6.3000: Signal Processing

Fourier Series – Complex Exponential Form

- complex numbers
- complex exponentials and their relation to sinusoids
- complex exponential form of Fourier series
- delay property of Fourier series

Tutorials: Tutorials begin today at 3:00 p.m.

Optional Lab Check-In: You must complete the check-in by 9:30 p.m. tonight (when common hours end) to receive credit. Otherwise, your lab grade will be entirely determined from your lab write-up. The check-in can only improve your grade!

Homework: Homework 1 is due by 2:00 p.m. on Thursday.

February 11, 2025

Fourier Series

Previously: representing periodic signals as weighted sums of sinusoids

Synthesis Equation

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t) \quad \text{where } \omega_o = \frac{2\pi}{T}$$

Analysis Equations

$$c_0 = \frac{1}{T}\int_T f(t)\,dt$$

$$c_k = \frac{2}{T}\int_T f(t)\cos(k\omega_o t)\,dt$$

$$d_k = \frac{2}{T}\int_T f(t)\sin(k\omega_o t)\,dt$$

Today: simplifying the math with complex numbers

Simplifying Math By Using Complex Numbers

Complex numbers simplify thinking about roots of numbers, polynomials:

- all numbers have two square roots, three cube roots, ...
- all polynomials of order n have n roots, some of which may be repeated.

Much simpler than the rules that govern purely real-valued formulations! For example, a cubic equation with real-valued coefficients can have 1 or 3 real-valued roots; a quartic equation can have 0, 2, or 4.

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Complex exponentials simplify working with trigonometric functions (Euler's formula, Leonhard Euler, 1748):

$$e^{j\theta} = \cos\theta + j\sin\theta$$

This single equation virtually eliminates a need for trig tables. Richard Feynman called this "the most remarkable formula in mathematics."

The special case $\theta = \pi$ leads to Euler's Identity:

$$e^{j\pi} + 1 = 0$$

which relates five fundamental constants in a single equation.

Note that we will normally use j (instead of i) to represent $\sqrt{-1}$.

Where Does Euler's Formula Come From?

Euler showed the relation between complex exponentials and sinusoids by solving the following differential equation two ways.

$$\frac{d^2 f(\theta)}{d\theta^2} + f(\theta) = 0$$

Let $f_1(\theta) = A\cos(\alpha\theta) + B\sin(\beta\theta)$

$$\frac{df_1(\theta)}{d\theta} = -\alpha A\sin(\alpha\theta) + \beta B\cos(\beta\theta)$$

$$\frac{d^2 f_1(\theta)}{d\theta^2} = -\alpha^2 A\cos(\alpha\theta) - \beta^2 B\sin(\beta\theta)$$

Let $\alpha = \beta = 1$

$$f_1(\theta) = A\cos\theta + B\sin\theta$$

Let $f_2(\theta) = Ce^{\gamma\theta}$

$$\frac{df_2(\theta)}{d\theta} = \gamma Ce^{\gamma\theta}$$

$$\frac{d^2 f_2(\theta)}{d\theta^2} = \gamma^2 Ce^{\gamma\theta}$$

Let $\gamma^2 = -1$

$$f_2(\theta) = Ce^{\pm j\theta}$$

If we arbitrarily take $f_2(\theta) = e^{j\theta}$, then $f_2(0) = 1$ and $f_2'(0) = j$.
To make $f_1(\theta) = f_2(\theta)$, A must be 1 and B must be j:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

This argument presumes the existance of a constant j whose square is -1 and that can be manipulated as an ordinary algebraic constant.

Where Does Euler's Formula Come From?

Euler's formula also follows from Maclaurin expansion of the exponential function, assuming the j behaves like any other algebraic constant.

Start with the expansion of the real-valued function:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

Assume that the same expansion holds for complex-valued arguments:

$$\begin{aligned} e^{j\theta} &= 1 + j\theta + \frac{j^2\theta^2}{2!} + \frac{j^3\theta^3}{3!} + \frac{j^4\theta^4}{4!} + \frac{j^5\theta^5}{5!} + \frac{j^6\theta^6}{6!} + \frac{j^7\theta^7}{7!} + \cdots \\ &= 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{j\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{j\theta^7}{7!} + \cdots \\ &= \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)}_{\cos\theta} + j \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)}_{\sin\theta} \end{aligned}$$

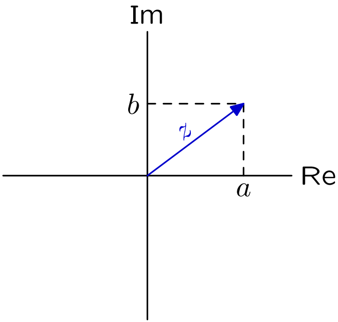
Euler's formula results by splitting the even and odd powers of θ.

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Geometric Interpretation

In 1799, Caspar Wessel was the first to describe complex numbers as points in the complex plane. Imaginary numbers had been in use since the 1500's.

$$z = a + jb$$



Complex numbers are fundamentally two dimensional. Unlike other constants (such as π), $j = \sqrt{-1}$ defines an entirely new (imaginary) dimension – and a new way to think about operations that involve complex numbers.

Algebraic Addition

Addition: the real part of a sum is the sum of the real parts, and the imaginary part of a sum is the sum of the imaginary parts.

Let z_1 and z_2 represent complex numbers:

$$z_1 = a_1 + jb_1$$

$$z_2 = a_2 + jb_2$$

Then

$$z_1 + z_2 = (a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$$

Geometric Addition

Rules for adding complex numbers are same as those for adding vectors.

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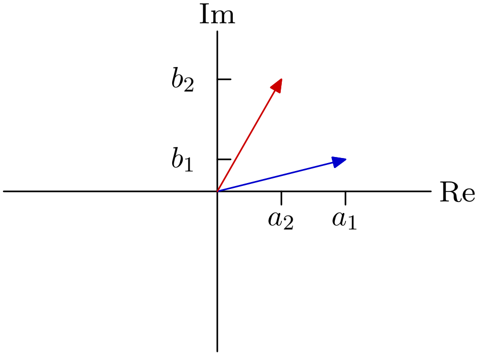

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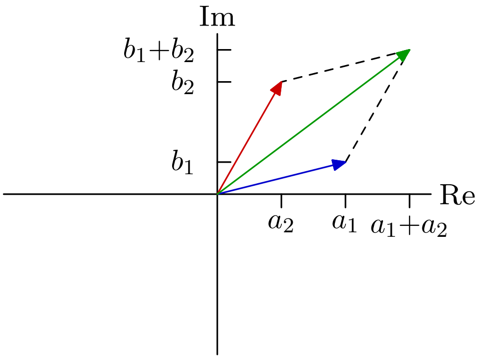


Algebraic Multiplication

Multiplication is more complicated.

Let z_1 and z_2 represent complex numbers:

$$z_1 = a_1 + jb_1$$

$$z_2 = a_2 + jb_2$$

Then

$$\begin{aligned} z_1 \times z_2 &= (a_1 + jb_1) \times (a_2 + jb_2) \\ &= a_1 \times a_2 + a_1 \times jb_2 + jb_1 \times a_2 + jb_1 \times jb_2 \\ &= (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + b_1 a_2) \end{aligned}$$

Although the rules of algebra still apply, the result is complicated:

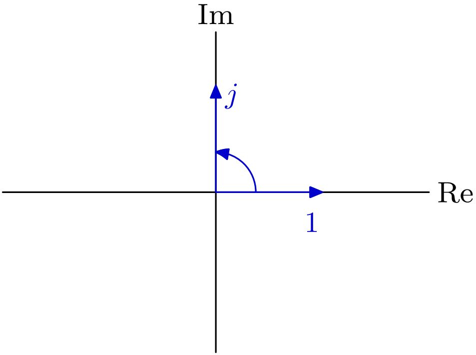
- the real part of a product is NOT the product of the real parts, and
- the imaginary part is NOT the product of the imaginary parts.

Geometric Multiplication

The two-dimensional view of complex numbers allows us to think about multiplication by an imaginary number as a **rotation**.

Multiplying by j

- **rotates 1 to j**,
- rotates j to -1,
- rotates -1 to $-j$, and
- rotates $-j$ to 1.



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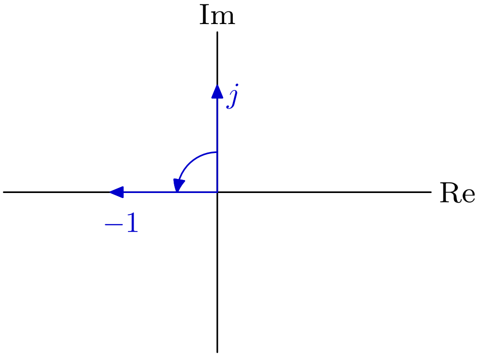

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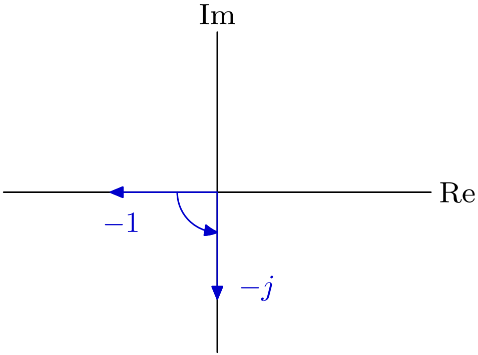

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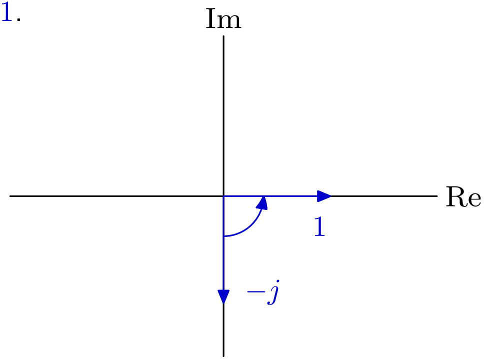

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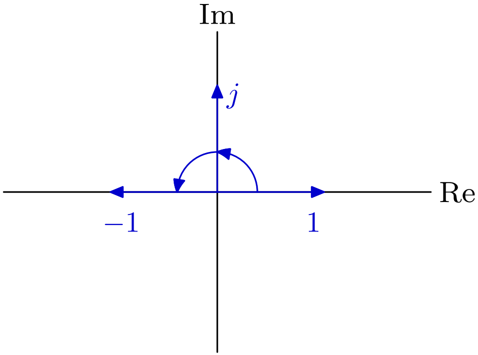


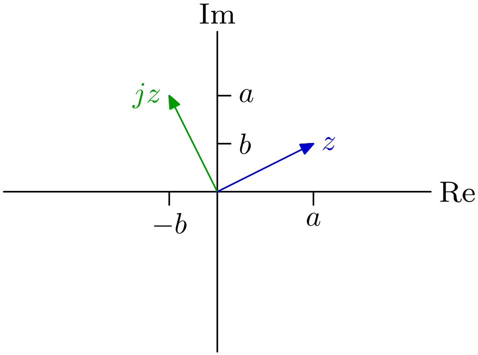
Multiplying by j rotates a vector by $\pi/2$.
Multiplying by $j^2 = -1$ rotates a vector by π.

Geometric Multiplication

Multiplying by j rotates an arbitrary complex number by $\pi/2$.

$$z = a + jb$$

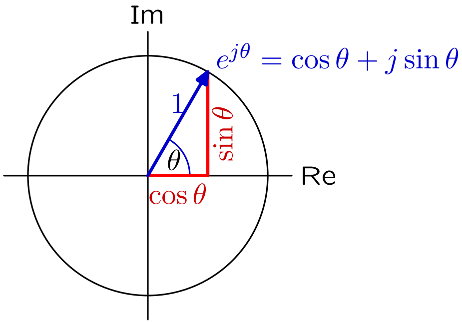
$$jz = ja - b$$



Geometric Interpretation of Euler's Formula

Euler's formula equates polar and rectangular descriptions of a unit vector at angle θ.

$$e^{j\theta} = \cos\theta + j\sin\theta$$

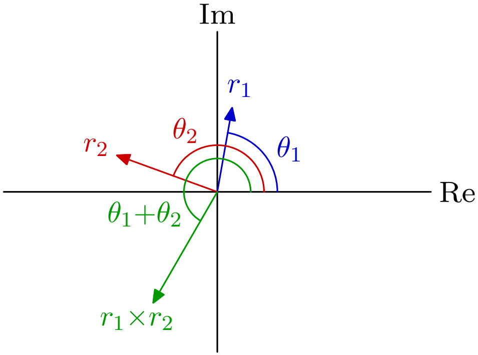


This construction provides

- a direct link between Euler's formula and the planar representation of complex numbers
- a new **polar** representation of complex numbers

Geometric Approach: Polar Form

The magnitude of the product of complex numbers is the **product** of their magnitudes. The angle of a product is the **sum** of the angles.



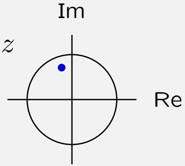
$$r_1e^{j\theta_1} \times r_2e^{j\theta_2} = r_1(\cos\theta_1 + j\sin\theta_1) \times r_2(\cos\theta_2 + j\sin\theta_2)$$

$$= r_1r_2(\underbrace{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}_{\cos(\theta_1+\theta_2)} + \underbrace{j\cos\theta_1\sin\theta_2 + j\sin\theta_1\cos\theta_2}_{j\sin(\theta_1+\theta_2)})$$

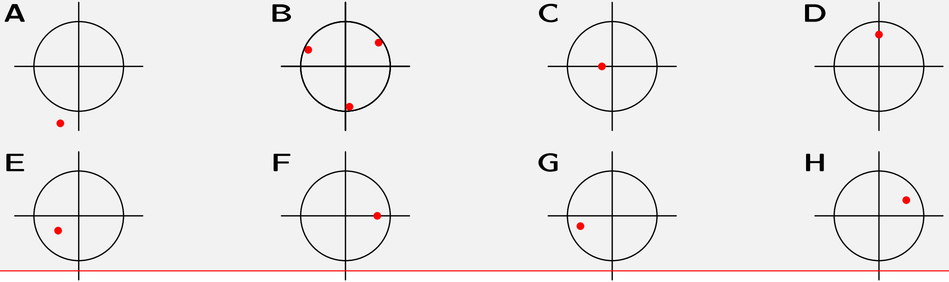
$$= r_1r_2e^{j(\theta_1+\theta_2)}$$

Check Yourself

Let z represent the complex number shown by a filled dot in the complex plane below, where the circle has radius 1.



Which if any of the following figures shows the value of jz?
Which if any of the following figures shows the value of $\text{Im}(z)$?
Which if any of the following figures shows the value of $1/z$?



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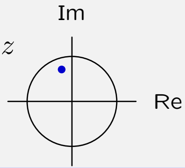

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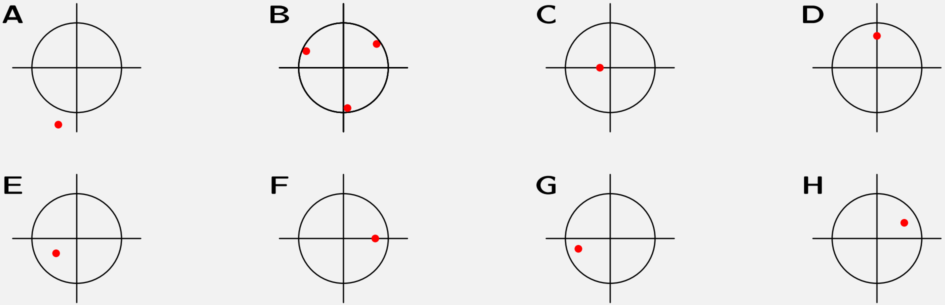

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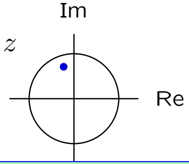
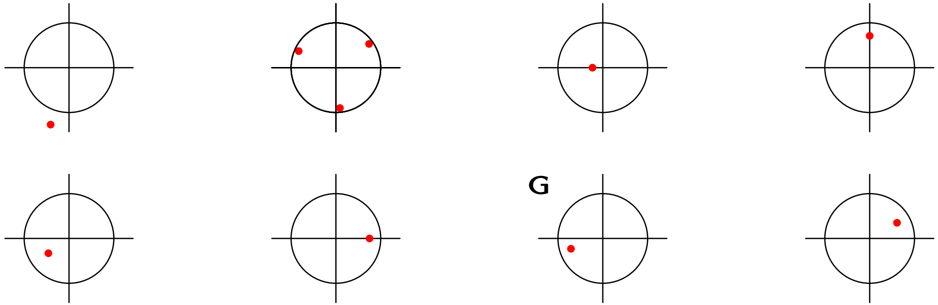


Which if any of the following figures shows the value of jz? **G**



$|jz| = |z|$ and $\angle(jz) = \angle z + \pi/2$

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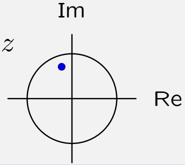

Which if any of the following figures shows the value of $\text{Im}(z)$?

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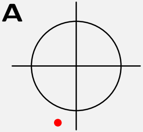
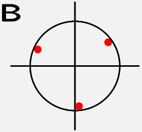
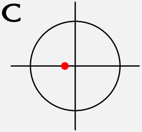


B



C



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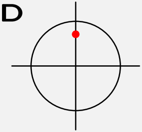
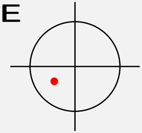
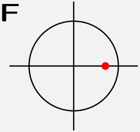


E



F



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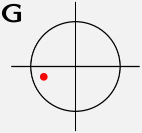
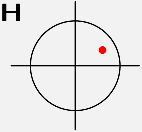


H



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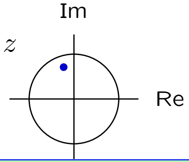
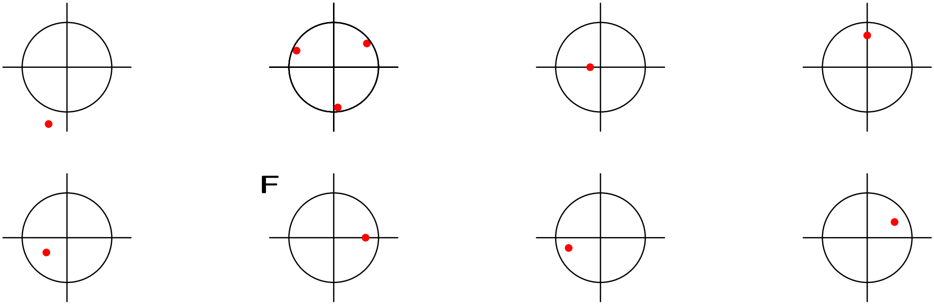


Which if any of the following figures shows the value of Im(z)? F



Im(z)$= \frac{z-z^*}{j2}$, which is a real number.

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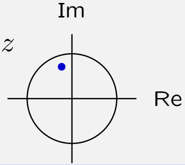

Which if any of the following figures shows the value of $1/z$?

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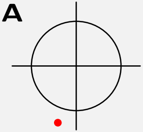

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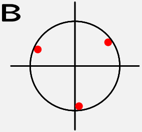

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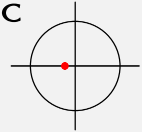

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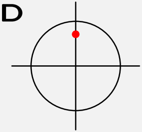

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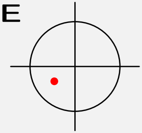

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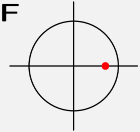

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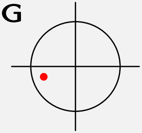

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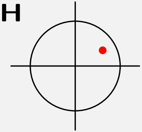

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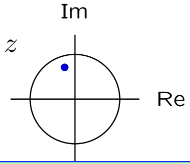

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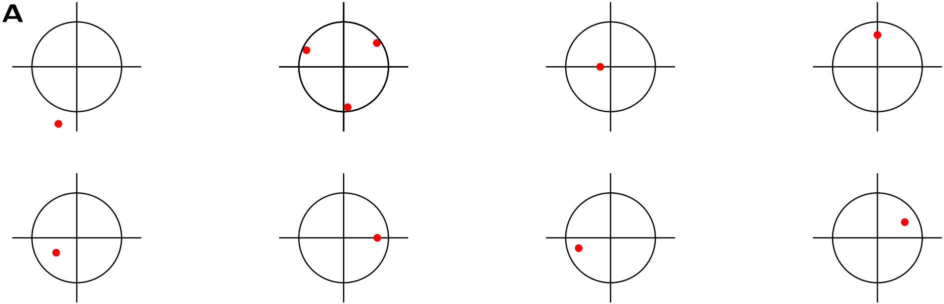

$|1/z| = 1/|z|$ and $\angle(1/z) = -\angle z$

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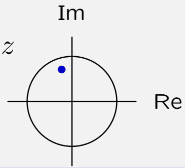
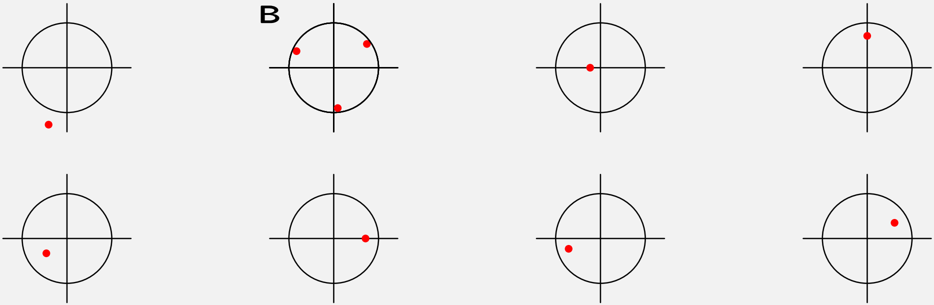


What simple function of z is shown in **B**?



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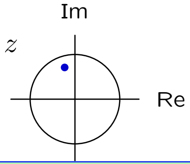

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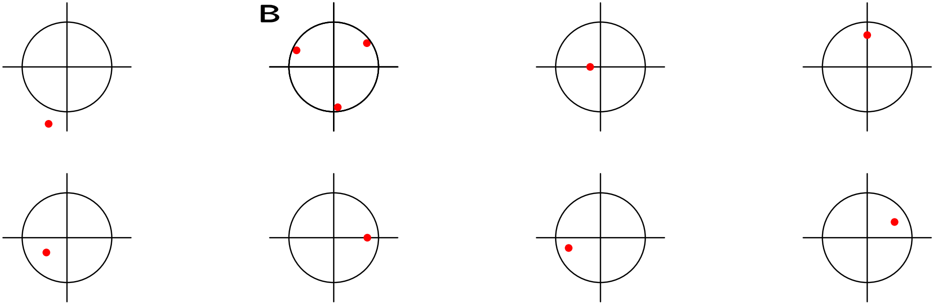
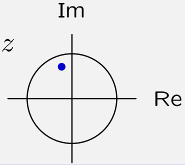


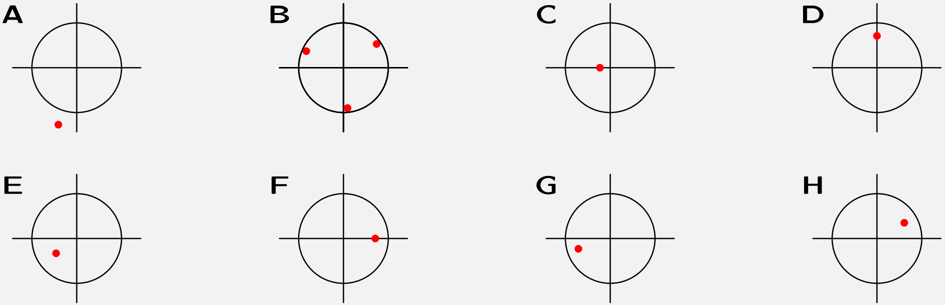
$\sqrt[3]{z}$

Check Yourself

Let z represent the complex number shown by a filled dot in the complex plane below, where the circle has radius 1.

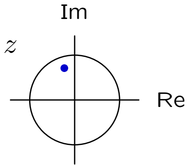


Which if any of the following figures shows the value of $1/(1-z)$?

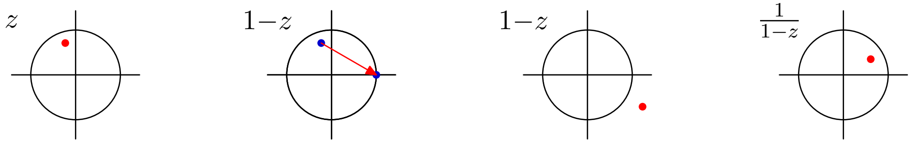


Check Yourself

Let z represent the complex number shown by a filled dot in the complex plane below, where the circle has radius 1.



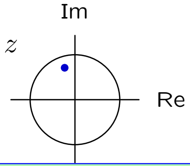
Which if any of the following figures shows the value of $1/(1-z)$?



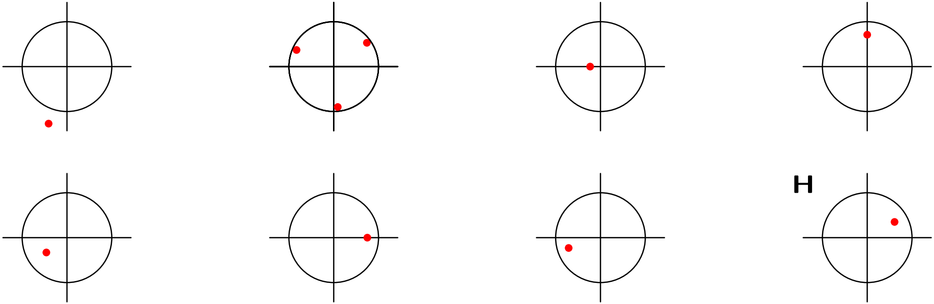
The reciprocal of $1-z$ has a magnitude of $\frac{1}{|1-z|}$ and an angle that is the negative of that of $1-z$, as shown in the right panel above.

Check Yourself

Let z represent the complex number shown by a filled dot in the complex plane below, where the circle has radius 1.



Which if any of the following figures shows the value of $1/(1-z)$? **H**



First find $1 - z$ then take the reciprocal.

Using Complex Numbers to Simplify Fourier Series

We have reviewed complex numbers and complex exponentials.

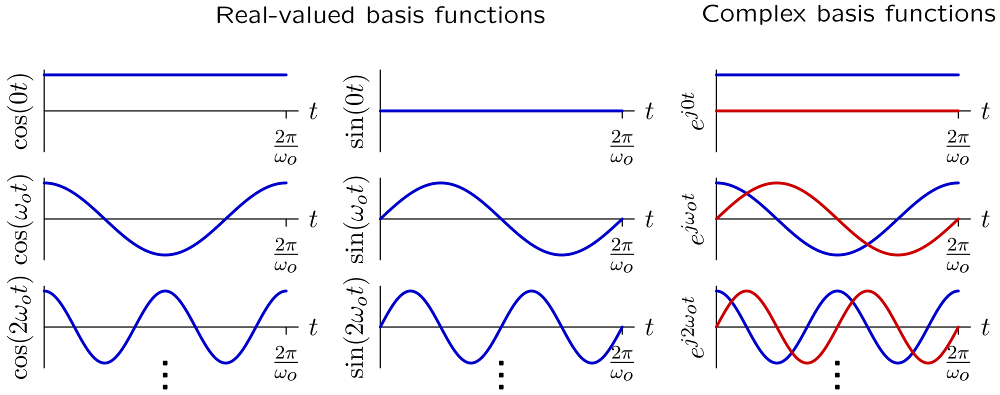
- complex numbers $\surd$
- complex exponentials and their relation to sinusoids $\surd$
- complex exponential form of Fourier series
- delay property of Fourier series

Next: develop a complex exponential form for Fourier series.

Simplifying Math By Using Complex Numbers

Euler's formula allows us to represent both sine and cosine basis functions with a single complex exponential:

$$f(t) = \sum \Big(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big) = \sum a_k e^{jk\omega_o t}$$



This halves the number of coefficients, but each is now complex-valued. More importantly, it replaces the trig functions with an exponential.

Fourier Series Directly From Complex Exponential Form

Assume that $f(t)$ is periodic in T and is composed of a weighted sum of harmonically related complex exponentials.

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_O kt}$$

We can "sift" out the component at $l\omega_o$ by multiplying both sides by $e^{-jl\omega_O t}$ and integrating over a period.

$$\int_T f(t) e^{-j\omega_O lt} dt = \int_T \sum_{k=-\infty}^{\infty} a_k e^{j\omega_O kt} e^{-j\omega_O lt} dt = \sum_{k=-\infty}^{\infty} a_k \int_T e^{j\omega_O (k-l)t} dt$$

$$= \begin{cases} T a_l & \text{if } l = k \\ 0 & \text{otherwise} \end{cases}$$

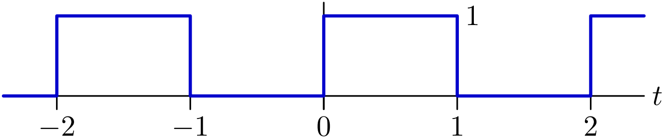
Solving for a_l provides an explicit formula for the coefficients:

$$a_k = \frac{1}{T} \int_T f(t) e^{-j\omega_O kt} dt\,; \quad \text{where } \omega_o = \frac{2\pi}{T}\,.$$

Fourier Analysis of a Square Wave

We previously used trig functions to find the Fourier series for $f(t)$ below:

$$f(t) = f(t+2)$$



$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

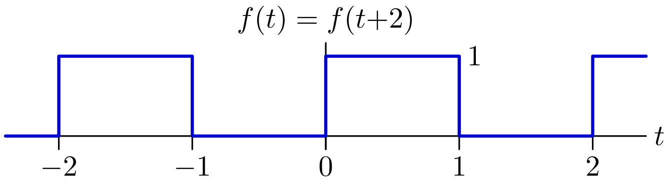
$$c_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega_o t)\,dt = \int_0^1 \cos(k\pi t)dt = \left.\frac{\sin(k\pi t)}{k\pi}\right|_0^1 = 0 \text{ for } k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega_o t)\,dt = \int_0^1 \sin(k\pi t)dt = -\left.\frac{\cos(k\pi t)}{k\pi}\right|_0^1 = \begin{cases} \frac{2}{k\pi} & k = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi}\sin(k\pi t)$$

Fourier Analysis of a Square Wave

Now try complex exponentials.



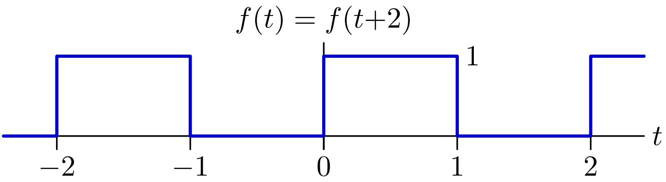
$$a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_o t}dt = \frac{1}{2}\int_0^1 e^{-jk\pi t}dt = \frac{1}{2}\left[\frac{e^{-jk\pi t}}{-jk\pi}\right]_0^1 = \begin{cases} \frac{1}{jk\pi} & \text{if } k \text{ is odd} \\ 0/0 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a_0 = \frac{1}{T}\int_T f(t)dt = \frac{1}{2}$$

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \frac{1}{2} + \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{1}{jk\pi} e^{jk\pi t}$$

Fourier Analysis of a Square Wave

Now try complex exponentials.



$$a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_0 t}dt = \frac{1}{2}\int_0^1 e^{-jk\pi t}dt = \frac{1}{2}\left[\frac{e^{-jk\pi t}}{-jk\pi}\right]_0^1 = \begin{cases} \frac{1}{jk\pi} & \text{if } k \text{ is odd} \\ 0/0 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a_0 = \frac{1}{T}\int_T f(t)dt = \frac{1}{2}$$

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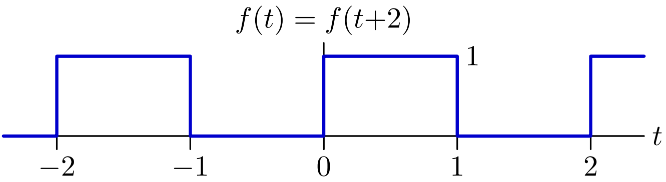
Trig functions have been replaced with exponential functions.

Fourier Analysis of a Square Wave

Now try complex exponentials.



$$a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_o t}dt = \frac{1}{2}\int_0^1 e^{-jk\pi t}dt = \frac{1}{2}\left[\frac{e^{-jk\pi t}}{-jk\pi}\right]_0^1 = \begin{cases} \frac{1}{jk\pi} & \text{if } k \text{ is odd} \\ 0/0 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a_0 = \frac{1}{T}\int_T f(t)dt = \frac{1}{2}$$

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \frac{1}{2} + \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{1}{jk\pi}e^{jk\pi t} = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi}\sin(k\pi t)$$

Same answer we obtained with trig functions.

Negative k

The complex exponential form of the series has positive and negative k's.

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

Only positive values of k are used in the trig form.

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t)$$

The negative k's are required by Euler's formula.

$$e^{jk\omega_o t} = \cos(k\omega_o t) + j\sin(k\omega_o t)$$

$$\cos(k\omega_o t) = \mathrm{Re}\{e^{jk\omega_o t}\} = \frac{1}{2}\left(e^{jk\omega_o t} + e^{-jk\omega_o t}\right)$$

$$\sin(k\omega_o t) = \mathrm{Im}\{e^{jk\omega_o t}\} = \frac{1}{2j}\left(e^{jk\omega_o t} - e^{-jk\omega_o t}\right)$$

The negative k do not indicate negative frequencies. They are the mathematical result of representing sinusoids with complex exponentials.

Fourier Series

Comparison of trigonometric and complex exponential forms.

Complex Exponential Form

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

$$a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_o t}dt$$

Trigonometric Form

$$f(t) = f(t+T) = c_o + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t)$$

$$c_0 = \frac{1}{T}\int_T f(t)\,dt$$

$$c_k = \frac{2}{T}\int_T f(t)\cos(k\omega_o t)\,dt; \qquad k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T}\int_T f(t)\sin(k\omega_o t)\,dt; \qquad k = 1, 2, 3, \ldots$$

Is the Complex Exponential Form Actually Easier?

Last time, we determined the effect of a half-period shift on the Fourier coefficients of the trig form. The result was a bit complicated.

Assume that $f(t)$ is periodic in time with period T:

$$f(t) = f(t+T)\,.$$

Let $g(t)$ represent a version of $f(t)$ shifted by half a period:

$$g(t) = f(t-T/2)\,.$$

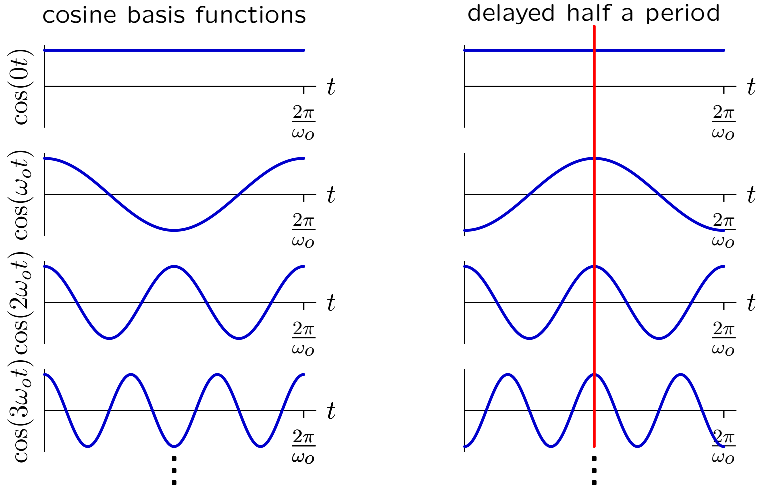
How many of the following statements correctly describe the effect of this shift on the Fourier series coefficients?

- cosine coefficients c_k are negated ✗
- sine coefficients d_k are negated ✗
- odd-numbered coefficients $c_1, d_1, c_3, d_3, \ldots$ are negated $\surd$
- sine and cosine coefficients are swapped: $c_k \to d_k$ and $d_k \to c_k$ ✗

Half-Period Shift

Shifting $f(t)$ shifts the underlying basis functions of it Fourier expansion.

$$f(t-T/2)=\sum_{k=0}^{\infty} c_k \cos\left(k\omega_o(t-T/2)\right)+\sum_{k=1}^{\infty} d_k \sin\left(k\omega_o(t-T/2)\right)$$

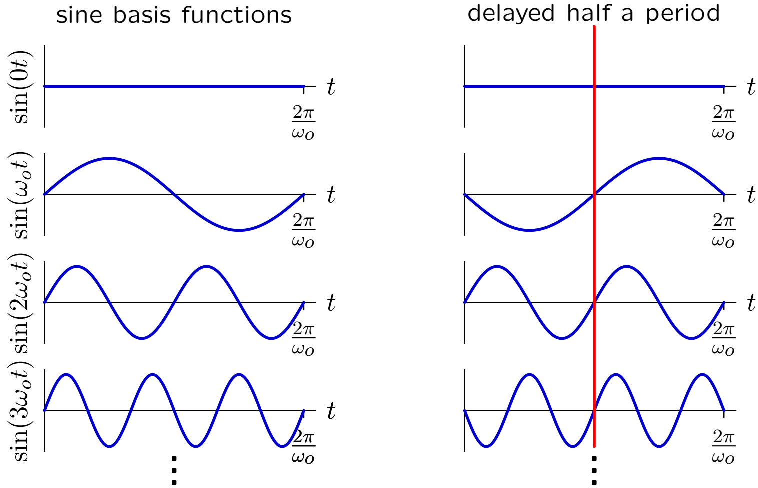


Half-period shift inverts c_k terms if k is odd. It has no effect if k is even.

Half-Period Shift

Shifting $f(t)$ shifts the underlying basis functions of it Fourier expansion.

$$f(t-T/2)=\sum_{k=0}^{\infty} c_k\cos\left(k\omega_o(t-T/2)\right)+\sum_{k=1}^{\infty} d_k\sin\left(k\omega_o(t-T/2)\right)$$

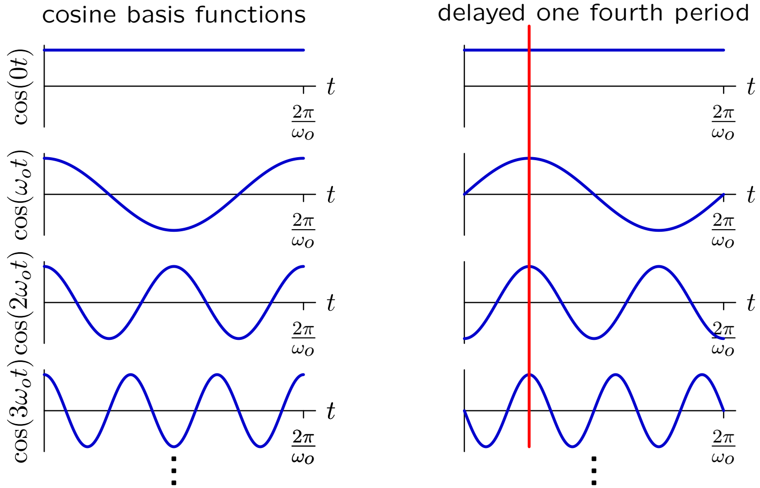


Half-period shift inverts d_k terms if and only if k is odd.

Quarter-Period Shift

Shifting by $T/4$ is **even more complicated**.

$$f(t - T/4) = \sum_{k=0}^{\infty} c_k \cos\left(k\omega_o(t - T/4)\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\omega_o(t - T/4)\right)$$

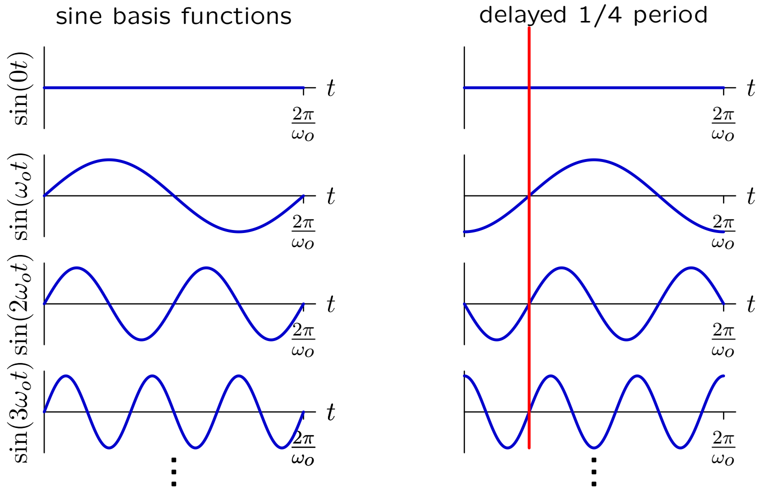


$\cos(\omega_o t) \rightarrow \sin(\omega_o t)$; $\cos(2\omega_o t) \rightarrow -\cos(2\omega_o t)$; $\cos(3\omega_o t) \rightarrow -\sin(3\omega_o t)$

Quarter-Period Shift

Shifting by $T/4$ is **even more complicated**.

$$f(t-T/4)=\sum_{k=0}^{\infty} c_k \cos\left(k\omega_o(t-T/4)\right)+\sum_{k=1}^{\infty} d_k \sin\left(k\omega_o(t-T/4)\right)$$



$$\sin(\omega_o t)\to-\cos(\omega_o t);\quad \sin(2\omega_o t)\to-\sin(2\omega_o t);\quad \sin(3\omega_o t)\to\cos(3\omega_o t)$$

Comparison of Half- and Quarter-Period Shifts

Let c_k and d_k represent the Fourier series coefficients for $f(t)$

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t)$$

and c'_k and d'_k represent those for a **half-period delay**.

$$g(t) = f(t - T/2) = c_0 + \sum_{k=1}^{\infty} c'_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d'_k \sin(k\omega_o t)$$

Then $c'_k = (-1)^k c_k$ and $d'_k = (-1)^k d_k$.

Comparison of Half- and Quarter-Period Shifts

Let c_k and d_k represent the Fourier series coefficients for $f(t)$

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t)$$

and c'_k and d'_k represent those for a **half-period delay**.

$$g(t) = f(t - T/2) = c_0 + \sum_{k=1}^{\infty} c'_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d'_k \sin(k\omega_o t)$$

Then $c'_k = (-1)^k c_k$ and $d'_k = (-1)^k d_k$.

Let c''_k and d''_k represent those for a **quarter-period delay**.

$$g(t) = f(t - T/4) = c_0 + \sum_{k=1}^{\infty} c''_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d''_k \sin(k\omega_o t)$$

Then

$$c''_k = \begin{cases} c_k & \text{if } k = 0, 4, 8, 12, \ldots \\ d_k & \text{if } k = 1, 5, 9, 13, \ldots \\ -c_k & \text{if } k = 2, 6, 10, 14, \ldots \\ -d_k & \text{if } k = 3, 7, 11, 15, \ldots \end{cases} \qquad d''_k = \begin{cases} d_k & \text{if } k = 0, 4, 8, 12, \ldots \\ -c_k & \text{if } k = 1, 5, 9, 13, \ldots \\ -d_k & \text{if } k = 2, 6, 10, 14, \ldots \\ c_k & \text{if } k = 3, 7, 11, 15, \ldots \end{cases}$$

Other Shifts Yield Even More Complicated Results

Let c_k and d_k represent the Fourier series coefficients for $f(t)$

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t)$$

and c_k''' and d_k''' represent those for an **eighth-period delay**.

$$g(t) = f(t-T/8) = c_0 + \sum_{k=1}^{\infty} c_k''' \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k''' \sin(k\omega_o t)$$

$$c_k''' = \begin{cases} c_k & \text{if } k = 0, 8, 16, 24, \ldots \\ \frac{\sqrt{2}}{2}(c_k + d_k) & \text{if } k = 1, 9, 17, 25, \ldots \\ d_k & \text{if } k = 2, 10, 18, 26, \ldots \\ \frac{\sqrt{2}}{2}(-c_k + d_k) & \text{if } k = 3, 11, 19, 27, \ldots \\ -c_k & \text{if } k = 4, 12, 20, 28, \ldots \\ \frac{\sqrt{2}}{2}(-c_k - d_k) & \text{if } k = 5, 13, 21, 29, \ldots \\ -d_k & \text{if } k = 6, 14, 22, 30, \ldots \\ \frac{\sqrt{2}}{2}(c_k - d_k) & \text{if } k = 7, 15, 23, 31, \ldots \end{cases} \qquad d_k''' = \ldots$$

Effects of Time Shifts on Complex Exponential Series

Delaying time by τ multiplies the complex exponential coefficients of a Fourier series by a constant $e^{-jk\omega_o\tau}$.

Let a_k represent the complex exponential series coefficients of $f(t)$ and a'_k represent the complex exponential series coefficents of $g(t) = f(t-\tau)$.

$$\begin{aligned}
a'_k &= \frac{1}{T}\int_T g(t)e^{-jk\omega_o t}dt \\
&= \frac{1}{T}\int_T f(t-\tau)e^{-jk\omega_o t}dt \\
&= \frac{1}{T}\int_T f(s)e^{-jk\omega_o (s+\tau)}ds \\
&= e^{-jk\omega_o\tau}\frac{1}{T}\int_T f(s)e^{-jk\omega_o s}ds \\
&= e^{-jk\omega_o\tau}a_k
\end{aligned}$$

Each coefficient a'_k in the series for $g(t)$ is a constant $e^{-jk\omega_o\tau}$ times the corresponding coefficient a_k in the series for $f(t)$.

Summary

We introduced the complex exponential form of Fourier series.

- complex numbers
- complex exponentials and their relation to sinusoids
- analysis and synthesis with complex exponentials
- delay property: much simpler with complex exponentials

Now you know ...

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \qquad \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \quad \text{where} \quad a_k = \frac{1}{T}\int_T f(t)e^{-jk\frac{2\pi}{T}t}dt$$

$$g(t) = f(t-\tau) = \sum_{k=-\infty}^{\infty} a'_k e^{jk\frac{2\pi}{T}t} \quad \text{where} \quad a'_k = e^{-jk\frac{2\pi}{T}\tau}a_k$$

Recitation: Reconvene in 5 minutes for recitation with Jing.

Tutorials: Head to your tutorial classroom.

Trigonometric Identies

```
sin(a+b) = sin(a) cos(b) + cos(a) sin(b)
sin(a-b) = sin(a) cos(b) - cos(a) sin(b)
cos(a+b) = cos(a) cos(b) - sin(a) sin(b)
cos(a-b) = cos(a) cos(b) + sin(a) sin(b)
tan(a+b) = (tan(a) + tan(b)) / (1 - tan(a) tan(b))
tan(a-b) = (tan(a) - tan(b)) / (1 + tan(a) tan(b))

sin(A) + sin(B) =  2 sin((A+B)/2) cos((A-B)/2)
sin(A) - sin(B) =  2 cos((A+B)/2) sin((A-B)/2)
cos(A) + cos(B) =  2 cos((A+B)/2) cos((A-B)/2)
cos(A) - cos(B) = -2 sin((A+B)/2) sin((A-B)/2)

sin(a+b) + sin(a-b) =  2 sin(a) cos(b)
sin(a+b) - sin(a-b) =  2 cos(a) sin(b)
cos(a+b) + cos(a-b) =  2 cos(a) cos(b)
cos(a+b) - cos(a-b) = -2 sin(a) sin(b)

2 cos(A) cos(B) = cos(A-B) + cos(A+B)
2 sin(A) sin(B) = cos(A-B) - cos(A+B)
2 sin(A) cos(B) = sin(A+B) + sin(A-B)
2 cos(A) sin(B) = sin(A+B) - sin(A-B)
```