6.3000: Signal Processing

Sinusoids and Series

- Series representations of discontinuous functions.
- Relations between time and frequency.
- Fourier analysis of a vibrating string.

Homework 1 is posted and will be due next Thursday (Feb 13) at 2pm.

We will have **office hours** today from 4-5pm in 34-302.

Optional Tutorials:
This semester, we will offer small, interactive tutorials as an optional alternative to recitations. See our website for more information.

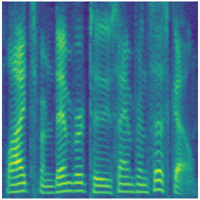
February 06, 2025

Last Time

Signals are functions that contain and convey information.

Examples:

- the MP3 representation of a sound
- the JPEG representation of a picture
- an MRI image of a brain

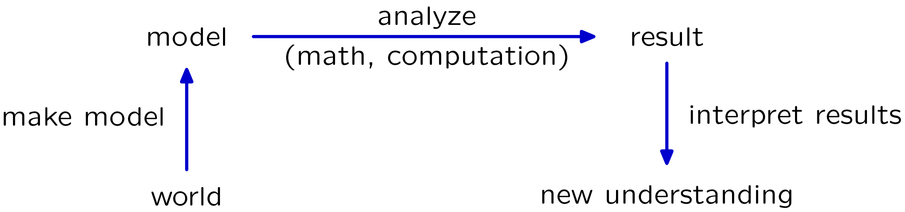

Signal Processing develops the use of signals as abstractions:

- **identifying** signals in physical, mathematical, computation contexts,
- **analyzing** signals to understand the information they contain, and
- **manipulating** signals to modify the information they contain.

Last Time

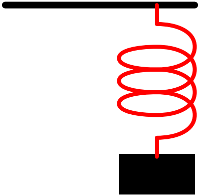
Signal Processing is **widely used** in science and engineering to ...

- **model** some aspect of the world,
- **analyze** the model and get a result, then
- **interpret** the result to gain a new or better understanding.



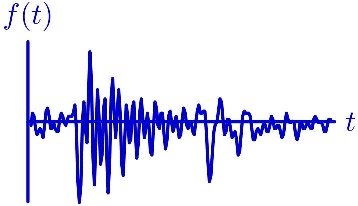
Signal Processing provides a common language across disciplines.

Example: Mass and Spring



Signals as Abstractions

Relation between a signal and the information contained in that signal.



Listen to the following four manipulated signals:

$$f_1(t),\ f_2(t),\ f_3(t),\ f_4(t).$$

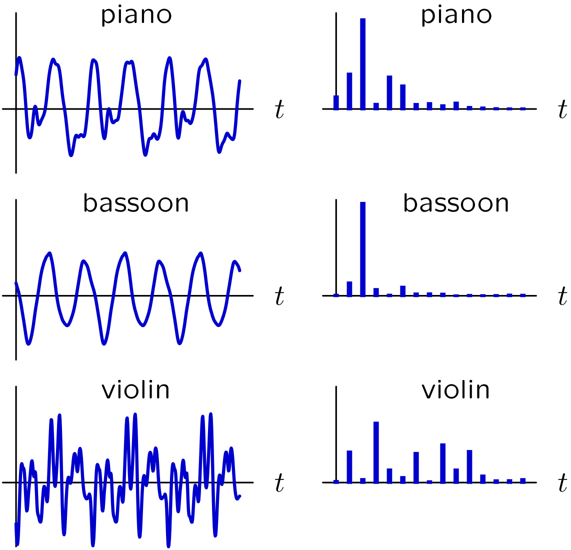
How many of the following relations are true?

- $f_1(t) = f(2t)$
- $f_2(t) = -f(t)$
- $f_3(t) = f(2t)$
- $f_4(t) = \frac{1}{3} f(t)$

* speech signal synthesized by Robert Donovan

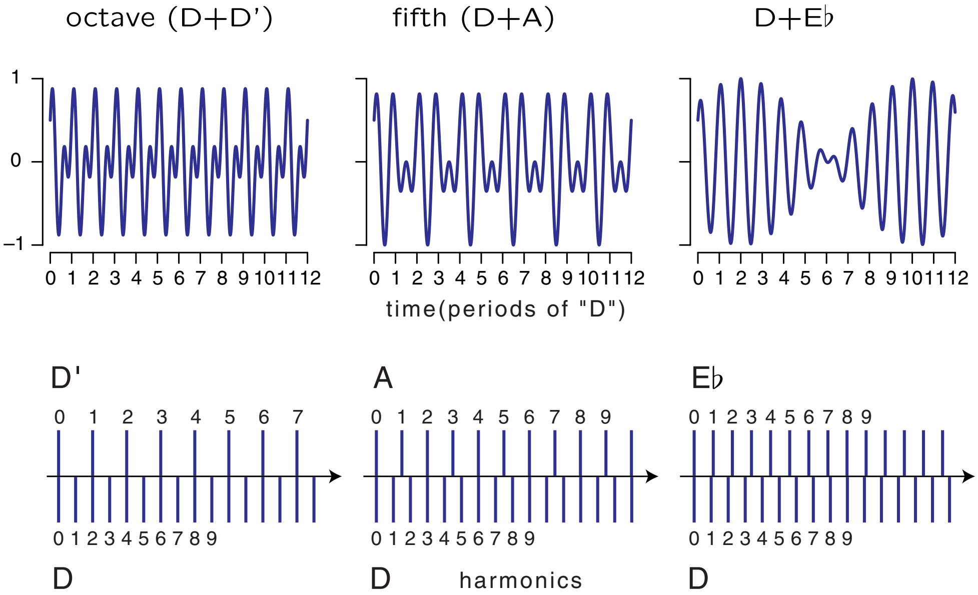
Last Time

A frequency representation provides a different view of a signal, and can help expose important properties of the signal.



Last Time

Time functions do a poor job of conveying consonance and dissonance.



Harmonic structure conveys consonance and dissonance better.

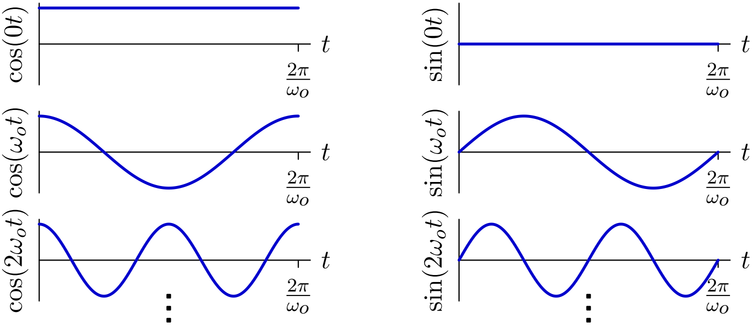
Fourier Series

Fourier series are weighted sums of harmonically related sinusoids.

$$f(t) = \sum_{k=0}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right)$$

where $\omega_o = 2\pi/T$ represents the fundamental frequency.

Basis functions:



Fourier Series

How do we find the coefficients c_k and d_k?

Key idea: Sift out the component of interest by

- multiplying by the corresponding basis function, and then
- integrating over a period.

This results in the following expressions for the Fourier series coefficients:

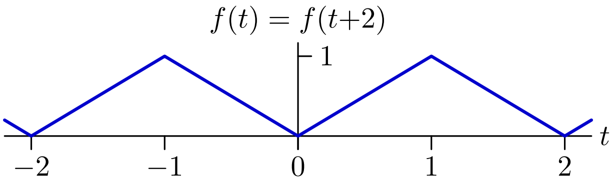
$$c_0 = \frac{1}{T}\int_T f(t)\,dt$$

$$c_k = \frac{2}{T}\int_T f(t)\cos(k\omega_o t)\,dt;\;\; k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T}\int_T f(t)\sin(k\omega_o t)\,dt;\;\; k = 1, 2, 3, \ldots$$

Example of Analysis

Find the Fourier series coefficients for the following triangle wave:



$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

$$c_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi k t}{T}\,dt = 2\int_0^1 t\cos(\pi k t)\,dt = \begin{cases} -\frac{4}{\pi^2 k^2} & k \text{ odd} \\ 0 & k = 2, 4, 6, \ldots \end{cases}$$

$$d_k = 0 \quad \text{(by symmetry)}$$

Fourier Synthesis

The previous example shows that the sum of an infinite number of sinusoids can approximate a piecewise linear function **with discontinuous slope**!

This result is a bit surprising since none of the basis functions have discontinuous slopes.

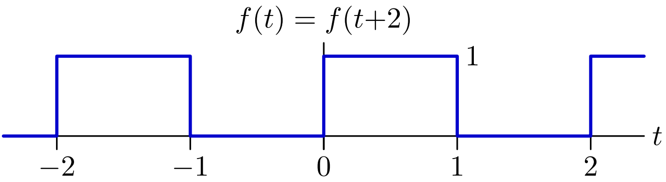
What about signals **with discontinuous values**?

Fourier defended the idea that such a series is meaningful.
Lagrange ridiculed the idea that discontinuities could be generated from a sum of continuous signals.

We can test this idea empirically – using computation.

Fourier Analysis of a Square Wave

Find the Fourier series coefficients for the following square wave:



$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

$$c_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega_o t)\,dt = \int_0^1 \cos(k\pi t)dt = \left.\frac{\sin(k\pi t)}{k\pi}\right|_0^1 = 0 \text{ for } k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega_o t)\,dt = \int_0^1 \sin(k\pi t)dt = -\left.\frac{\cos(k\pi t)}{k\pi}\right|_0^1 = \begin{cases} \frac{2}{k\pi} & k = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases}$$

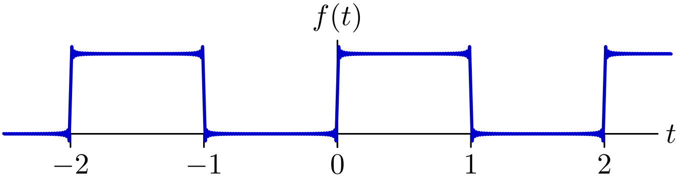
Fourier Synthesis of a Square Wave

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)$$

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{99} \frac{2}{k\pi} \sin(k\pi t)$$

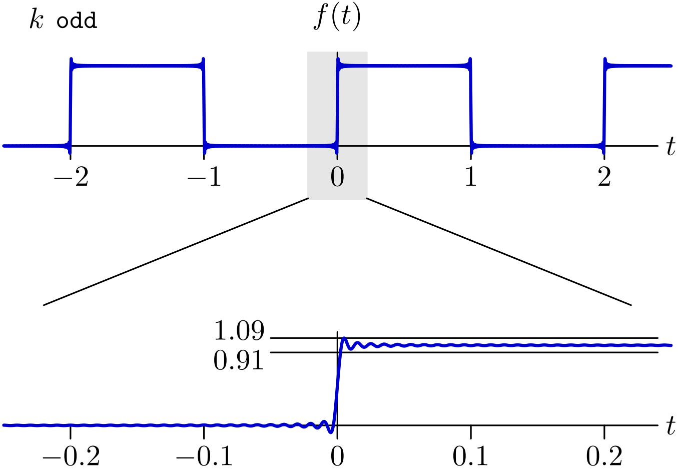


The synthesized function approaches original as number of terms increases.

Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{199} \frac{2}{k\pi} \sin(k\pi t)$$



Increasing the number of terms does not decrease the peak overshoot, but it does shrink the region of time that is occupied by the overshoot.

Convergence of Fourier Series

If there is a **step discontinuity** in $f(t)$ at $t = t_0$, then the Fourier series for $f(t_0)$ converges to the average of the limits of $f(t)$ as t approaches t_0 from the left and from the right.

Let $f_K(t)$ represent the **partial sum** of the Fourier series using just N terms:

$$f_K(t) = a_0 + \sum_{k=0}^{K} \Big(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big)$$

As $K \to \infty$,

- the maximum difference between $f(t)$ and $f_K(t)$ converges to $\approx 9\%$ of $|f(t_0^+) - f(t_0^-)|$ and
- the region over which the absolute value of the difference exceeds any small number ϵ shrinks to zero.

We refer to this type of overshoot as **Gibb's Phenomenon**.

So who was right? Fourier or Lagrange?

Both. The series representation of a discontinuous function converges, but not uniformly.

Properties of Fourier Series

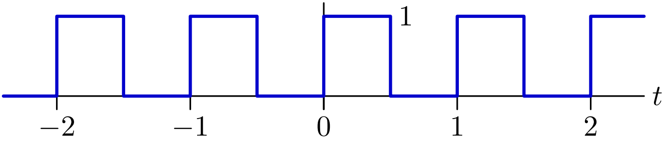
How do changes in time affect their frequency representation?

$\rightarrow$ investigate **properties** of Fourier representations

Properties of Fourier Series: Scaling Time

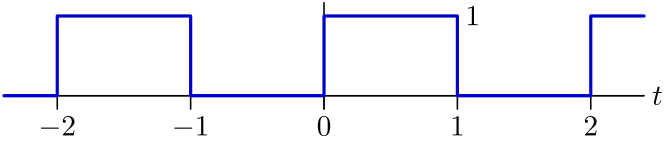
Find the Fourier series coefficients for the following square wave:

$$g(t) = g(t+1)$$



We could repeat the process used to find the Fourier coefficients for $f(t)$.

$$f(t) = f(t+2)$$



Alternatively, we can take advantage of the relation between $f(t)$ and $g(t)$:

$$g(t) = f(2t)$$

Scaling Time

We already know the Fourier series expansion of $f(t)$:

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi} \sin(k\pi t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi} \sin(k\omega_o t)$$

$$d_k = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{2}{k\pi} & k = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad c_k = 0$$

where $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$.

Check Yourself

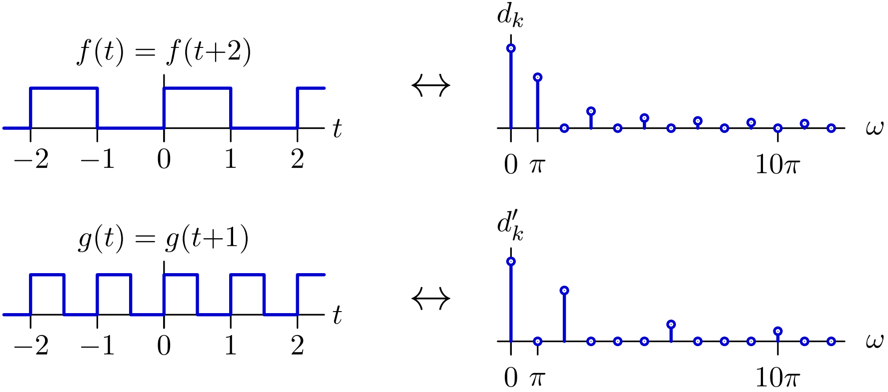
Let d_k represent the Fourier series coefficients for $f(t)$ and let d'_k represent those for $g(t) = f(2t)$.

Which of the following relations are true?

- $d'_k = 2d_k$: amplitudes double
- $d'_k = d_{2k}$: harmonic indicies half
- $d'_k = d_{k/2}$: harmonic indicies double
- $d'_k = 2d_{k/2}$: amplitudes and harmonic indices double
- $d'_k = d_k$: no change

Scaling Time

Plot the Fourier series coefficients on a frequency scale.



Compressing the time axis has stretched the ω axis.

What is the Effect of Shifting Time?

Assume that $f(t)$ is periodic in time with period T:

$$f(t) = f(t+T)\,.$$

Let $g(t)$ represent a version of $f(t)$ shifted by half a period:

$$g(t) = f(t-T/2)\,.$$

How many of the following statements correctly describe the effect of this shift on the Fourier series coefficients.

- cosine coefficients c_k are negated
- sine coefficients d_k are negated
- odd-numbered coefficients $c_1, d_1, c_3, d_3, \ldots$ are negated
- sine and cosine coefficients are swapped: $c_k \rightarrow d_k$ and $d_k \rightarrow c_k$

Why Focus on Fourier Series?

What's so special about sines and cosines?

Sinusoidal functions have interesting **mathematical properties**.
$\rightarrow$ harmonically related sinusoids are **orthogonal** to each other over $[0, T]$.

Orthogonality: $f(t)$ and $g(t)$ are orthogonal over $0 \leq t \leq T$ if

$$\int_T f(t)g(t)\,dt = 0$$

Example: Calculate this integral for the k^{th} and l^{th} harmonics of $\cos(\omega_o t)$.

$$\int_T \cos(k\omega_o t)\cos(l\omega_o t)\,dt$$

We can use trigonometry to express the product of the two cosines as the sum of cosines of the sum and difference frequencies:

$$\int_T \Big(\frac{1}{2}\cos((k{+}l)\omega_o t) + \frac{1}{2}\cos((k{-}l)\omega_o t)\Big)\,dt$$

The sum and difference frequencies are also harmonics of ω_o, so their integral over T is zero (provided $k \neq l$).

Why Focus on Fourier Series?

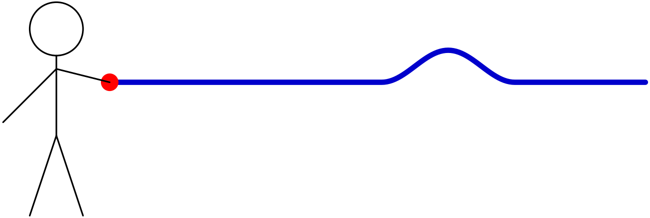
What's so special about sines and cosines?

Sinusoidal functions have interesting **mathematical properties**.
$\rightarrow$ harmonically related sinusoids are **orthogonal** to each other over $[0, T]$.

Sines and cosines also play important roles in **physics** –
especially the physics of waves.

Physical Example: Vibrating String

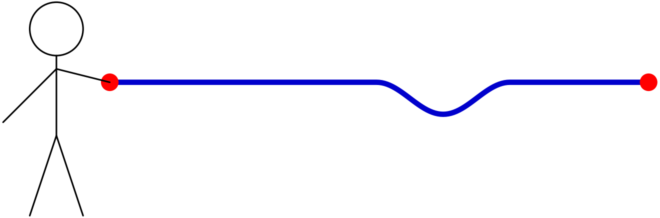
A taut string supports wave motion.



The speed of the wave depends on the tension on and mass of the string.

Physical Example: Vibrating String

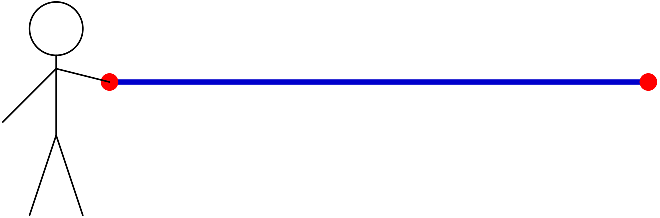
The wave will reflect off a rigid boundary.



The amplitude of the reflected wave is opposite that of the incident wave.

Physical Example: Vibrating String

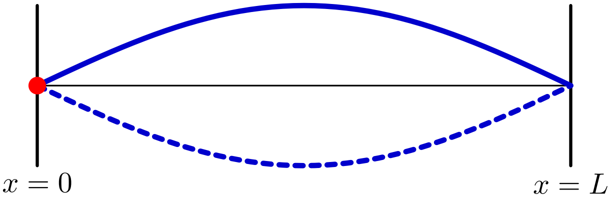
Reflections can interfere with excitations.



The interference can be constructive or destructive depending on the frequency of the excitation.

Physical Example: Vibrating String

We get constructive interference if round-trip travel time equals the period.

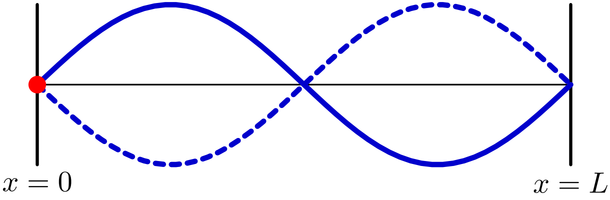


Round-trip travel time $= \dfrac{2L}{v} = T$

$$\omega_o = \frac{2\pi}{T} = \frac{2\pi}{2L/v} = \frac{\pi v}{L}$$

Physical Example: Vibrating String

We also get constructive interference if round-trip travel time is $2T$.

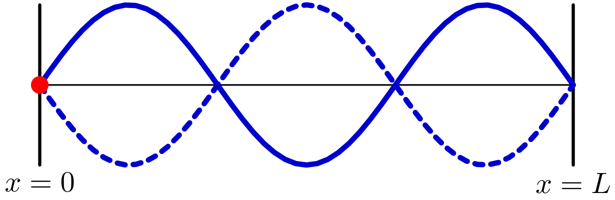


Round-trip travel time $= \dfrac{2L}{v} = 2T$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{L/v} = \frac{2\pi v}{L} = 2\omega_o$$

Physical Example: Vibrating String

In fact, we also get constructive interference if round-trip travel time is kT.



Round-trip travel time $= \dfrac{2L}{v} = kT$

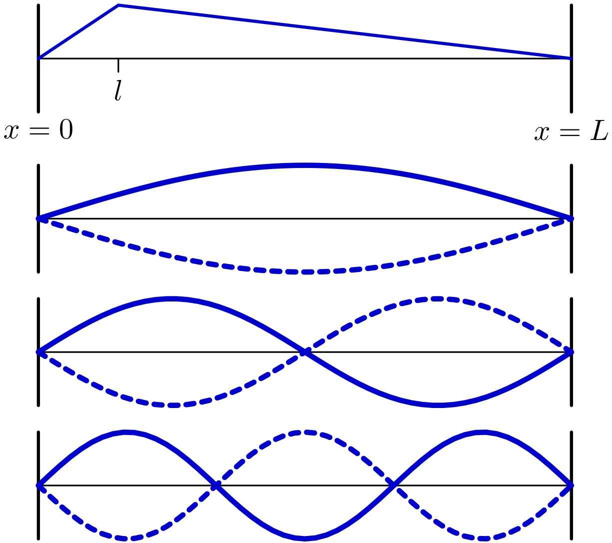
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2L/kv} = \frac{k\pi v}{L} = k\omega_o$$

Only certain frequencies (harmonics of $\omega_o = \pi v/L$) persist.

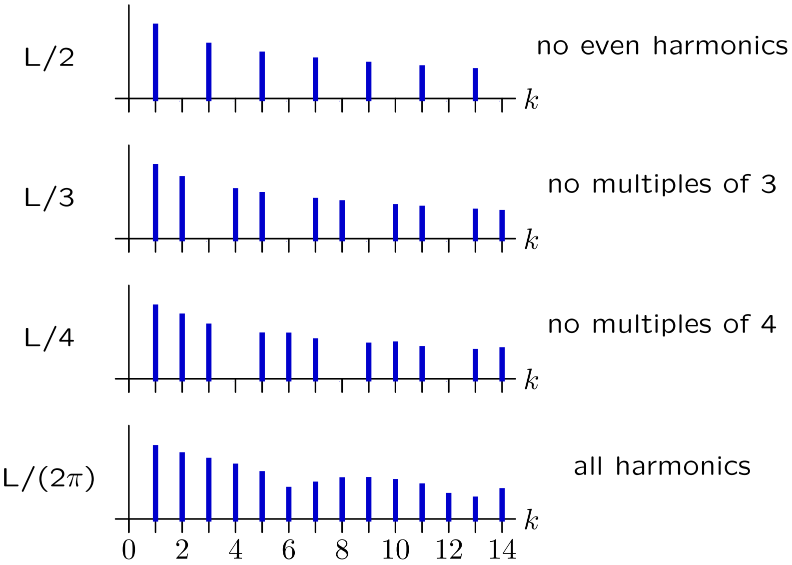
This is the basis of stringed instruments.

Physical Example: Vibrating String

More complicated motions can be expressed as a sum of normal modes using Fourier series. Here the string is "plucked" at $x = l$.



Physical Example: Vibrating String



Differences in harmonic structure generate differences in timbre.

Summary

- We examined the convergence of Fourier series.
  - Functions with discontinuous slopes well represented.
  - Functions with discontinuous values generate ripples
    $\rightarrow$ Gibb's phenomenon.
- We investigated several **properties** of Fourier series.
  - scaling time
  - shifting time
  - We will find that there are **many** others
- We saw how Fourier series are useful for modeling a vibrating string.