

# 6.3000: Signal Processing

## Signal Processing

- Overview of Subject
- Signals: Definitions, Examples, and Operations
- Time and Frequency Representations
- Fourier Series

Lecture slides are available at the course website:

<http://mit.edu/6.3000>

*February 04, 2025*

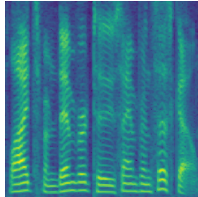
## 6.3000: Signal Processing

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**Signals** are functions that contain and convey information.

Examples:

- the MP3 representation of a sound
- the JPEG representation of a picture
- an MRI image of a brain



**Signal Processing** develops the use of signals as abstractions:

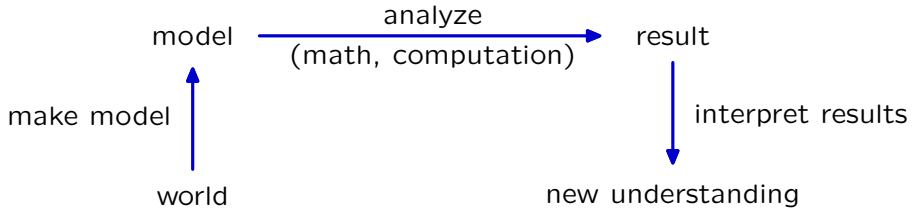
- **identifying** signals in physical, mathematical, computation contexts,
- **analyzing** signals to understand the information they contain, and
- **manipulating** signals to modify the information they contain.

## 6.3000: Signal Processing

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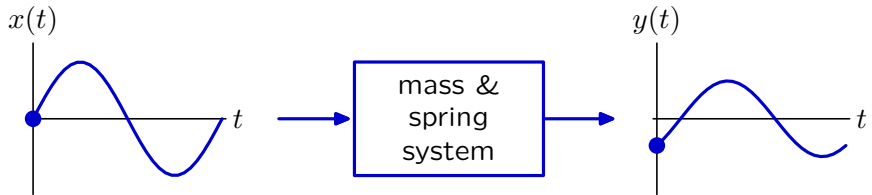
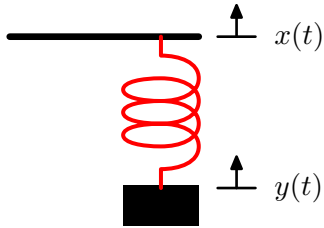
Signal Processing is **widely used** in science and engineering to ...

- **model** some aspect of the world,
- **analyze** the model and get a result, then
- **interpret** the result to gain a new or better understanding.



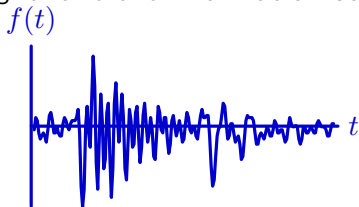
## Example: Mass and Spring

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## Check Yourself

Relation between a signal and the information contained in that signal.



Listen to the following four manipulated signals:

$$f_1(t), f_2(t), f_3(t), f_4(t).$$

How many of the following relations are true?

- $f_1(t) = f(2t)$
- $f_2(t) = -f(t)$
- $f_3(t) = f(2t)$
- $f_4(t) = \frac{1}{3}f(t)$

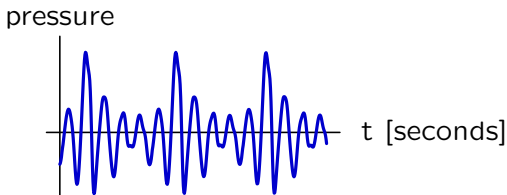
\* speech signal synthesized by Robert Donovan

## Musical Sounds as Signals

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Signals are functions that contain and convey information.

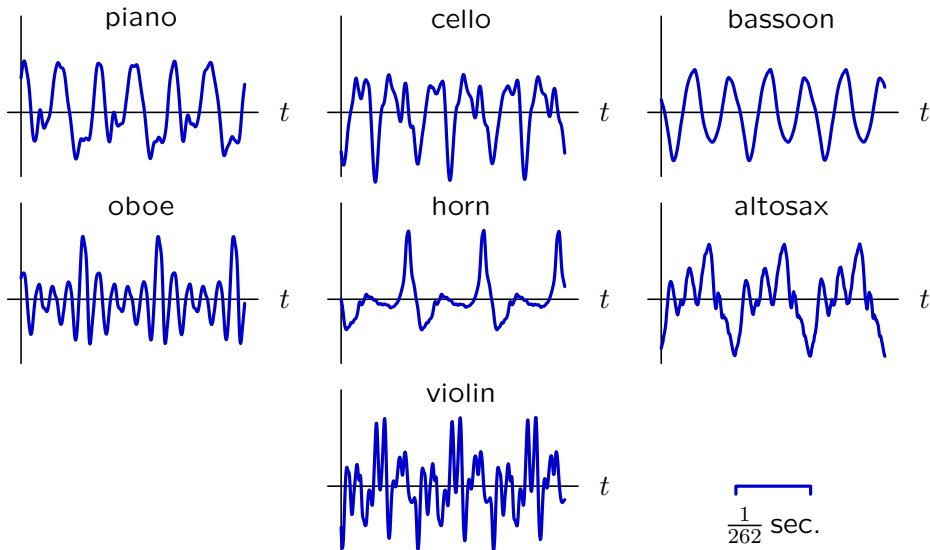
Example: a musical sound can be represented as a function of time.



Although this time function is a complete description of the sound, it does not expose many of the important properties of the sound.

## Musical Sounds as Signals

Even though these sounds have the same pitch, they sound different.



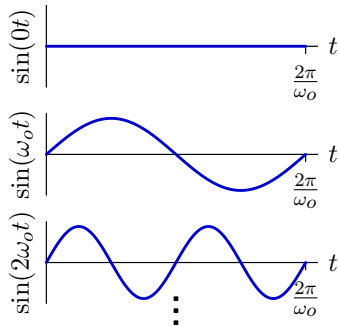
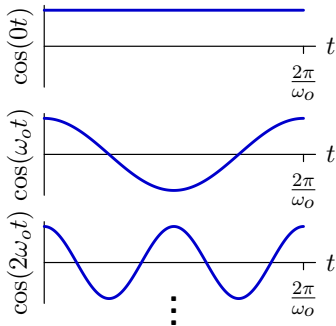
It's not clear how the differences relate to properties of the signals.

(audio clips from <http://theremin.music.uiowa.edu>)

## Musical Signals as Sums of Sinusoids

One way to characterize differences between these signals is express each as a sum of sinusoids.

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos k\omega_0 t + d_k \sin k\omega_0 t)$$



Since these sounds are (nearly) periodic, the frequencies of the dominant sinusoids are (nearly) integer multiples of a **fundamental** frequency  $\omega_0$ .

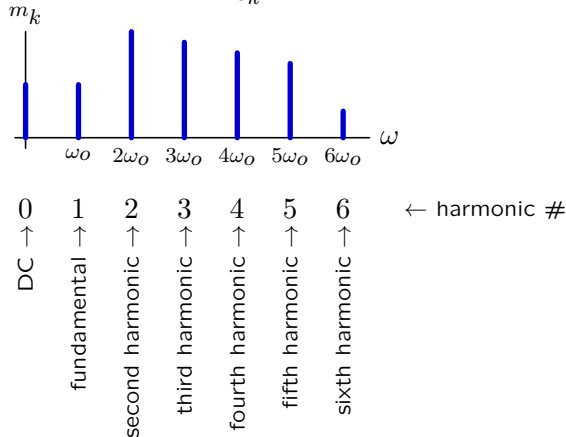


## Harmonic Structure

The sum of sinusoids describes the distribution of energy across frequencies.

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos k\omega_o t + d_k \sin k\omega_o t) = \sum_{k=0}^{\infty} m_k \cos (k\omega_o t + \phi_k)$$

where  $m_k^2 = c_k^2 + d_k^2$  and  $\tan \phi_k = \frac{d_k}{c_k}$ .

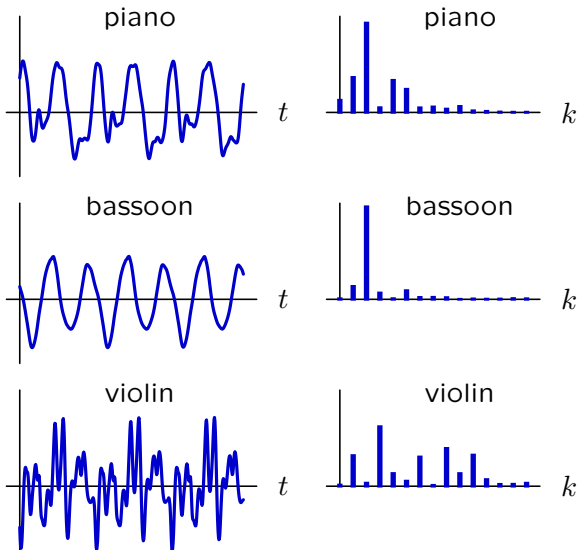


This distribution represents the **harmonic structure** of the signal.

## Harmonic Structure

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The harmonic structures of notes from different instruments are different.

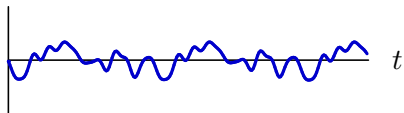


Some musical qualities are more easily seen in time, others in frequency.

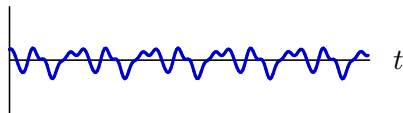
## Consonance and Dissonance

Which of the following pairs is least consonant?

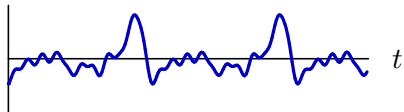
A1



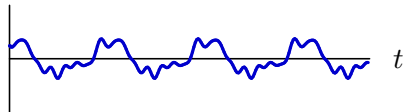
A2



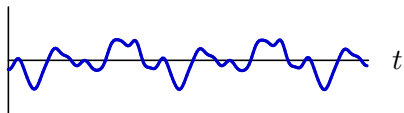
B1



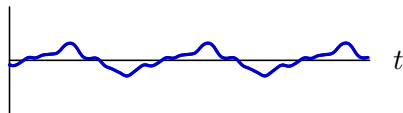
B2



C1



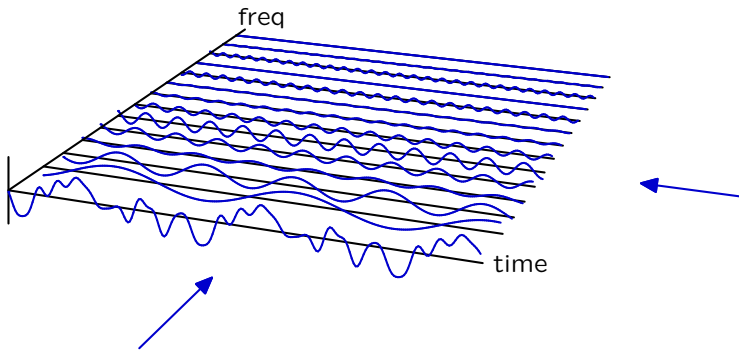
C2



Obvious from the sounds ... less obvious from the waveforms.

## Express Each Signal as a Sum of Sinusoids

$$f(t) = \sum_{k=0}^{\infty} m_k \cos(k\omega_o t + \phi_k)$$
$$= m_1 \cos(\omega_o t + \phi_1) + m_2 \cos(2\omega_o t + \phi_2) + m_3 \cos(3\omega_o t + \phi_3) + \dots$$

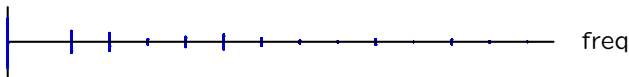


Two views: as a function of time and as a function of frequency

## Express Each Signal as a Sum of Sinusoids

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$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} m_k \cos(k\omega_o t + \phi_k) \\ &= m_1 \cos(\omega_o t + \phi_1) + m_2 \cos(2\omega_o t + \phi_2) + m_3 \cos(3\omega_o t + \phi_3) + \dots \end{aligned}$$

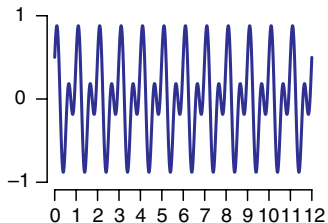


The signal  $f(t)$  can be expressed as a discrete set of frequency components.

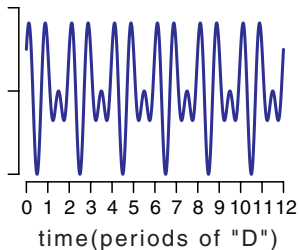
# Musical Sounds as Signals

Time functions do a poor job of conveying consonance and dissonance.

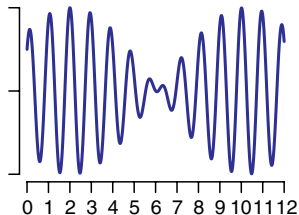
octave (D+D')



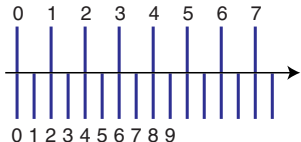
fifth (D+A)



D+E<sub>b</sub>

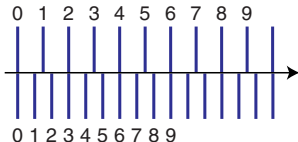


D'



D

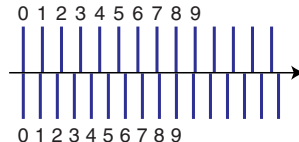
A



D

harmonics

E<sub>b</sub>



D

Harmonic structure conveys consonance and dissonance better.

# Fourier Representations of Signals

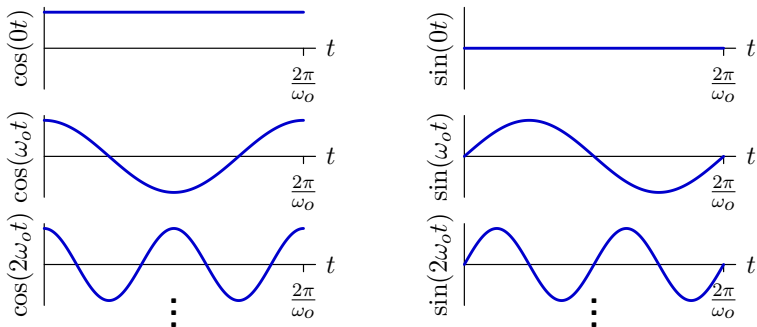
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**Fourier series** are sums of harmonically related sinusoids.

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t))$$

where  $\omega_o = 2\pi/T$  represents the fundamental frequency.

Basis functions:



Q1: Under what conditions can we write  $f(t)$  as a Fourier series?

Q2: How do we find the coefficients  $c_k$  and  $d_k$ .

## Fourier Representations of Signals

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Under what conditions can we write  $f(t)$  as a Fourier series?

Fourier series can only represent **periodic** signals.

Definition: a signal  $f(t)$  is periodic in  $T$  if

$$f(t) = f(t+T)$$

for all  $t$ .

Note: if a signal is periodic in  $T$  it is also periodic in  $2T, 3T, \dots$

The smallest positive number  $T_o$  for which  $f(t) = f(t + T_o)$  for all  $t$  is sometimes called the **fundamental period**.

If a signal does not satisfy  $f(t) = f(t+T)$  for any value of  $T$ , then the signal is **aperiodic**.



## Calculating Fourier Coefficients

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How do we find the coefficients  $c_k$  and  $d_k$  for all  $k$ ?

Key idea: simplify by integrating over the period  $T$  of the fundamental.

Start with the general form:

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t))$$

Integrate both sides over  $T$ :

$$\begin{aligned} \int_0^T f(t) dt &= \int_0^T c_0 dt + \int_0^T \left( \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)) \right) dt \\ &= Tc_0 + \sum_{k=1}^{\infty} \left( c_k \int_0^T \cos(k\omega_0 t) dt + d_k \int_0^T \sin(k\omega_0 t) dt \right) = Tc_0 \end{aligned}$$

All but the first term integrates to zero, leaving

$$c_0 = \frac{1}{T} \int_0^T f(t) dt.$$

This  $k=0$  term represents the average (“DC”) value.

## Calculating Fourier Coefficients

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Isolate the  $c_l$  term by multiplying both sides by  $\cos(l\omega_o t)$  before integrating.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t))$$

$$\begin{aligned} \int_0^T f(t) \cos(l\omega_o t) dt &= \int_0^T c_0 \cos(l\omega_o t) dt \\ &+ \sum_{k=1}^{\infty} \int_0^T c_k \left( \frac{1}{2} \cos((k-l)\omega_o t) + \frac{1}{2} \cos((k+l)\omega_o t) \right) dt \\ &+ \sum_{k=1}^{\infty} \int_0^T d_k \left( \frac{1}{2} \sin((k-l)\omega_o t) + \frac{1}{2} \sin((k+l)\omega_o t) \right) dt \end{aligned}$$

If  $k = l$ , then  $\sin((k-l)\omega_o t) = 0$  and the integral is 0.

All of the other  $d_k$  terms are harmonic sinusoids that integrate to 0.

The only non-zero term on the right side is  $\frac{T}{2} c_l$ .

We can solve to get an expression for  $c_l$  as

$$c_l = \frac{2}{T} \int_0^T f(t) \cos(l\omega_o t) dt$$

## Calculating Fourier Coefficients

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Analogous reasoning allows us to calculate the  $d_k$  coefficients, but this time multiplying by  $\sin(l\omega_0 t)$  before integrating.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t))$$

$$\begin{aligned} \int_0^T f(t) \sin(l\omega_0 t) dt &= \int_0^T c_0 \sin(l\omega_0 t) dt \\ &+ \sum_{k=1}^{\infty} \int_0^T c_k \cos(k\omega_0 t) \sin(l\omega_0 t) dt \\ &+ \sum_{k=1}^{\infty} \int_0^T d_k \sin(k\omega_0 t) \sin(l\omega_0 t) dt \end{aligned}$$

A single term remains after integrating, allowing us to solve for  $d_l$  as

$$d_l = \frac{2}{T} \int_0^T f(t) \sin(l\omega_0 t) dt$$

## Calculating Fourier Coefficients

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Summarizing ...

If  $f(t)$  is expressed as a Fourier series

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t))$$

the Fourier coefficients are given by

$$c_0 = \frac{1}{T} \int_T f(t) dt$$

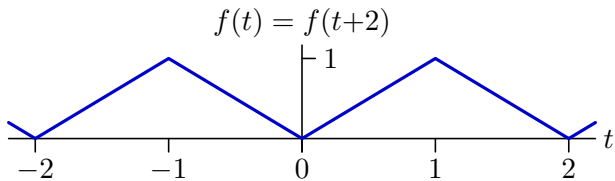
$$c_k = \frac{2}{T} \int_T f(t) \cos(k\omega_0 t) dt; \quad k = 1, 2, 3, \dots$$

$$d_k = \frac{2}{T} \int_T f(t) \sin(k\omega_0 t) dt; \quad k = 1, 2, 3, \dots$$

## Example of Analysis

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Find the Fourier series coefficients for the following triangle wave:



$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{2} \int_0^2 f(t) dt = \frac{1}{2}$$

$$c_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos \frac{2\pi kt}{T} dt = 2 \int_0^1 t \cos(\pi kt) dt = \begin{cases} -\frac{4}{\pi^2 k^2} & k \text{ odd} \\ 0 & k = 2, 4, 6, \dots \end{cases}$$

$$d_k = 0 \quad (\text{by symmetry})$$

## Example of Synthesis

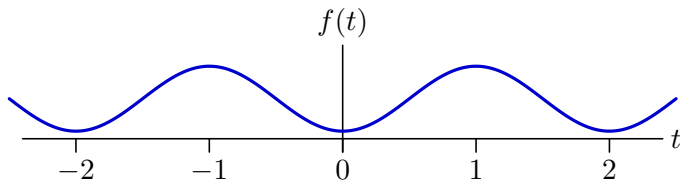
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Generate  $f(t)$  from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 - \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



## Example of Synthesis

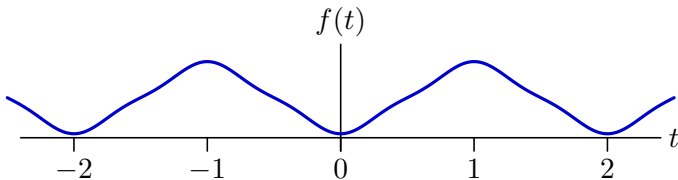
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$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^3 \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



## Example of Synthesis

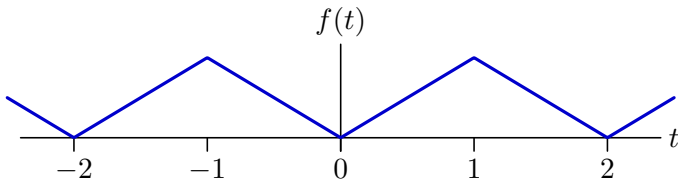
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$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{99} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



The synthesized function approaches original as number of terms increases.



## Two Views of the Same Signal

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The harmonic expansion provides an alternative view of the signal.

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)) = \sum_{k=0}^{\infty} m_k \cos(k\omega_0 t + \phi_k)$$

We can view the musical signal

- as a function of time  $f(t)$ , or
- as a sum of harmonics.

Both views are useful. For example,

- the peak sound pressure is more easily seen in  $f(t)$ , while
- consonance is more easily analyzed by comparing harmonics.

This type of harmonic analysis is an example of **Fourier Analysis**, which is a major theme of this subject.

**Next Time:** understanding Fourier series and their properties.

## Trig Table

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$$\sin(a+b) = \sin(a) \cos(b) + \cos(a) \sin(b)$$

$$\sin(a-b) = \sin(a) \cos(b) - \cos(a) \sin(b)$$

$$\cos(a+b) = \cos(a) \cos(b) - \sin(a) \sin(b)$$

$$\cos(a-b) = \cos(a) \cos(b) + \sin(a) \sin(b)$$

$$\tan(a+b) = (\tan(a)+\tan(b))/(1-\tan(a) \tan(b))$$

$$\tan(a-b) = (\tan(a)-\tan(b))/(1+\tan(a) \tan(b))$$

$$\sin(A) + \sin(B) = 2 \sin((A+B)/2) \cos((A-B)/2)$$

$$\sin(A) - \sin(B) = 2 \cos((A+B)/2) \sin((A-B)/2)$$

$$\cos(A) + \cos(B) = 2 \cos((A+B)/2) \cos((A-B)/2)$$

$$\cos(A) - \cos(B) = -2 \sin((A+B)/2) \sin((A-B)/2)$$

$$\sin(a+b) + \sin(a-b) = 2 \sin(a) \cos(b)$$

$$\sin(a+b) - \sin(a-b) = 2 \cos(a) \sin(b)$$

$$\cos(a+b) + \cos(a-b) = 2 \cos(a) \cos(b)$$

$$\cos(a+b) - \cos(a-b) = -2 \sin(a) \sin(b)$$

$$2 \cos(A) \cos(B) = \cos(A-B) + \cos(A+B)$$

$$2 \sin(A) \sin(B) = \cos(A-B) - \cos(A+B)$$

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