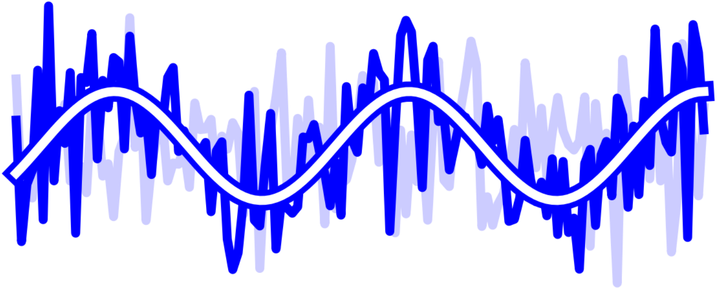
6.300: Signal Processing

Quiz Review

- Quiz #2 is in Walker (50-340) on Tuesday, 04/15 at 2:00 p.m.
- You may bring two 8.5" × 11" double-sided pages of notes.
- The review during class on Thursday, 04/10 will emphasize problem-solving strategies. Look over these slides on your own.



Slides by Titus K. Roesler (tkr@mit.edu)

The Story So Far

Signals

02/04 Signal Processing
02/06 Fourier Series (Sinusoids)
02/11 Fourier Series (Exponentials)
02/13 Discretization (Sampling and Quantization)
02/20 Discrete-Time Fourier Series
02/25 Continuous-Time Fourier Transform
02/27 Discrete-Time Fourier Transform
03/04 Quiz #1

Systems

03/06 Systems
03/11 Impulse Response and Convolution
03/13 Frequency Response and Filtering

The Story So Far

Discrete Fourier Transform

03/18 Discrete Fourier Transform
03/20 DFT: Frequency Resolution and Circular Convolution
04/01 Short-Time Fourier Transforms
04/03 Fast Fourier Transform (FFT)

Applications and Extensions

04/08 Communications Systems
04/10 Quiz Review
04/15 Quiz #2

(There are many more applications and extensions yet to come.)

Calculus Analogy

You wouldn't want to walk into a calculus quiz without knowing

$$\frac{d\sin(\theta)}{d\theta} = \cos(\theta)$$

by heart, right? Going back to the derivation wastes precious time!

$$\lim_{\phi \to 0} \frac{\sin(\theta + \phi) - \sin(\theta)}{\phi}$$

$$\lim_{\phi \to 0} \frac{\sin(\theta)\cos(\phi) + \sin(\phi)\cos(\theta) - \sin(\theta)}{\phi}$$

$$\lim_{\phi \to 0} \underbrace{\left[\frac{\sin(\phi)}{\phi}\right]}_{1 \text{ as } \phi \to 0} \cos(\theta) + \lim_{\phi \to 0} \underbrace{\left[\frac{\cos(\phi) - 1}{\phi}\right]}_{0 \text{ as } \phi \to 0} \sin(\theta)$$

Calculus Analogy

To do well on a calculus quiz,
you probably need to know at least a few things by heart.

common functions and their derivatives

$$\frac{d(t^n)}{dt} = nt^{n-1} \qquad \frac{d(e^{\lambda t})}{dt} = \lambda e^{\lambda t} \qquad \frac{d\sin(t)}{dt} = \cos(t)$$

differentiation rules

$$\frac{d[c_1 f(t) + c_2 g(t)]}{dt} = c_1 \frac{df}{dt} + c_2 \frac{dg}{dt} \qquad \frac{dg(f(t))}{dt} = \frac{dg}{df} \cdot \frac{df}{dt}$$

Calculus Analogy

To do well on a signal processing quiz,
you probably need to know at least a few things by heart.

common signals and their Fourier transforms

$$\delta[n-n_0] \iff e^{-j\Omega n_0} \qquad e^{j\Omega_0 n} \iff 2\pi\delta\big((\Omega-\Omega_0) \bmod 2\pi\big)$$

Fourier properties

$$c_1x_1[n]+c_2x_2[n] \iff c_1X_1(\Omega)+c_2X_2(\Omega)$$
$$x[n-n_0] \iff e^{-j\Omega n_0}X(\Omega)$$
$$e^{j\Omega_0 n}x[n] \iff X(\Omega-\Omega_0)$$

Fourier Transforms (CT)

$$\begin{aligned} \textbf{time domain} &\iff \textbf{frequency domain} \\ \delta(t) &\iff 1 \\ \delta(t-t_0) &\iff e^{-j\omega t_0} \\ 1 &\iff 2\pi\delta(\omega) \\ e^{j\omega_0 t} &\iff 2\pi\delta(\omega-\omega_0) \end{aligned}$$

Duality: Notice the common trend in transform pairs.

$$\begin{aligned} x(t) &\iff X(\omega) \\ X(t) &\iff 2\pi x(-\omega) \end{aligned}$$

Fourier Transforms (DT)

All discrete-time Fourier transforms are 2π-periodic.

time domain	$\Longleftrightarrow$	frequency domain
$\delta[n]$	$\Longleftrightarrow$	1
$\delta[n-n_0]$	$\Longleftrightarrow$	$e^{-j\Omega n_0}$
1	$\Longleftrightarrow$	$2\pi\delta(\Omega \bmod 2\pi)$
$e^{j\Omega_0 n}$	$\Longleftrightarrow$	$2\pi\delta\big((\Omega-\Omega_0) \bmod 2\pi\big)$

Duality: Not so easy with discrete-time Fourier transforms.
$x[n]$ is discrete in time, but $X(\Omega)$ is continuous in frequency!

Some Fourier Properties

time domain	$\Longleftrightarrow$	frequency domain
$c_1x_1[n] + c_2x_2[n]$	$\Longleftrightarrow$	$c_1X_1(\Omega) + c_2X_2(\Omega)$
$(x_1 * x_2)[n]$	$\Longleftrightarrow$	$X_1(\Omega)X_2(\Omega)$
$x_1[n]x_2[n]$	$\Longleftrightarrow$	$\frac{1}{2\pi}\Big(X_1 * X_2\Big)(\Omega)$
$x[-n]$	$\Longleftrightarrow$	$X(-\Omega)$
$x[nM]$	$\Longleftrightarrow$	$X(\frac{\Omega}{M})$
$x[n - n_0]$	$\Longleftrightarrow$	$e^{-j\Omega n_0}X(\Omega)$
$e^{j\Omega_0 n}x[n]$	$\Longleftrightarrow$	$X(\Omega - \Omega_0)$
$n\,x[n]$	$\Longleftrightarrow$	$j\frac{d}{d\Omega}X(\Omega)$
$\frac{d}{dt}x(t)$	$\Longleftrightarrow$	$j\omega X(\omega)$

Check Yourself

Determine the Fourier transforms of the signals listed below. (Apply Fourier properties to reduce the number of calculations.)

(a) $x_1(t) = e^{-t}u(t)$

(b) $x_2(t) = e^{-|t|}$

(c) $x_3(t) = 2e^{-|t|}\cos(t)$

(d) $x_4(t) = 4e^{-|t|}\cos^2(t)$

Check Yourself

(a) $x_1(t) = e^{-t}u(t)$

Directly compute the Fourier transform.

$$X_1(\omega) = \int_0^{\infty} e^{-(1+j\omega)t} dt$$
$$= \frac{1}{1+j\omega}$$

Check Yourself

(b) $x_2(t) = e^{-|t|}$

$$x_2(t) = \tfrac{1}{2}x_1(t) + \tfrac{1}{2}x_1(-t)$$

$$X_2(\omega) = \tfrac{1}{2}X_1(\omega) + \tfrac{1}{2}X_1(-\omega)$$

$$= \frac{1}{1+\omega^2}$$

Check Yourself

(c) $x_3(t) = 2e^{-|t|}\cos(t)$

$$
\begin{aligned}
x_3(t) &= x_2(t) \cdot 2\cos(t) \\
&= x_2(t) \cdot \left(e^{jt} + e^{-jt}\right) \\
X_3(\omega) &= X_2(\omega) * \left[\delta(\omega - 1) + \delta(\omega + 1)\right] \\
&= X_2(\omega - 1) + X_2(\omega + 1) \\
&= \frac{1}{1 + (\omega - 1)^2} + \frac{1}{1 + (\omega + 1)^2}
\end{aligned}
$$

Check Yourself

(d) $x_4(t) = 4e^{-|t|}\cos^2(t)$

$$\begin{aligned}
x_4(t) &= x_3(t)\cdot 2\cos(t) \\
&= x_3(t)\cdot\left(e^{jt}+e^{-jt}\right) \\
X_4(\omega) &= X_3(\omega) * \left[\delta(\omega-1)+\delta(\omega+1)\right] \\
&= X_3(\omega-1)+X_3(\omega+1) \\
&= \frac{1}{1+(\omega-2)^2}+\frac{2}{1+\omega^2}+\frac{1}{1+(\omega+2)^2}
\end{aligned}$$

Linearity and Time-Invariance

Linearity

$$x_1[n] \rightarrow \boxed{\text{linear system}} \rightarrow y_1[n]$$

$$x_2[n] \rightarrow \boxed{\text{linear system}} \rightarrow y_2[n]$$

$$c_1 x_1[n] + c_2 x_2[n] \rightarrow \boxed{\text{linear system}} \rightarrow c_1 y_1[n] + c_2 y_2[n]$$

Time-Invariance

$$x[n] \rightarrow \boxed{\text{time-invariant system}} \rightarrow y[n]$$

$$x[n - n_0] \rightarrow \boxed{\text{time-invariant system}} \rightarrow y[n - n_0]$$

Check Yourself

Are the following systems **linear** and **time-invariant**?
(Recall: Together, additivity and homogeneity imply linearity.)

$$y[n] = \frac{1}{3}x[n-1] + \frac{1}{3}x[n] + \frac{1}{3}x[n+1]$$

$$y[n] = Mx[n] + B \quad \text{for constants } M \text{ and } B$$

$$y(t) = \int_0^t x(\tau)\,d\tau$$

Check Yourself

$$y[n] = \frac{1}{3}x[n-1] + \frac{1}{3}x[n] + \frac{1}{3}x[n+1]$$

Linear? By inspection, yes!

$$x[n] = c_1 x_1[n] + c_2 x_2[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow y[n] = c_1 y_1[n] + c_2 y_2[n]$$

Time-invariant? By inspection, yes!

$$x[n-n_0] \rightarrow \boxed{\textbf{LTI}} \rightarrow y[n-n_0]$$

Check Yourself

$$y[n] = Mx[n] + B \quad \text{for constants } M \text{ and } B$$

Linear? If $M \neq 0$ and $B = 0$, yes.

$$x[n] = c_1 x_1[n] + c_2 x_2[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow y[n] = c_1 y_1[n] + c_2 y_2[n]$$

Time-invariant? Yes.

$$x[n-n_0] \rightarrow \boxed{\textbf{LTI}} \rightarrow y[n-n_0]$$

Check Yourself

$$y(t) = \int_0^t x(\tau) d\tau$$

Linear? Yes. Integration is a linear operation.

$$\int_0^t c_1 x_1(\tau) + c_2 x_2(\tau) d\tau = c_1 \int_0^t x_1(\tau) d\tau + c_2 \int_0^t x_2(\tau) d\tau$$

Time-invariant? No.

$$y(t - t_0) = \int_0^{t-t_0} x(\tau) d\tau \neq \int_{-t_0}^{t-t_0} x(\tau') d\tau' \text{ for } \tau' = \tau - t_0$$

LTI Systems

Three representations for LTI systems:

- difference equation (DT) or differential equation (CT)
- unit-sample response (DT) or impulse response (CT)
- frequency response

LTI Systems

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Difference Equations and Differential Equations

Impose time-domain constraints on the input and output.

$$y[n] = \tfrac{1}{2}y[n-1] + x[n]$$

$$\frac{dy(t)}{dt} = x(t) - \tfrac{1}{2}y(t)$$

LTI Systems

Three representations for LTI systems:

- difference equation (DT) or differential equation (CT)
- unit-sample response (DT) or impulse response (CT)
- frequency response

Unit-Sample Response

Characterize a system by a single time-domain signal.

$$\delta[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow h[n]$$

$$x[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow \sum_k h[k]x[n-k]$$

Convolution

Convolving $x[n]$ with $\delta[n-n_0]$ time-shifts $x[n]$.

$$h[n] = \delta[n-n_0] \implies (x * h)[n] = x[n-n_0]$$

Convolving $x[n]$ with a sum of scaled and time-shifted δ signals produces a sum of scaled and time-shifted $x[n]$.

$$h[n] = \sum_k h[k]\delta[n-k]$$

$$(x * h)[n] = \sum_k \underbrace{h[k]}_{\text{scale}} \underbrace{x[n-k]}_{\text{time-shift}}$$

Convolution

$(x * h)[n]$ is a superposition of scaled and time-shifted $x[n]$.

$$\vdots$$

$$\begin{aligned}(x * h)[n] = {} & h[0]\, x[n] + \\ & h[1]\, x[n-1] + \\ & h[2]\, x[n-2] + \end{aligned}$$

$$\vdots$$

Convolution is Commutative

$(x * h)[n]$ is a superposition of scaled and time-shifted $h[n]$.

$$\vdots$$

$$\begin{aligned}(x * h)[n] = \; & x[0]\,h[n] + \\ & x[1]\,h[n-1] + \\ & x[2]\,h[n-2] +\end{aligned}$$

$$\vdots$$

Convolution

n	$=$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
$x[n]$	$=$	1	1	1	1	0	0	0	0
$h[n]$	$=$	1	2	3	0	0	0	0	0

n	$=$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
$h[0]\,x[n-0]$	$=$	1	1	1	1	0	0	0	0
$h[1]\,x[n-1]$	$=$	0	2	2	2	2	0	0	0
$h[2]\,x[n-2]$	$=$	0	0	3	3	3	3	0	0
$(x*h)[n]$	$=$	**1**	**3**	**6**	**6**	**5**	**3**	**0**	**0**

Check Yourself

Consider an LTI system with unit-sample response $h[n]$.

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

Suppose that the input to the system is $x[n]$.

$$x[n] = \cos\left(\frac{2\pi}{3}n\right)$$

Determine a closed-form expression for the output $y[n]$.

Check Yourself

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

$$y[n] = (x * h)[n] = x[n] + x[n-1] + x[n-2]$$

$x[n] = \cos\left(\frac{2\pi}{3}n\right)$ is periodic in $N = 3$ samples.

$$x[0] = 1 \qquad x[1] = -\tfrac{1}{2} \qquad x[2] = -\tfrac{1}{2}$$

$$x[n] + x[n-1] + x[n-2] = 0 \text{ for all } n$$

Check Yourself

Alternatively, think of the frequency response.

$$\begin{aligned} H(\Omega) &= 1 + e^{-j\Omega} + e^{-j2\Omega} \\ &= e^{j\Omega}\left(e^{-j\Omega} + 1 + e^{-j\Omega}\right) \\ &= e^{j\Omega}\left(1 + 2\cos(\Omega)\right) \end{aligned}$$

$$x[n] = \cos\left(\tfrac{2\pi}{3}\right) \iff X(\Omega) = \tfrac{1}{2}e^{j\frac{2\pi}{3}n} + \tfrac{1}{2}e^{-j\frac{2\pi}{3}n}$$

$$H\left(\tfrac{2\pi}{3}\right) = 0 \implies Y(\Omega) = 0 \iff y[n] = 0$$

LTI Systems

Three representations for LTI systems:

- difference equation (DT) or differential equation (CT)
- unit-sample response (DT) or impulse response (CT)
- frequency response

Frequency Response

Complex exponentials are eigenfunctions of LTI systems!

Characterize a system by how it shapes a signal's spectrum.

$$e^{j\Omega n} \rightarrow \boxed{\textbf{LTI}} \rightarrow H(\Omega)e^{j\Omega n}$$

$$X(\Omega) \rightarrow \boxed{\textbf{LTI}} \rightarrow H(\Omega)X(\Omega)$$

Eigenfunctions (if you're interested)

An eigenvalue-eigenvector pair $(\lambda, \boldsymbol{v})$ satisfy the eigenequation.

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$$

Likewise, eigenvalue-eigenfunction pairs satisfy eigenequations.

$$\frac{d}{dt}\{e^{\lambda t}\} = \lambda e^{\lambda t} \qquad \underbrace{\mathcal{R}\{\lambda^n\}}_{\text{right shift}} = \lambda^{-1}\lambda^n$$

Exponential functions $e^{\lambda t}$ are eigenfunctions of the d/dt operator.
eigenvalues $\lambda = j\omega \implies$ Eigenfunctions are CTFT basis functions!

Geometric sequences λ^n are eigenfunctions of the $\mathcal{R}$ (shift) operator.
eigenvalues $\lambda = e^{j\Omega} \implies$ Eigenfunctions are DTFT basis functions!

Eigenfunctions (if you're interested)

Let $P(\boldsymbol{A})$ denote a polynomial in $\boldsymbol{A}$. $P(\boldsymbol{A})$ has the same eigenvectors $\boldsymbol{v}_k$, but the corresponding eigenvalues are $P(\lambda_k)$.

$$P(\boldsymbol{A})\boldsymbol{v} = P(\lambda)\boldsymbol{v}$$

Likewise ...

$$P\left(\frac{d}{dt}\right)e^{\lambda t} = P(\lambda)e^{\lambda t}$$

$$P(\mathcal{R})\lambda^n = P(\lambda^{-1})\lambda^n$$

Expressing a signal in a basis of eigenfunctions facilitates analysis.

(e.g., The homogeneous solution to a linear differential equation with constant coefficients is a linear combination of eigenfunctions that lie in the null space of the polynomial differential operator.)

Eigenfunctions (if you're interested)

How do we interpret $\boldsymbol{Ax} = \boldsymbol{b}$?

- express $\boldsymbol{x} = \sum_k c_k \boldsymbol{v}_k$ in basis spanned by eigenvectors of $\boldsymbol{A}$
- scale each eigenvector $\boldsymbol{v}_k$ by the eigenvalue λ_k
- $\boldsymbol{b} = \sum_k c_k \lambda_k \boldsymbol{v}_k$

How do we interpret $x[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow y[n]$?

- express $x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$ in eigenfunction basis
- scale each eigenfunction $e^{j\Omega n}$ by the eigenvalue $H(\Omega)$
- $y[n] = \frac{1}{2\pi}\int_{2\pi} Y(\Omega)e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{2\pi} H(\Omega)X(\Omega)e^{j\Omega n}d\Omega$

Check Yourself

Consider an LTI system with unit-sample response $h[n]$.

$$h[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(\tfrac{1}{3}\right)^n u[n]$$

Suppose that the input to the system is $x[n]$.

$$x[n] = (-1)^n$$

Determine a closed-form expression for the output $y[n]$.

Check Yourself

$$h[n] = (\tfrac{1}{2})^n u[n] + (\tfrac{1}{3})^n u[n]$$

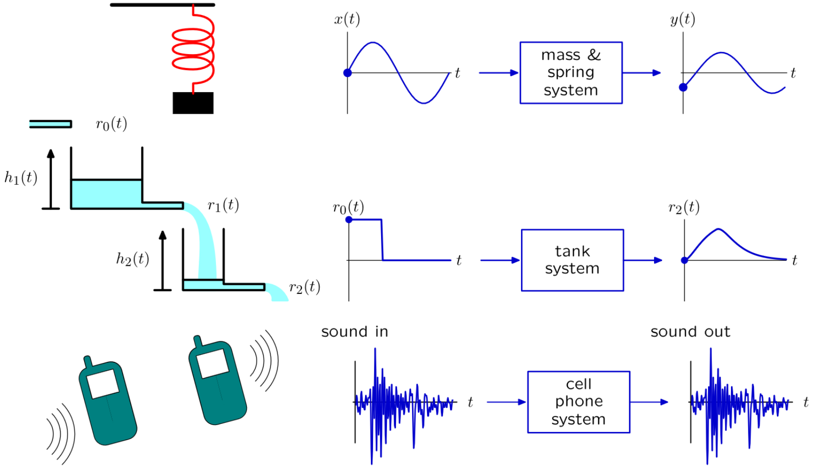
$$H(\Omega) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} + \frac{1}{1-\frac{1}{3}e^{-j\Omega}}$$

$$x[n] = (-1)^n = e^{j\pi n} \implies \Omega = \pi$$

$$H(\pi) = \frac{1}{1+\frac{1}{2}} + \frac{1}{1+\frac{1}{3}} = \frac{2}{3} + \frac{3}{4} = \frac{17}{12}$$

$$x[n] = (-1)^n \rightarrow \boxed{\textbf{LTI}} \rightarrow y[n] = \tfrac{17}{12}(-1)^n$$

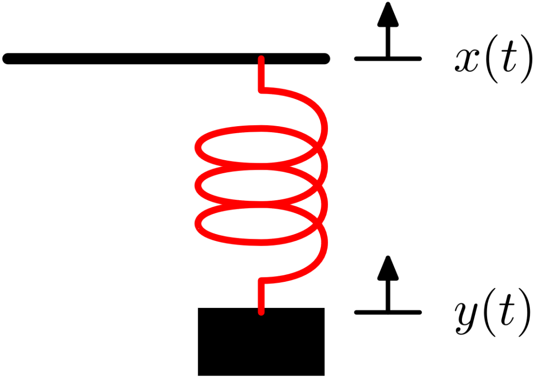
Signals and Systems



(Graphic: Denny Freeman)

Signals and Systems

Example: Mass on a Spring



(Graphic: Denny Freeman)

Signals and Systems

Example: Mass on a Spring

- **signals:** position $x(t)$ and position $y(t)$
- **parameters:** mass M and spring constant K

$$M\frac{d^2y(t)}{dt^2} = K\big(x(t) - y(t)\big)$$

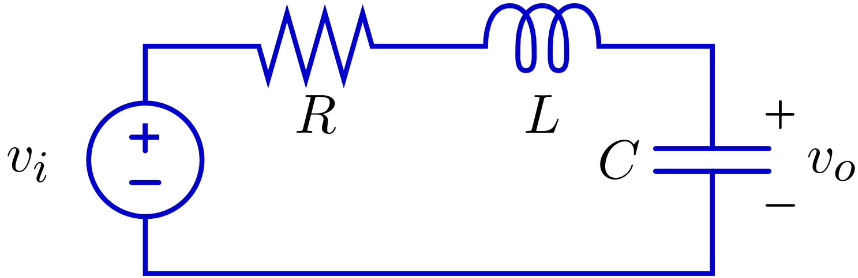
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\omega_0^2}{\omega_0^2 - \omega^2} \qquad \omega_0 = \sqrt{\frac{K}{M}}$$

$$\cos(\omega t) \rightarrow \boxed{\textbf{LTI}} \rightarrow |H(\omega)|\cos\big(\omega t + \angle H(\omega)\big)$$

very responsive to sinusoidal oscillations at $\omega \approx \omega_0$

Signals and Systems

Example: Series RLC Circuit



(Graphic: Denny Freeman)

Signals and Systems

Example: Series RLC Circuit

- **signals:** input voltage $v_i(t)$ and output voltage $v_o(t)$
- **parameters:** resistance R, inductance L, and capacitance C

$$C\frac{d^2v_o(t)}{dt^2} = \frac{1}{L}\left(v_i(t) - RC\frac{dv_o(t)}{dt} - v_o(t)\right)$$

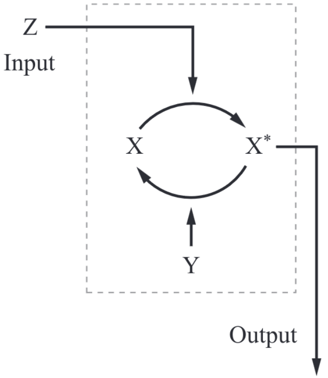
$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{\omega_0^2}{\omega_0^2 + \frac{1}{\tau}j\omega - \omega^2} \qquad \omega_0 = \sqrt{\frac{1}{LC}} \qquad \tau = \frac{L}{R}$$

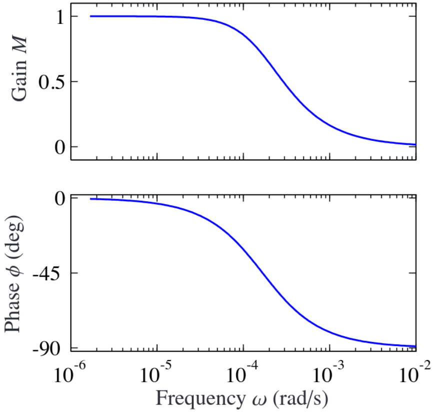
$$\cos(\omega t) \rightarrow \boxed{\textbf{LTI}} \rightarrow |H(\omega)|\cos\big(\omega t + \angle H(\omega)\big)$$

damped harmonic oscillator

Signals and Systems

Example: Phosphorylation Cycle





(*Biomolecular Feedback Systems*, D. Del Vecchio and R. M. Murray)

Signals and Systems

Example: Phosphorylation Cycle

- **signals:** kinase $x(t)$ and phosphorylated substrate $y(t)$
- **parameters:** production rate β and decay rate γ

$$\frac{dy(t)}{dt} = \beta x(t) - \gamma y(t) \iff j\omega Y(\omega) = \beta X(\omega) - \gamma Y(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\beta}{\gamma + j\omega}$$

$$\left|H(\omega)\right| = \frac{\beta}{\sqrt{\gamma^2 + \omega^2}} \qquad \angle H(\omega) = -\tan^{-1}\left(\frac{\omega}{\gamma}\right)$$

low-pass filter: unresponsive to rapidly-varying stimuli

Check Yourself

Difference Equation → Unit-Sample Response

Determine the unit-sample response $h[n]$ for the following linear constant-coefficient difference equation. Assume that the system is initially at rest: For $n < 0$, $x[n] = y[n] = 0$.

$$y[n] = \frac{1}{2}y[n-1] + x[n]$$

Check Yourself

We could set $x[n]=\delta[n]$ and notice that the response $y[n] = h[n]$ is a decaying geometric sequence. Alternatively, we could determine the frequency response $H(\Omega)$ by computing the DTFT of the difference equation. The unit-sample response $h[n]$ is the inverse DTFT of the frequency response $H(\Omega)$.

$$Y(\Omega)=\tfrac{1}{2}e^{-j\Omega}Y(\Omega)+X(\Omega) \iff H(\Omega)=\frac{Y(\Omega)}{X(\Omega)}=\frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

$$h[n]=\frac{1}{2\pi}\int_{2\pi}H(\Omega)e^{j\Omega n}\,d\Omega=\frac{1}{2\pi}\int_{2\pi}\frac{e^{j\Omega n}}{1-\frac{1}{2}e^{-j\Omega}}\,d\Omega=\left(\tfrac{1}{2}\right)^n u[n]$$

Check Yourself

Frequency Response → Unit-Sample Response

Determine the unit-sample response $h[n]$ corresponding to the frequency response $H(\Omega)$.

$$H(\Omega) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} + \frac{e^{-j2\Omega}}{1-\frac{1}{3}e^{-j\Omega}}$$

Check Yourself

Write $H(\Omega) = H_1(\Omega) + H_2(\Omega)$. Compute the inverse discrete-time Fourier transforms of $H_1(\Omega)$ and $H_2(\Omega)$ separately. The corresponding unit-sample response $h[n]$ is the superposition of the inverse DTFTs: $h[n] = h_1[n] + h_2[n]$.

$$H_1(\Omega) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} \iff h_1[n] = \left(\tfrac{1}{2}\right)^n u[n]$$

$$H_2(\Omega) = \frac{e^{-j2\Omega}}{1 - \frac{1}{3}e^{-j\Omega}} \iff h_2[n] = \left(\tfrac{1}{3}\right)^{n-2} u[n-2]$$

$$h[n] = h_1[n] + h_2[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(\tfrac{1}{3}\right)^{n-2} u[n-2]$$

Check Yourself

Frequency Response → Difference Equation

Determine a linear difference equation with constant coefficients with frequency response $H(\Omega)$.

$$H(\Omega) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{e^{-j2\Omega}}{1 - \frac{1}{3}e^{-j\Omega}}$$

Check Yourself

Express $H(\Omega)$ with a common denominator. Pattern-match.

$$H(\Omega)=\frac{(1-\frac{1}{3}e^{-j\Omega})+e^{-j2\Omega}(1-\frac{1}{2}e^{-j\Omega})}{(1-\frac{1}{2}e^{-j\Omega})(1-\frac{1}{3}e^{-j\Omega})}$$

$$\frac{Y(\Omega)}{X(\Omega)}=\frac{1-\frac{1}{3}e^{-j\Omega}+e^{-j2\Omega}-\frac{1}{2}e^{-j3\Omega}}{1-\frac{5}{6}e^{-j\Omega}+\frac{1}{6}e^{-j2\Omega}}$$

$$\left(1-\frac{5}{6}e^{-j\Omega}+\frac{1}{6}e^{-j2\Omega}\right)Y(\Omega)=\left(1-\frac{1}{3}e^{-j\Omega}+e^{-j2\Omega}-\frac{1}{2}e^{-j3\Omega}\right)X(\Omega)$$

$$y[n]-\frac{5}{6}y[n-1]+\frac{1}{6}y[n-2]=x[n]-\frac{1}{3}x[n-1]+x[n-2]-\frac{1}{2}x[n-3]$$

Check Yourself

Differential Equation → Impulse Response

Determine the impulse response $h(t)$ for the following linear ordinary differential equation with constant coefficients. Assume that the system is initially at rest: For $t < 0$, $x(t) = y(t) = 0$.

$$\frac{dy(t)}{dt} = x(t) - \frac{1}{2}y(t)$$

Check Yourself

Fourier transforms turn differential equations into algebraic equations. First, determine the frequency response $H(\omega)$ by computing the Fourier transform of the differential equation. The impulse response $h(t)$ is the inverse Fourier transform of the frequency response $H(\omega)$.

$$j\omega Y(\omega) = X(\omega) - \tfrac{1}{2} Y(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{1}{\frac{1}{2} + j\omega}$$

$$H(\omega) = \int_{-\infty}^{\infty} \underbrace{e^{-\frac{1}{2}t} u(t)}_{h(t)} e^{-j\omega t} dt$$

Check Yourself

Frequency Response → Differential Equation

Determine a linear ordinary differential equation with constant coefficients with frequency response $H(\omega)$.

$$H(\omega) = \frac{1 - j\omega}{1 - 4\omega^2}$$

Check Yourself

Multiplication by $j\omega$ in the frequency domain corresponds to differentiation with respect to t in the time domain.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 - j\omega}{1 + 4(j\omega)^2}$$

$$y(t) + 4\,\frac{d^2y(t)}{dt^2} = x(t) - \frac{dx(t)}{dt}$$

LTI Systems

Three representations for LTI systems:

- difference equation (DT) or differential equation (CT)
- unit-sample response (DT) or impulse response (CT)
- frequency response

Communications Systems

Amplitude Modulation

$$x(t) \rightarrow \boxed{\textbf{AM}} \rightarrow y(t) = x(t)\cos(\omega_c t)$$

Is an amplitude modulator a linear system?
Is an amplitude modulator a time-invariant system?

Communications Systems

Amplitude Modulation

$$x(t) \rightarrow \boxed{\textbf{AM}} \rightarrow y(t) = x(t)\cos(\omega_c t)$$

Is an amplitude modulator a linear system?
Is an amplitude modulator a time-invariant system?

Linear? Yes.

$$(c_1 x_1(t)+c_2 x_2(t))\cos(\omega_c t) = c_1 x_1(t)\cos(\omega_c t)+c_2 x_2(t)\cos(\omega_c t)$$

Time-invariant? No! The carrier $\cos(\omega_c t)$ is time-varying.
The system generates new non-zero frequencies in the output!

Communications Systems

Amplitude Modulation

Transmission: Multiply $x(t)$ by sinusoidal carrier signal $c(t)$ and transmit the amplitude-modulated signal $y(t) = x(t)c(t)$.

Reception: Recover $x(t)$ from the amplitude-modulated signal $y(t)$ by multiplying by the carrier $c(t)$ and then low-pass filtering.

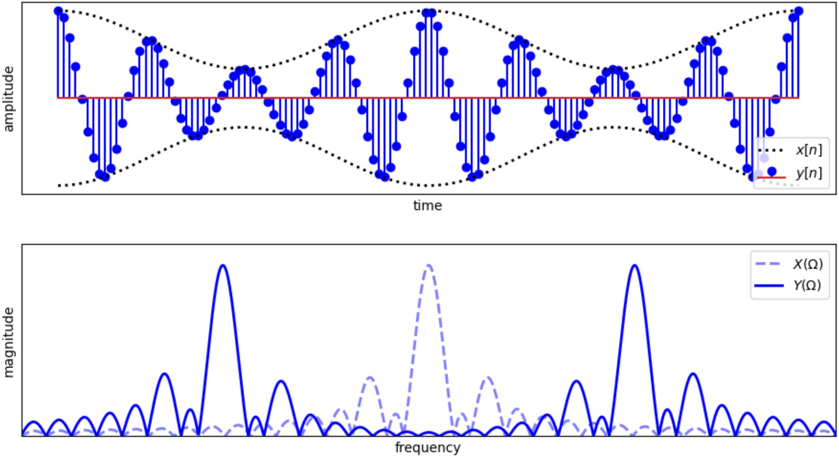
$$c(t) = \cos(\omega_c t) = \tfrac{1}{2}e^{j\omega_c t} + \tfrac{1}{2}e^{-j\omega_c t}$$

$$y(t) = x(t)c(t) \iff Y(\omega) = \tfrac{1}{2\pi}\big(X * C\big)(\omega)$$

$$Y(\omega) = \underbrace{\tfrac{1}{2}X(\omega - \omega_c) + \tfrac{1}{2}X(\omega + \omega_c)}_{\text{copies of } X(\omega) \text{ shifted outward by } \omega_c}$$

Communications Systems

Examine a finite-length window of the signal $y[n] = x[n]\cos(\Omega_c n)$.



More Modulation

We examined amplitude modulation in class. Perhaps you've heard of frequency modulation (FM) or phase modulation (PM) — but you don't need to know these for the quiz, per se.

Sinusoidal Modulation

$$y(t) = A\cos(\omega t + \phi)$$

- amplitude (AM) — time-varying amplitude $A = A(t)$
- frequency (FM) — time-varying frequency $\omega = \omega(t)$
- phase (PM) — time-varying phase $\phi = \phi(t)$

Communications: Match the signal to the channel medium by encoding the message in the carrier signal.

The Summary So Far

- Fourier transform pairs and properties
- linearity and time-invariance
- difference equations (DT) and differential equations (CT)
- unit-sample response (DT) and impulse response (CT)
- frequency response
- convolution and filtering
- modulation and communications systems

Question of the Day #1

Consider the unit-sample response $h[n]$.

$$h[n] = 2\left(\frac{1}{3}\right)^n u[n] + 5\left(\frac{1}{7}\right)^n u[n]$$

Suppose we want to express the frequency response in the form

$$H(\Omega) = \frac{A_1}{1 - p_1 e^{-j\Omega}} + \frac{A_2}{1 - p_2 e^{-j\Omega}}$$

where A_1, A_2, p_1, and p_2 are constants.

Determine values for A_1, A_2, p_1, and p_2.

Next: Discrete Fourier Transform

Discrete Fourier Transform

Quotes

'After this class, I intend to type "fft" when I need to, and try to forget the rest.'

Even if all you do after this class is type

$$\texttt{fft}(\cdots)$$

once in a while, you better know what you're doing!

- "DFT? What's that? You mean, FFT?"
- "What's with all these non-zero frequencies?"
- "Zero-padding gives me arbitrarily-good frequency resolution."
- "The FFT only works if the signal length N is a power of 2."

DT Fourier Representations

The **DTFS** is for periodic signals. No real-world periodic signals!

- finite summation over n (infinite-length periodic signals)
- frequency variable k of discrete domain

The **DTFT** may only be computed in theory.

- infinite summation over n (infinite-length aperiodic signals)
- frequency variable Ω of continuous domain

The **DFT** can be computed in practice.

- finite summation over n (finite-length aperiodic signals)
- frequency variable k of discrete domain

The **FFT** refers to a family of algorithms for computing the DFT.

The **STFT** is a "moving-window Fourier transform."

- For practical computation, use the DFT.

Discrete Fourier Transform

The DFT is a discrete-time, discrete-frequency Fourier transform.

- finite-length signals $x_w[n] = x[n]w[n]$
- discrete in time (n) N time-samples
- discrete in frequency (k) N frequency-samples

Discrete Fourier Transform

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n} \qquad \textbf{analysis}$$

$$x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\frac{2\pi}{N}n} \qquad \textbf{synthesis}$$

Discrete Fourier Transform

DFT vs. Discrete-Time Fourier Series (DTFS)

The length-N DFT is equivalent to the discrete-time Fourier series of an N-periodic extension of the windowed signal $x_w[n] = x[n]w[n]$.

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x_w[n \bmod N] e^{-jk\frac{2\pi}{N}n}$$

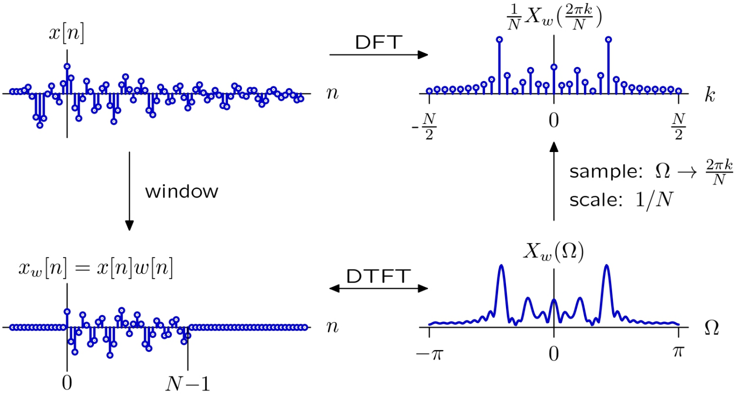
DFT vs. Discrete-Time Fourier Transform (DTFT)

$$X[k] = \tfrac{1}{N} X_w\!\left(\tfrac{2\pi}{N} k\right)$$

DFT frequency resolution: $\frac{f_s}{N}$ hertz or $\frac{2\pi}{N}$ radians

Relation Between DFT and DTFT

Graphical depiction of relation between DFT and DTFT.



While sampling and scaling are important, it is the **windowing** that most affects frequency content.

(Graphic: Denny Freeman)

Window Functions

Multiplying $x[n]$ by the window function $w[n]$ corresponds to convolving the DTFT of $x[n]$ with the DTFT of $w[n]$.

Windowing

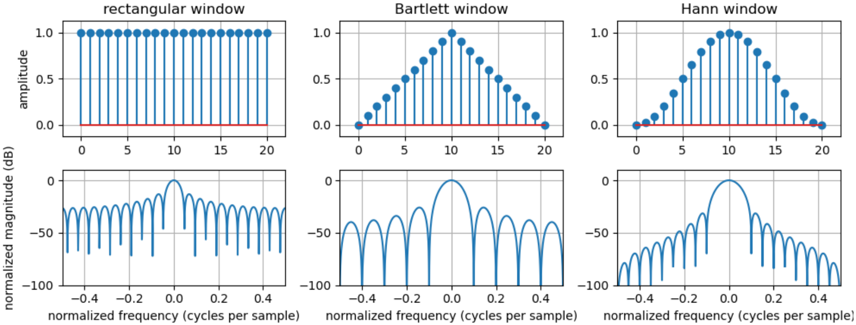
$$x_w[n] = x[n]w[n] \iff X_w(\Omega) = \frac{1}{2\pi}\big(X * W\big)(\Omega)$$

long time-domain $w[n] \iff$ narrow frequency-domain $W(\Omega)$

There are many window functions.

SciPy: Bartlett, Bartlett-Hann, Blackman, Blackman-Harris, Bohman, box-car, cosine, discrete prolate spheroidal sequences, Dolph-Chebyshev, exponential, flat-top, Gaussian, generalized Hamming, Hamming, Hann, Kaiser, Kaiser-Bessel, Lanczos, Nutall, Parzen, Taylor, triangular, Tukey, ...

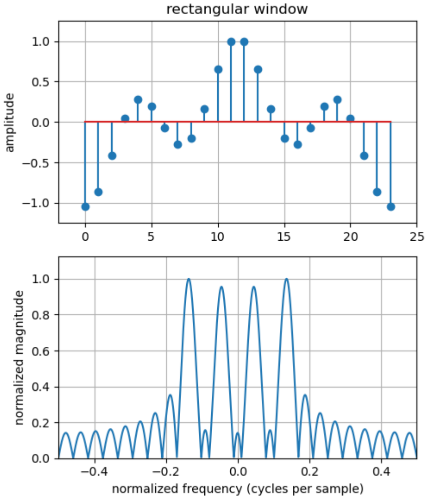
Window Functions



The window to use depends on the task at hand.

- What's most important? Narrow mainlobe? Low sidelobes?

Window Functions



DFT: Circular Convolution

Multiplication of N-point DFTs in the frequency domain corresponds to circular convolution in the time domain.

$$\begin{aligned}(x \circledast h)[n] &= N\,\mathrm{DFT}_N^{-1}\{X_N[k]H_N[k]\} \\ &= \sum_{m=0}^{N-1} x[m]h\Big[(n-m) \bmod N\Big]\end{aligned}$$

Circular convolution seems complicated, but it is really simple. You do need to know how to do regular convolution, though.

Circular Convolution

- Compute the regular (non-circular) convolution.
- Wrap the result into a length-N interval.
- Periodically extend this length-N interval.

Circular Convolution

n	=	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
$x[n]$	=	1	1	1	1	0	0	0	0
$h[n]$	=	1	2	3	0	0	0	0	0

n	=	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
$(x * h)[n]$	=	1	3	6	6	5	3	0	0
$(x \circledast h)_6[n]$	=	1	3	6	6	5	3	1	3
$(x \circledast h)_5[n]$	=	4	3	6	6	5	4	3	6
$(x \circledast h)_4[n]$	=	6	6	6	6	6	6	6	6

Check Yourself

Suppose that $x[n] = 0$ and $h[n] = 0$ for $n \notin \{0, 1, 2, 3, \ldots, 9\}$.

$$y[n] = \underbrace{\text{DTFT}^{-1}\{X(\Omega)H(\Omega)\}}_{(x*h)[n]} \qquad z[n] = \underbrace{\text{DFT}_5^{-1}\{X(\tfrac{2\pi}{5}k)H(\tfrac{2\pi}{5}k)\}}_{(x \circledast h)[n]}$$

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
$y[n]$	4	3	7	7	0	A	B	C	D	E
$z[n]$	4	3	14	13	1	4	3	14	13	1

Determine appropriate values for the constants A, B, C, D, and E. Give a few choices of $x[n]$ and $h[n]$ that produce $y[n]$.

Check Yourself

Suppose that $x[n] = 0$ and $h[n] = 0$ for $n \notin \{0, 1, 2, 3, \ldots, 9\}$.

$$y[n] = \underbrace{\mathrm{DTFT}^{-1}\{X(\Omega)H(\Omega)\}}_{(x*h)[n]} \qquad z[n] = \underbrace{\mathrm{DFT}_5^{-1}\{X(\tfrac{2\pi}{5}k)H(\tfrac{2\pi}{5}k)\}}_{(x \circledast h)[n]}$$

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
$y[n]$	4	3	7	7	0	A	B	C	D	E
$z[n]$	4	3	14	13	1	4	3	14	13	1

Determine appropriate values for the constants A, B, C, D, and E. Give a few choices of $x[n]$ and $h[n]$ that produce $y[n]$.

$A = 0 \qquad B = 0 \qquad C = 7 \qquad D = 6 \qquad E = 1$

Short-Time Fourier Transforms

Think of short-time Fourier transforms as "moving-window Fourier transforms." We analyze how a signal's spectrum changes over time.

Any Fourier transform can be a short-time Fourier transform.

Short-Time CTFT: $$X(\omega, \tau) = \int_{-\infty}^{\infty} x(t) \underbrace{w(t-\tau)}_{\text{window}} e^{-j\omega t} dt$$

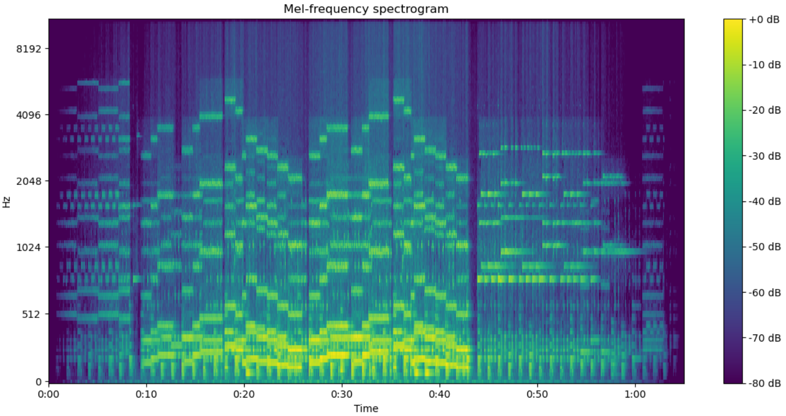
Short-Time DTFT: $$X(\Omega, m] = \sum_{n=-\infty}^{\infty} x[n] \underbrace{w[n-m]}_{\text{window}} e^{-j\Omega n}$$

Window Functions

$$x_w[n] = x[n]w[n] \iff X_w(\Omega) = \frac{1}{2\pi}\big(X * W\big)(\Omega)$$

Spectrograms

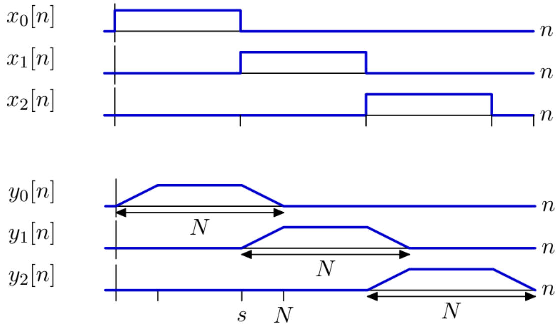
Examine the (magnitude)^2 of a signal's time-varying spectrum.



(spectrogram of "Les Patineurs" performed on Hammond organ)

Overlap-Add Method

How can we process long signals block-by-block? Divide the input $x[n]$ into blocks — each of length s. Convolve each block with $h[n]$.



The output is $y[n] = y_0[n] + y_1[n] + y_2[n] + \cdots$ Hence *overlap-add.*

(Graphic: Denny Freeman)

Fast Fourier Transform (FFT)

Gauss, circa 1805: "…truly, that method greatly reduces the tediousness of mechanical calculations … "

Radix-2 Decimation-in-Time Algorithm

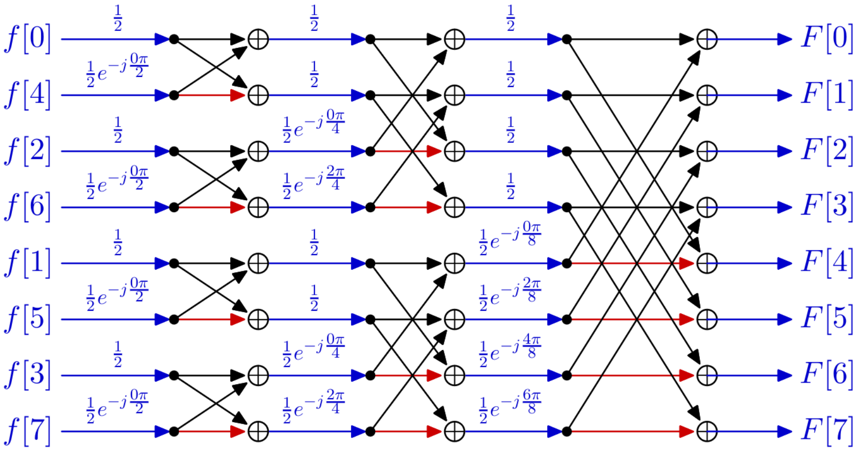
- Split a length-N DFT into a sum of two length-($N/2$) DFTs.

$$X_N[k] = \frac{1}{2}\left(X_{N/2}^{\text{even}}[k] + W_N^k X_{N/2}^{\text{odd}}[k]\right)$$

$$W_N = e^{-j\frac{2\pi}{N}} \quad (N^{\text{th}} \text{ root of unity, or "twiddle factor"})$$

- Repeat (↑) until $N/2 = 1$, when we can't divide by 2 anymore.
- The DFT of a length-1 signal is the signal itself: $X[0] = x[0]$.

FFT: Decimation in Time



(Graphic: Denny Freeman)

Summary

- Fourier transform pairs and properties
- linearity and time-invariance
- difference equations (DT) and differential equations (CT)
- unit-sample response (DT) and impulse response (CT)
- frequency response
- convolution and filtering
- modulation and communications systems
- discrete Fourier transform (DFT)
- window functions
- circular convolution
- short-time Fourier transforms
- fast Fourier transform (FFT)

“Signals and Systems” Subjects

You may be interested in the following subjects.
(These certainly aren’t the only “signals and systems” subjects!)

6.200	Electrical Circuits: Modeling and Design ($\mathbb{U}$)
6.300	Signal Processing ($\mathbb{U}$)
6.301	Signals, Systems, and Inference ($\mathbb{U}$)
6.302	Fundamentals of Music Processing ($\mathbb{U}/\mathbb{G}$)
6.310	Dynamical System Modeling and Control Design ($\mathbb{U}/\mathbb{G}$)
6.430	Introduction to Computer Vision ($\mathbb{U}$)
6.480	Biomedical Systems: Modeling and Inference ($\mathbb{U}$)
6.700	Discrete-Time Signal Processing ($\mathbb{G}$)
6.741	Principles of Digital Communication ($\mathbb{U}/\mathbb{G}$)
6.880	Biomedical Signal and Image Processing ($\mathbb{U}/\mathbb{G}$)
6.862	Spoken Language Processing ($\mathbb{U}/\mathbb{G}$)
6.C27	Computational Imaging: Physics and Algorithms ($\mathbb{U}/\mathbb{G}$)

Question of the Day #2

Describe how the discrete Fourier transform is related to

- the discrete-time Fourier series and
- the discrete-time Fourier transform.

Why do we care about the discrete Fourier transform, anyway?
(No participation credit if you say, "Nobody cares.")

Discrete Fourier Transform

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} \qquad \textbf{analysis}$$

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\frac{2\pi}{N}n} \qquad \textbf{synthesis}$$