6.3000: Signal Processing

Signal Processing

- Overview of Subject
- Signals: Definitions, Examples, and Operations
- Time and Frequency Representations
- Fourier Series

Lecture slides are available at the course website:

`http://mit.edu/6.3000`

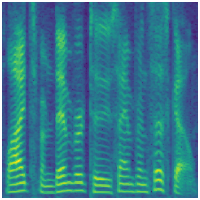
February 04, 2025

6.3000: Signal Processing

Signals are functions that contain and convey information.

Examples:

- the MP3 representation of a sound
- the JPEG representation of a picture
- an MRI image of a brain

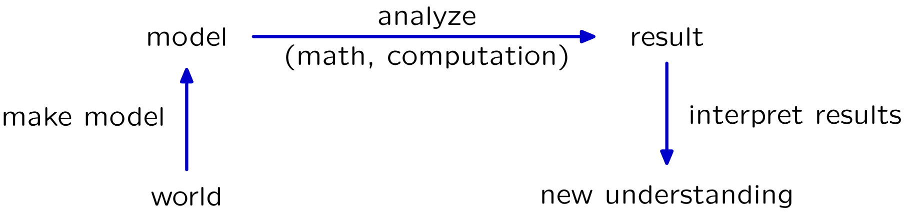

Signal Processing develops the use of signals as abstractions:

- **identifying** signals in physical, mathematical, computation contexts,
- **analyzing** signals to understand the information they contain, and
- **manipulating** signals to modify the information they contain.

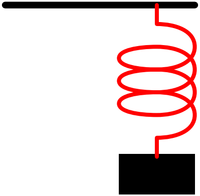
6.3000: Signal Processing

Signal Processing is **widely used** in science and engineering to ...

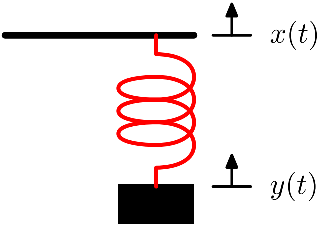
- **model** some aspect of the world,
- **analyze** the model and get a result, then
- **interpret** the result to gain a new or better understanding.

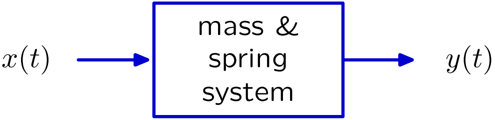


Example: Mass and Spring

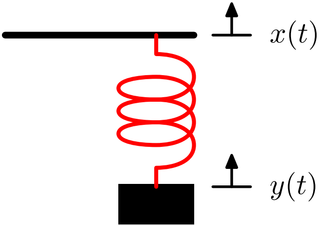


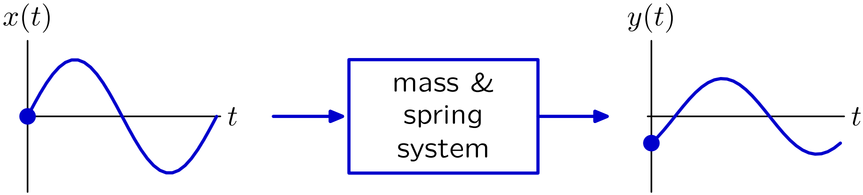
Example: Mass and Spring





Example: Mass and Spring

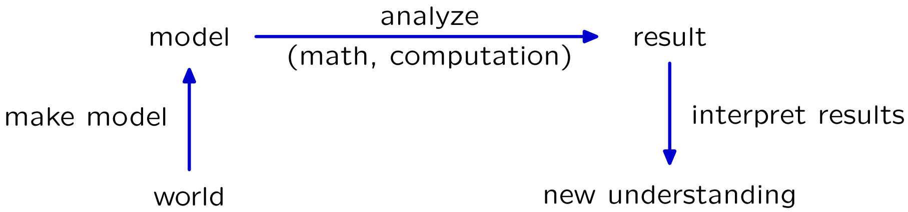




6.3000: Signal Processing

Signal Processing is **widely used** in science and engineering to ...

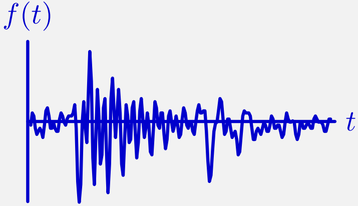
- **model** some aspect of the world,
- **analyze** the model, and
- **interpret** results to gain a new or better understanding.



Signal Processing provides a common language across disciplines.

Check Yourself

Relation between a signal and the information contained in that signal.



Listen to the following four manipulated signals:

$$f_1(t),\ f_2(t),\ f_3(t),\ f_4(t).$$

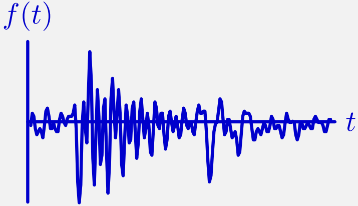
How many of the following relations are true?

- $f_1(t) = f(2t)$
- $f_2(t) = -f(t)$
- $f_3(t) = f(2t)$
- $f_4(t) = \frac{1}{3}f(t)$

* speech signal synthesized by Robert Donovan

Check Yourself

Relation between a signal and the information contained in that signal.



Listen to the following four manipulated signals:

$$f_1(t),\ f_2(t),\ f_3(t),\ f_4(t).$$

How many of the following relations are true? 2

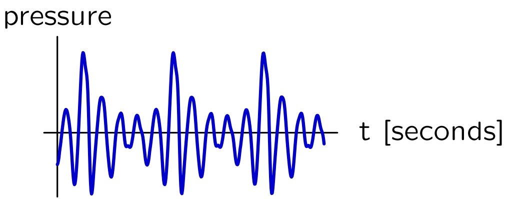
- $f_1(t) = f(2t)$ √
- $f_2(t) = -f(t)$ ×
- $f_3(t) = f(2t)$ ×
- $f_4(t) = \frac{1}{3}f(t)$ √

* speech signal synthesized by Robert Donovan

Musical Sounds as Signals

Signals are functions that contain and convey information.

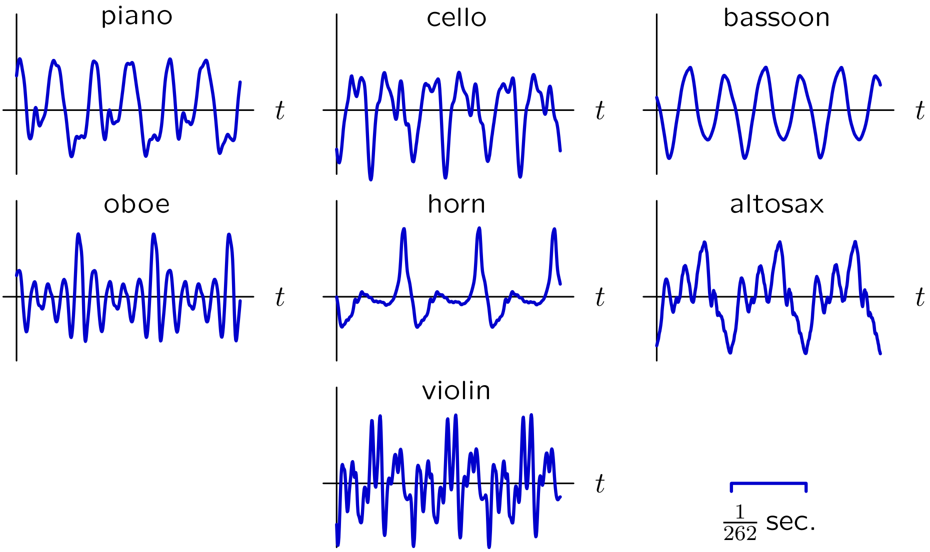
Example: a musical sound can be represented as a function of time.



Although this time function is a complete description of the sound, it does not expose many of the important properties of the sound.

Musical Sounds as Signals

Even though these sounds have the same pitch, they sound different.



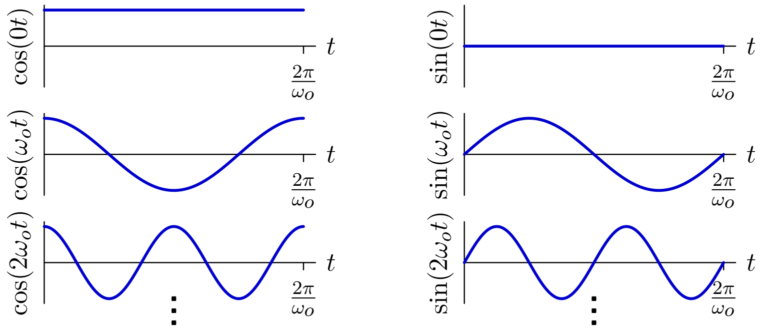
It's not clear how the differences relate to properties of the signals.

(audio clips from from http://theremin.music.uiowa.edu)

Musical Signals as Sums of Sinusoids

One way to characterize differences between these signals is express each as a sum of sinusoids.

$$f(t) = \sum_{k=0}^{\infty} \left(c_k \cos k\omega_o t + d_k \sin k\omega_o t \right)$$



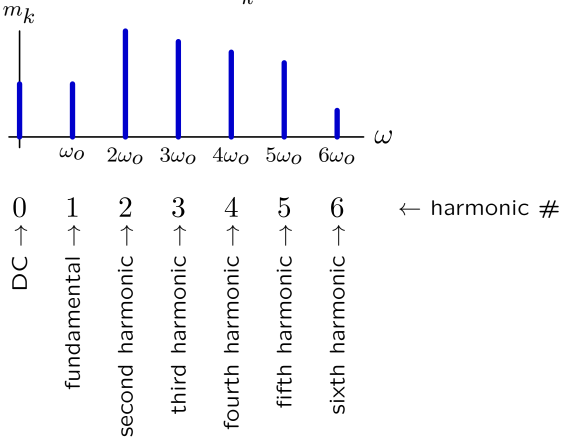
Since these sounds are (nearly) periodic, the frequencies of the dominant sinusoids are (nearly) integer multiples of a **fundamental** frequency ω_o.

Harmonic Structure

The sum of sinusoids describes the distribution of energy across frequencies.

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos k\omega_o t + d_k \sin k\omega_o t) = \sum_{k=0}^{\infty} m_k \cos(k\omega_o t + \phi_k)$$

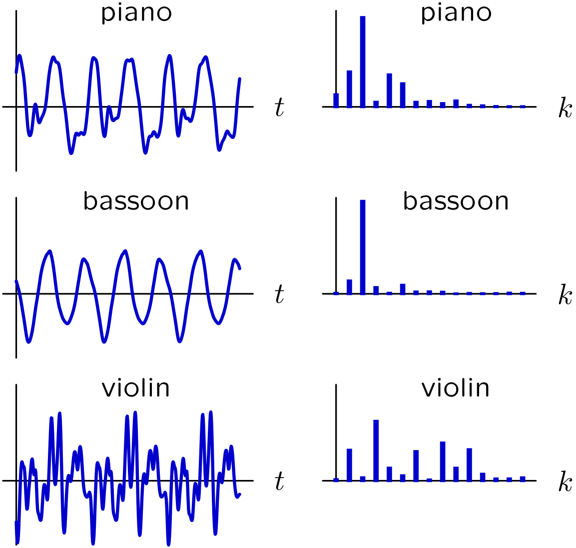
where $m_k^2 = c_k^2 + d_k^2$ and $\tan \phi_k = \dfrac{d_k}{c_k}$.



This distribution represents the **harmonic structure** of the signal.

Harmonic Structure

The harmonic structures of notes from different instruments are different.

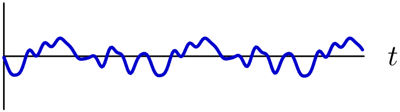


Some musical qualities are more easily seen in time, others in frequency.

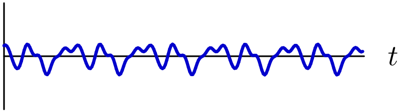
Consonance and Dissonance

Which of the following pairs is least consonant?

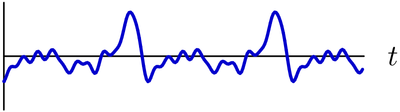
A1



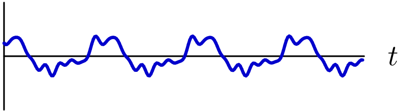
A2



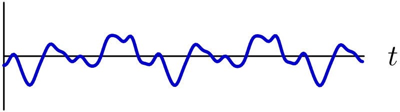
B1



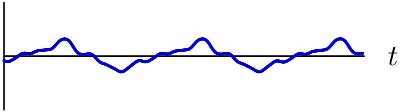
B2



C1



C2

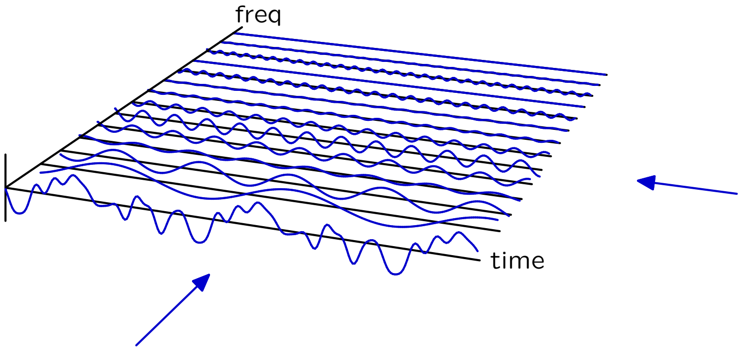


Obvious from the sounds ... less obvious from the waveforms.

Express Each Signal as a Sum of Sinusoids

$$f(t) = \sum_{k=0}^{\infty} m_k \cos(k\omega_o t + \phi_k)$$

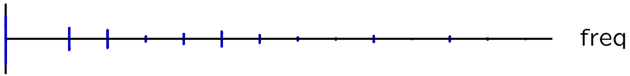
$$= m_1 \cos(\omega_o t + \phi_1) + m_2 \cos(2\omega_o t + \phi_2) + m_3 \cos(3\omega_o t + \phi_3) + \cdots$$



Two views: as a function of time and as a function of frequency

Express Each Signal as a Sum of Sinusoids

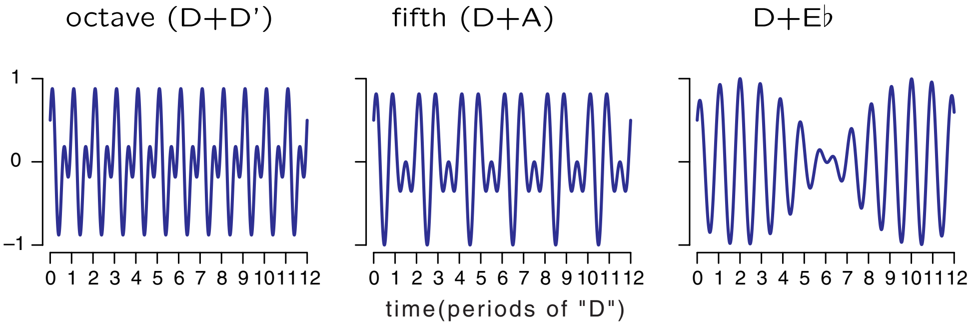
$$f(t) = \sum_{k=0}^{\infty} m_k \cos(k\omega_o t + \phi_k)$$
$$= m_1 \cos(\omega_o t + \phi_1) + m_2 \cos(2\omega_o t + \phi_2) + m_3 \cos(3\omega_o t + \phi_3) + \cdots$$

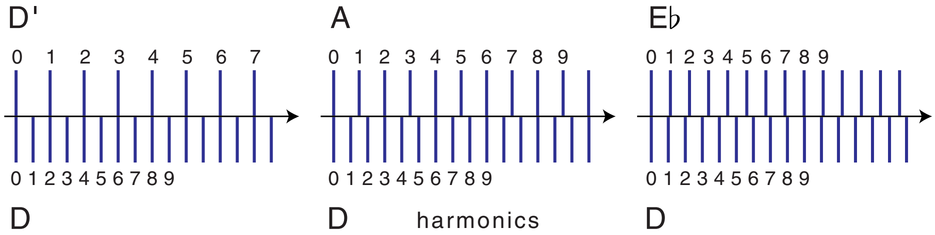


The signal $f(t)$ can be expressed as a discrete set of frequency components.

Musical Sounds as Signals

Time functions do a poor job of conveying consonance and dissonance.





Harmonic structure conveys consonance and dissonance better.

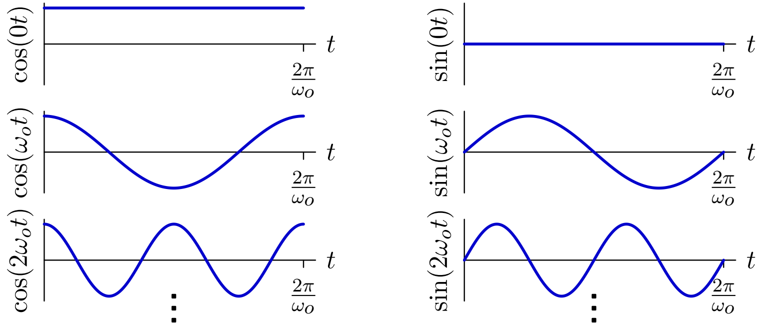
Fourier Representations of Signals

Fourier series are sums of harmonically related sinusoids.

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t))$$

where $\omega_o = 2\pi/T$ represents the fundamental frequency.

Basis functions:



Q1: Under what conditions can we write $f(t)$ as a Fourier series?

Q2: How do we find the coefficients c_k and d_k.

Fourier Representations of Signals

Under what conditions can we write $f(t)$ as a Fourier series?

Fourier series can only represent **periodic** signals.

Definition: a signal $f(t)$ is periodic in T if

$$f(t) = f(t+T)$$

for all t.

Note: if a signal is periodic in T it is also periodic in $2T$, $3T$, ...

The smallest positive number T_o for which $f(t) = f(t + T_o)$ for all t is sometimes called the **fundamental period**.

If a signal does not satisfy $f(t) = f(t+T)$ for any value of T, then the signal is **aperiodic**.

Calculating Fourier Coefficients

How do we find the coefficients c_k and d_k for all k?

Key idea: simplify by integrating over the period T of the fundamental. Start with the general form:

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right)$$

Integrate both sides over T:

$$\int_0^T f(t)\,dt = \int_0^T c_0\,dt + \int_0^T \left(\sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right) \right) dt$$

$$= Tc_0 + \sum_{k=1}^{\infty} \left(c_k \int_0^T \cos(k\omega_o t)\,dt + d_k \int_0^T \sin(k\omega_o t)\,dt \right) = Tc_0$$

All but the first term integrates to zero, leaving

$$c_0 = \frac{1}{T} \int_0^T f(t)\,dt.$$

This $k=0$ term represents the average (“DC”) value.

Calculating Fourier Coefficients

Isolate the c_l term by multiplying both sides by $\cos(l\omega_o t)$ before integrating.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right)$$

$$\begin{aligned}\int_0^T f(t) \cos(l\omega_o t)\, dt = & \int_0^T c_0 \cos(l\omega_o t)\, dt \\ & + \sum_{k=1}^{\infty} \int_0^T c_k \cos(k\omega_o t) \cos(l\omega_o t)\, dt \\ & + \sum_{k=1}^{\infty} \int_0^T d_k \sin(k\omega_o t) \cos(l\omega_o t)\, dt\end{aligned}$$

A product of sinusoids can be expressed as sum and difference frequencies.

$$\cos(k\omega_o t) \cos(l\omega_o t) = \frac{1}{2} \cos((k-l)\omega_o t) + \frac{1}{2} \cos((k+l)\omega_o t)$$

$$\sin(k\omega_o t) \cos(l\omega_o t) = \frac{1}{2} \sin((k-l)\omega_o t) + \frac{1}{2} \sin((k+l)\omega_o t)$$

Calculating Fourier Coefficients

Isolate the c_l term by multiplying both sides by $\cos(l\omega_o t)$ before integrating.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right)$$

$$\begin{aligned}\int_0^T f(t)\cos(l\omega_o t)\,dt &= \int_0^T c_0 \cos(l\omega_o t)\,dt \\ &+ \sum_{k=1}^{\infty}\int_0^T c_k \left(\frac{1}{2}\cos((k-l)\omega_o t) + \frac{1}{2}\cos((k+l)\omega_o t)\right) dt \\ &+ \sum_{k=1}^{\infty}\int_0^T d_k \left(\frac{1}{2}\sin((k-l)\omega_o t) + \frac{1}{2}\sin((k+l)\omega_o t)\right) dt\end{aligned}$$

A product of sinusoids can be expressed as sum and difference frequencies.

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Calculating Fourier Coefficients

Isolate the c_l term by multiplying both sides by $\cos(l\omega_o t)$ before integrating.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right)$$

$$\begin{aligned}\int_0^T f(t)\cos(l\omega_o t)\,dt = &\int_0^T c_0 \cos(l\omega_o t)\,dt \quad \mathbf{0}\\ &+ \sum_{k=1}^{\infty}\int_0^T c_k \left(\frac{1}{2}\cos((k-l)\omega_o t) + \frac{1}{2}\cos((k+l)\omega_o t)\right)\,dt\\ &+ \sum_{k=1}^{\infty}\int_0^T d_k \left(\frac{1}{2}\sin((k-l)\omega_o t) + \frac{1}{2}\sin((k+l)\omega_o t)\right)\,dt\end{aligned}$$

The c_0 term is zero because the integral of $\cos(l\omega_o t)$ over T is zero.

Calculating Fourier Coefficients

Isolate the c_l term by multiplying both sides by $\cos(l\omega_o t)$ before integrating.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right)$$

$$\begin{aligned}\int_0^T f(t)\cos(l\omega_o t)\,dt = &\int_0^T c_0 \cos(l\omega_o t)\,dt \quad \rightarrow \mathbf{0} \\ &+ \sum_{k=1}^{\infty}\int_0^T c_k \left(\frac{1}{2}\cos((k-l)\omega_o t) + \frac{1}{2}\cos((k+l)\omega_o t)\right)dt \quad \rightarrow \mathbf{\tfrac{T}{2}c_l},\ \mathbf{0} \\ &+ \sum_{k=1}^{\infty}\int_0^T d_k \left(\frac{1}{2}\sin((k-l)\omega_o t) + \frac{1}{2}\sin((k+l)\omega_o t)\right)dt\end{aligned}$$

If $k = l$, then $\cos((k-l)\omega_o t) = 1$ and the integral is $\frac{T}{2}c_l$.
All of the other $\cos((k-l)\omega_o t)$ terms in the sum integrate to zero.
All of the $\cos((k+l)\omega_o t)$ terms in the integrate to zero.

Calculating Fourier Coefficients

Isolate the c_l term by multiplying both sides by $\cos(l\omega_o t)$ before integrating.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right)$$

$$\int_0^T f(t)\cos(l\omega_o t)\,dt = \int_0^T \cancelto{0}{c_0 \cos(l\omega_o t)}\,dt$$

$$+ \sum_{k=1}^{\infty} \int_0^T c_k \left(\frac{1}{2} \cancelto{\frac{T}{2}c_l}{\cos((k-l)\omega_o t)} + \frac{1}{2} \cancelto{0}{\cos((k+l)\omega_o t)} \right) dt$$

$$+ \sum_{k=1}^{\infty} \int_0^T d_k \left(\frac{1}{2} \cancelto{0}{\sin((k-l)\omega_o t)} + \frac{1}{2} \cancelto{0}{\sin((k+l)\omega_o t)} \right) dt$$

If $k = l$, then $\sin((k-l)\omega_o t = 0$ and the integral is 0.
All of the other d_k terms are harmonic sinusoids that integrate to 0.

The only non-zero term on the right side is $\frac{T}{2}c_l$.
We can solve to get an expression for c_l as

$$c_l = \frac{2}{T}\int_0^T f(t)\cos(l\omega_o t)\,dt$$

Calculating Fourier Coefficients

Analogous reasoning allows us to calculate the d_k coefficients, but this time multiplying by $\sin(l\omega_o t)$ before integrating.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right)$$

$$\begin{aligned}\int_0^T f(t)\sin(l\omega_o t)\,dt &= \int_0^T c_0 \sin(l\omega_o t)\,dt \\ &+ \sum_{k=1}^{\infty}\int_0^T c_k \cos(k\omega_o t)\sin(l\omega_o t)\,dt \\ &+ \sum_{k=1}^{\infty}\int_0^T d_k \sin(k\omega_o t)\sin(l\omega_o t)\,dt\end{aligned}$$

A single term remains after integrating, allowing us to solve for d_l as

$$d_l = \frac{2}{T}\int_0^T f(t)\sin(l\omega_o t)\,dt$$

Calculating Fourier Coefficients

Summarizing ...

If $f(t)$ is expressed as a Fourier series

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t))$$

the Fourier coefficients are given by

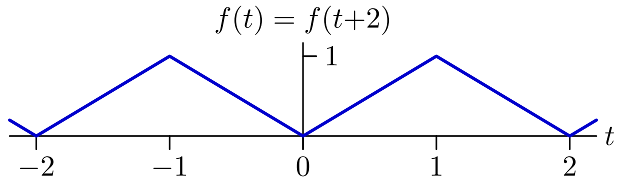
$$c_0 = \frac{1}{T} \int_T f(t)\, dt$$

$$c_k = \frac{2}{T} \int_T f(t) \cos(k\omega_o t)\, dt;\ \ k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T} \int_T f(t) \sin(k\omega_o t)\, dt;\ \ k = 1, 2, 3, \ldots$$

Example of Analysis

Find the Fourier series coefficients for the following triangle wave:



$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

$$c_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi kt}{T}\,dt = 2\int_0^1 t\cos(\pi kt)\,dt = \begin{cases} -\frac{4}{\pi^2 k^2} & k \text{ odd} \\ 0 & k = 2, 4, 6, \ldots \end{cases}$$

$d_k = 0$ (by symmetry)

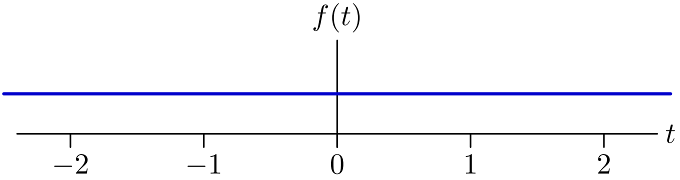
Example of Synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 - \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{0} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



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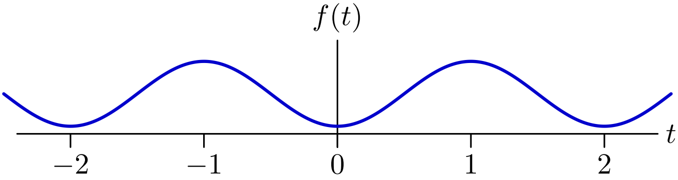
$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{1} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



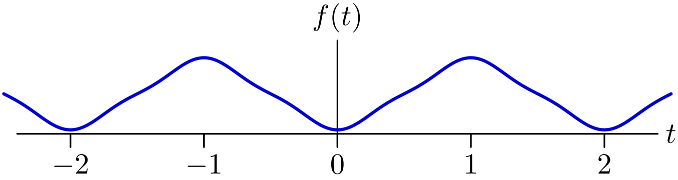
Example of Synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 - \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{3} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



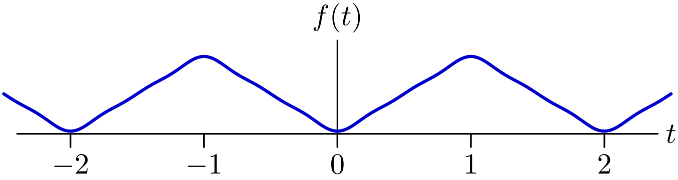
Example of Synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 - \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{5} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



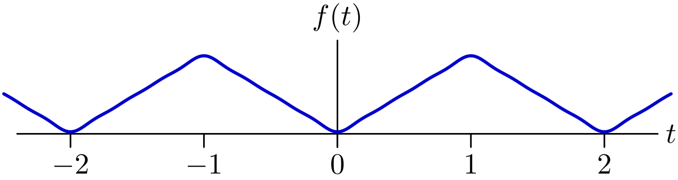
Example of Synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 - \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{7} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



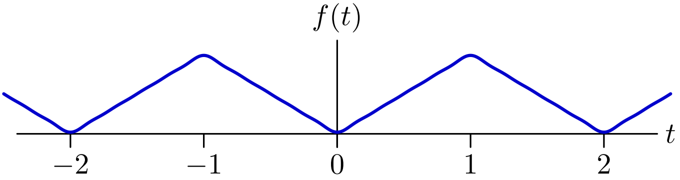
Example of Synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 - \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{9} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



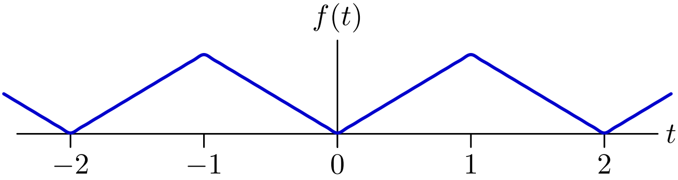
Example of Synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 - \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{19} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



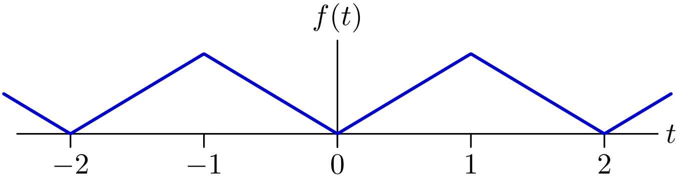
Example of Synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 - \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \texttt{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \texttt{ odd}}}^{99} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



The synthesized function approaches original as number of terms increases.

Two Views of the Same Signal

The harmonic expansion provides an alternative view of the signal.

$$f(t) = \sum_{k=0}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right) = \sum_{k=0}^{\infty} m_k \cos(k\omega_o t + \phi_k)$$

We can view the musical signal

- as a function of time $f(t)$, or
- as a sum of harmonics.

Both views are useful. For example,

- the peak sound pressure is more easily seen in $f(t)$, while
- consonance is more easily analyzed by comparing harmonics.

This type of harmonic analysis is an example of **Fourier Analysis**, which is a major theme of this subject.

Trig Table

```
sin(a+b) = sin(a) cos(b) + cos(a) sin(b)
sin(a-b) = sin(a) cos(b) - cos(a) sin(b)
cos(a+b) = cos(a) cos(b) - sin(a) sin(b)
cos(a-b) = cos(a) cos(b) + sin(a) sin(b)
tan(a+b) = (tan(a)+tan(b))/(1-tan(a) tan(b))
tan(a-b) = (tan(a)-tan(b))/(1+tan(a) tan(b))

sin(A) + sin(B) =  2 sin((A+B)/2) cos((A-B)/2)
sin(A) - sin(B) =  2 cos((A+B)/2) sin((A-B)/2)
cos(A) + cos(B) =  2 cos((A+B)/2) cos((A-B)/2)
cos(A) - cos(B) = -2 sin((A+B)/2) sin((A-B)/2)

sin(a+b) + sin(a-b) =  2 sin(a) cos(b)
sin(a+b) - sin(a-b) =  2 cos(a) sin(b)
cos(a+b) + cos(a-b) =  2 cos(a) cos(b)
cos(a+b) - cos(a-b) = -2 sin(a) sin(b)

2 cos(A) cos(B) = cos(A-B) + cos(A+B)
2 sin(A) sin(B) = cos(A-B) - cos(A+B)
2 sin(A) cos(B) = sin(A+B) + sin(A-B)
2 cos(A) sin(B) = sin(A+B) - sin(A-B)
```