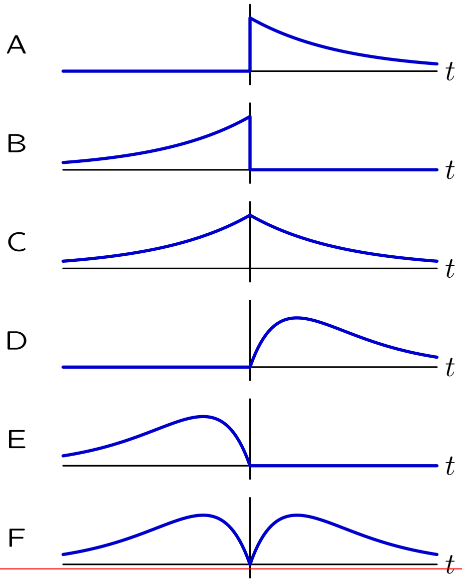
$$f(t) = e^{-t}\,u(t)$$
$$g(t) = e^{t}\,u(-t)$$

$(f * f)(t)$

$(g * g)(t)$

$(f * g)(t)$

$(g * f)(t)$



Summary

The response of a discrete-time, LTI system to an input $x[n]$ can be computed by convolving the input with the system's **unit sample response**.

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]$$

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \equiv (x * h)[n]$$

The response of a continuous-time, LTI system to an input $x(t)$ can be computed by convolving the input with the system's **impulse response**.

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \equiv (x * h)(t)$$

Convolution allows us to represent a system by a **single signal!**

Question of the Day

Let

$$f[n] = \begin{cases} n+1 & \text{if } 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$g[n] = (f * f)[n]$$

Determine $g[3]$.