6.3000: Signal Processing

Discrete Fourier Transform 2

- Frequency Resolution
- Circular Convolution

March 20, 2025

Last Time

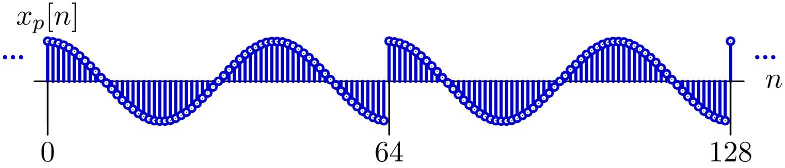
Define the Discrete Fourier Transform (DFT).

Compare the DFT to other Fourier representations.

	analysis	**synthesis**
DTFS:	$X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\frac{2\pi k}{N}n}$	$x[n] = \sum_{k=\langle N \rangle} X[k] e^{j\frac{2\pi k}{N}n}$
DTFT:	$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$	$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$
DFT:	$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$	$x[n] = \sum_{n=\langle N \rangle} X[k] e^{j\frac{2\pi k}{N}n}$

Relation Between DFT and DTFS

The 64-point DFT of $x_2[n] = \cos\frac{3\pi n}{64}$



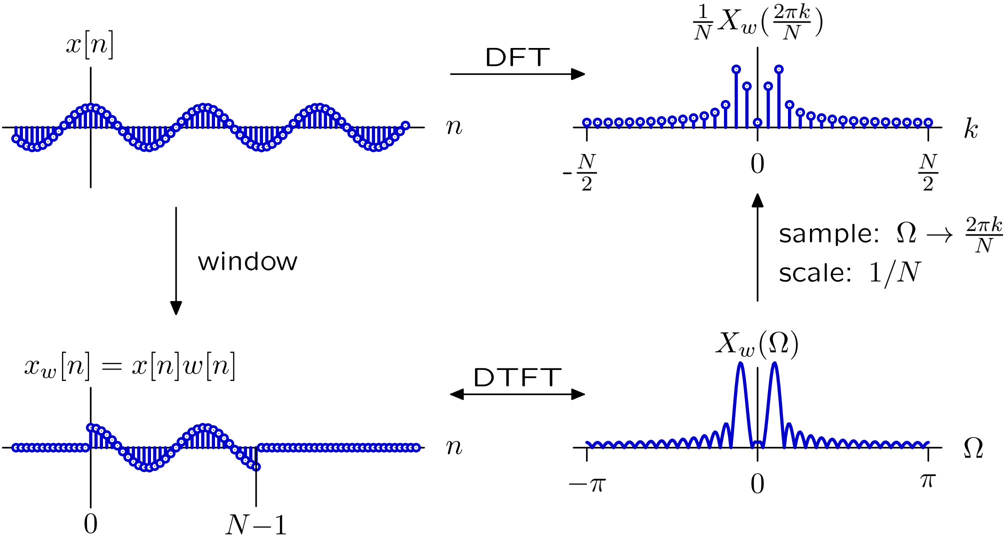
is equal to the 64-point DTFS of the **periodic extension** of $x_2[n]$.

$$x_p[n] = x_2[n \bmod 64]$$

From this perspective, the large number of non-zero frequency components in the DFT of x_2 are needed to generate the step discontinuity at $n = 64$.

Relation Between DFT and DTFT

Graphical depiction of relation between DFT and DTFT.

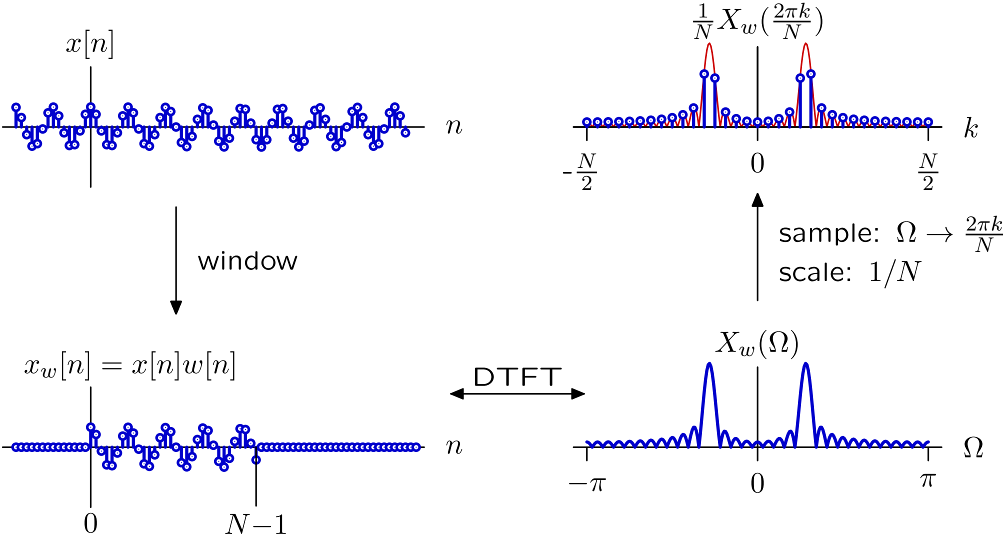


While sampling and scaling are important, it is the **windowing** that most affects frequency content.

Relation Between DFT and DTFT

Decreasing the analysis window N decreases frequency resolution.

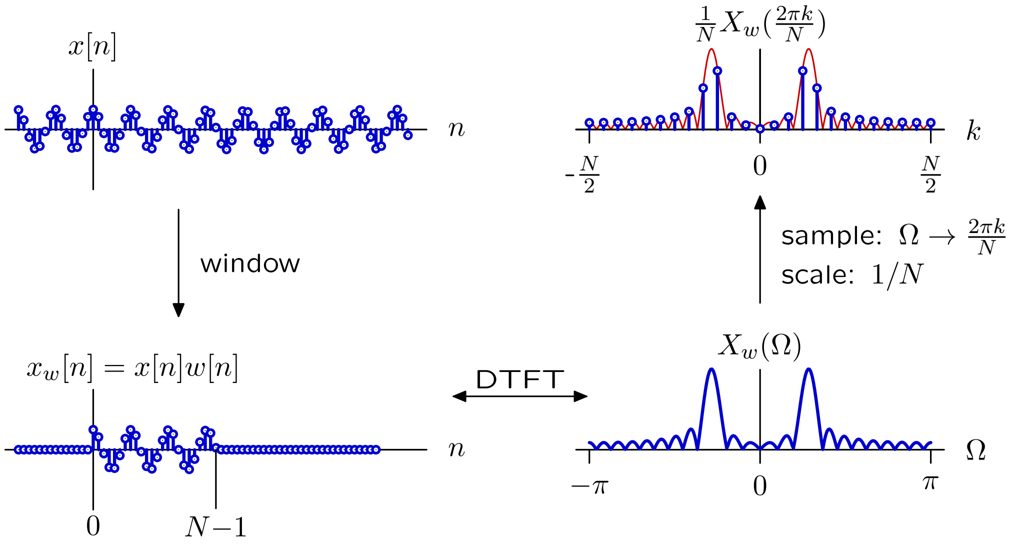
$N = 32$



Relation Between DFT and DTFT

Decreasing the analysis window N decreases frequency resolution.

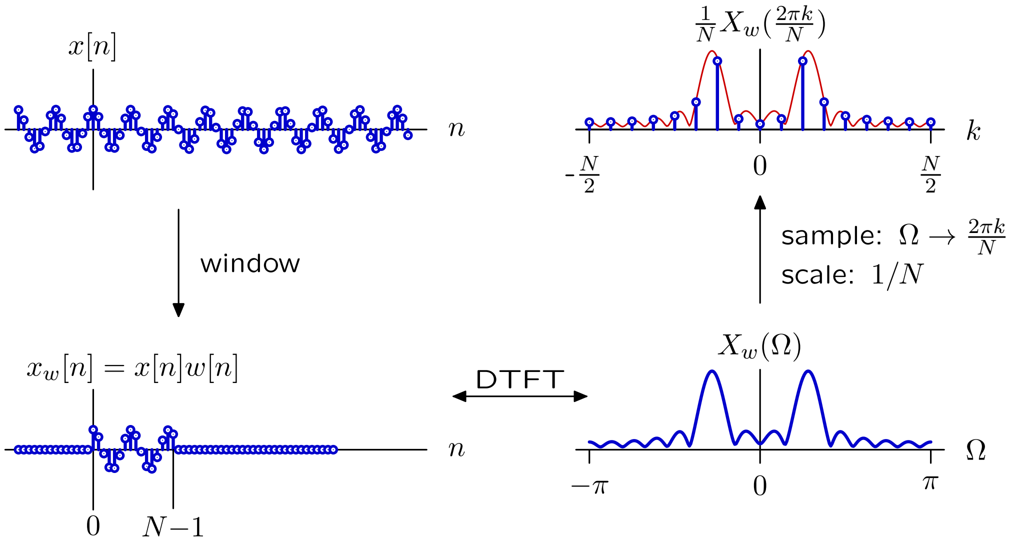
$N = 24$



Relation Between DFT and DTFT

Decreasing the analysis window N decreases frequency resolution.

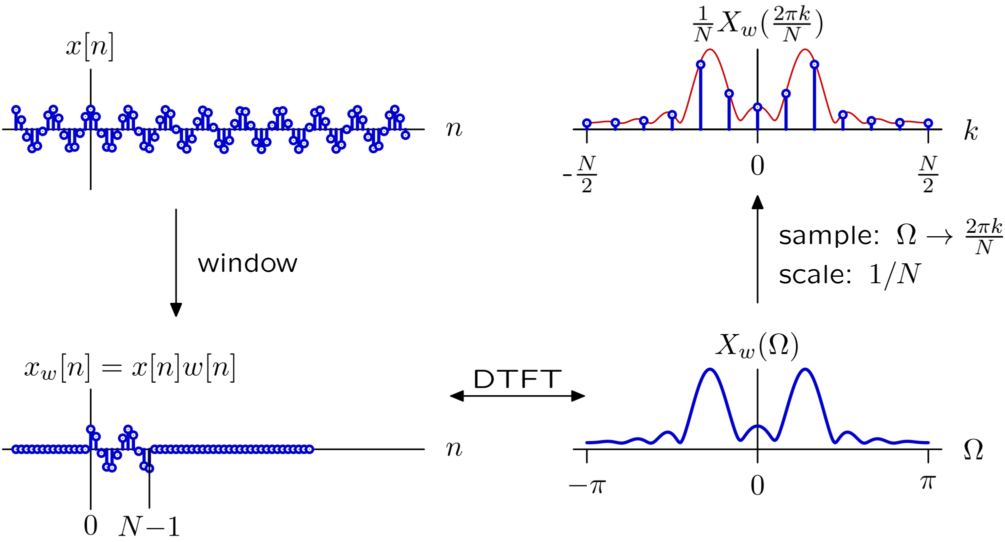
$N = 16$



Relation Between DFT and DTFT

Decreasing the analysis window N decreases frequency resolution.

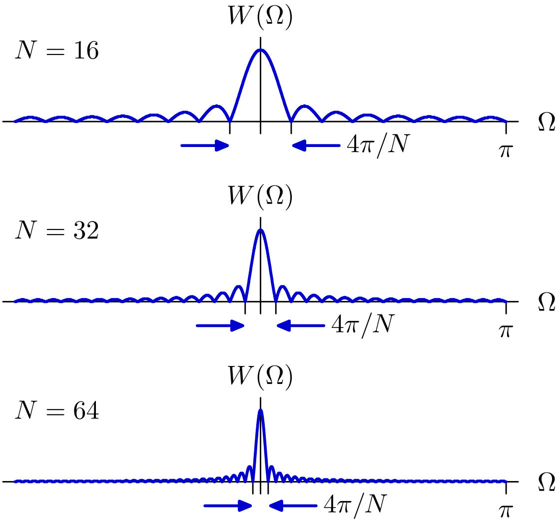
$N = 12$



Frequency Resolution

Frequency blurring is fundamental to the way the DFT works.

Longer windows provide finer frequency resolution.

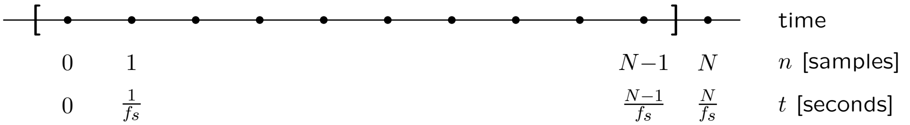


The width of the central lobe is inversely related to window length.

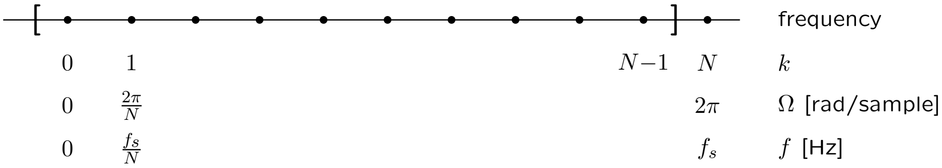
Length of Analysis Window N

The DFT provides a new parameter (N) to customize performance.

The time window is divided into N samples numbered $n = 0$ to $N-1$.



Discrete frequencies are similarly numbered as $k = 0$ to $N-1$.



N determines both the length of the window in time and the frequency resolution of the result.

Length of Analysis Window N

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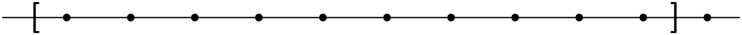
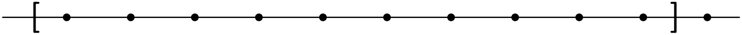


0	1	$N-1$	N	n [samples]
0	$\frac{1}{fs}$	$\frac{N-1}{fs}$	$\frac{N}{fs}$	t [seconds]

Discrete frequencies are similarly numbered as $k = 0$ to $N-1$.



0	1	$N-1$	N	k
0	$\frac{2\pi}{N}$		2π	Ω [rad/sample]
0	$\frac{fs}{N}$		f_s	f [Hz]

Which is better: big or small values of N?

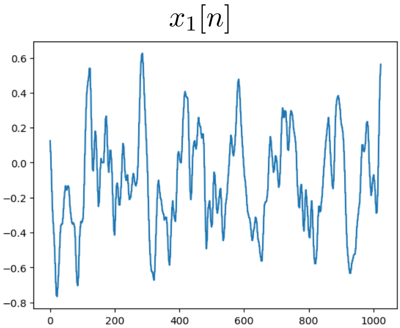
Frequency Resolution

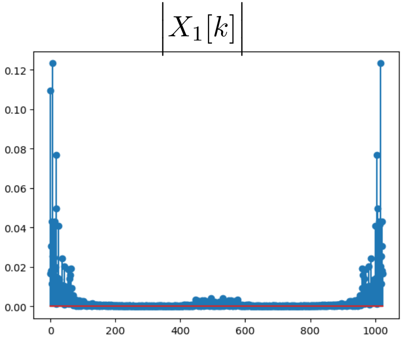
Example: Determine the frequency content of the following sound.

cello: DEb3.wav ($f_s = 44,100\,$Hz)

Frequency Resolution

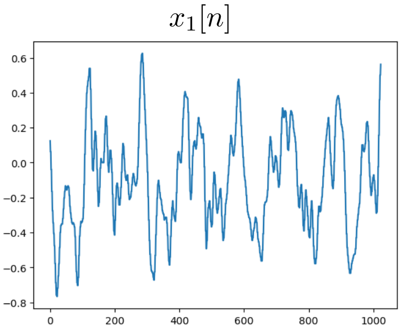
Extract 1024 samples and calculate DFT.

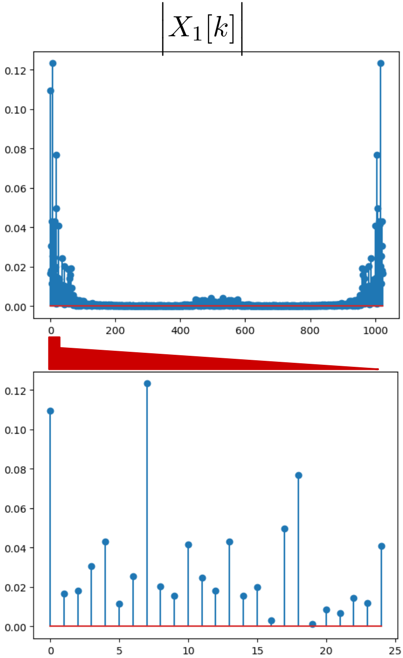




Frequency Resolution

Information about pitch is at low frequencies. Zoom in on $k = 0$ to 24.

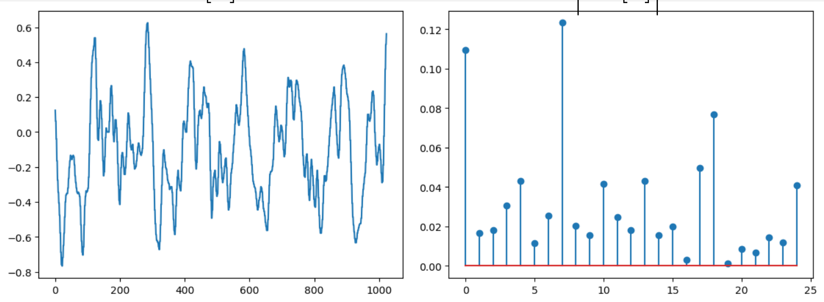




Check Yourself

The magnitude of the DFT is largest at $k = 7$ ($f_s = 44100\,\text{Hz}$).

$x_1[n]$ $\left|X_1[k]\right|$

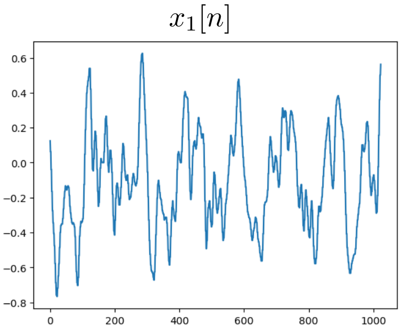


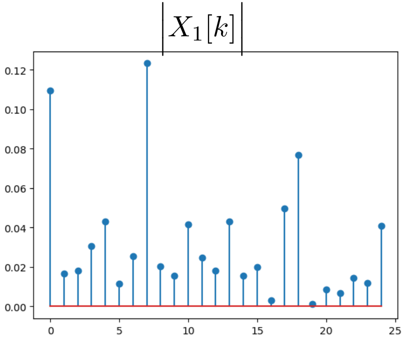
What is the corresponding frequency in Hz?

1. 293.66 Hz
2. 301.46 Hz
3. 146.83 Hz
4. 150.73 Hz
5. None of the above

Check Yourself

The magnitude of the DFT is largest at $k = 7$.





What is the corresponding frequency in Hz?

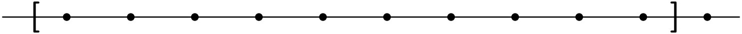
Use proportional reasoning:

$$\frac{f_o}{f_s} = \frac{k_o}{N}$$

Length of Analysis Window N

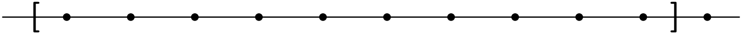
The DFT provides a new parameter (N) to customize performance.

The time window is divided into N samples numbered $n = 0$ to $N-1$.



0	1	$N-1$	N	n [samples]
0	$\frac{1}{fs}$	$\frac{N-1}{fs}$	$\frac{N}{fs}$	t [seconds]

Discrete frequencies are similarly numbered as $k = 0$ to $N-1$.

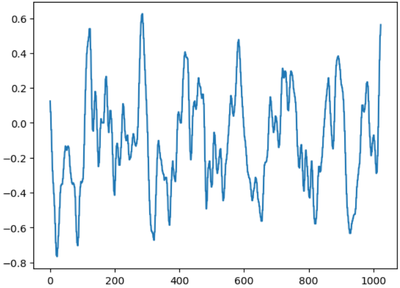


0	1	$N-1$	N	k
0	$\frac{2\pi}{N}$		2π	Ω [rad/sample]
0	$\frac{f_s}{N}$		f_s	f [Hz]

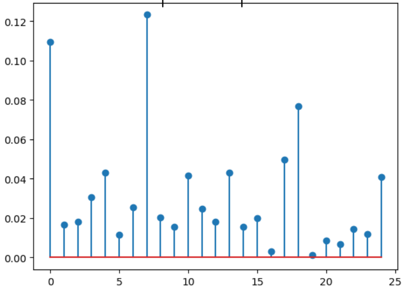
Check Yourself

The magnitude of the DFT is largest at $k = 7$.

$x_1[n]$



$\left|X_1[k]\right|$



What is the corresponding frequency in Hz?

Use proportional reasoning:

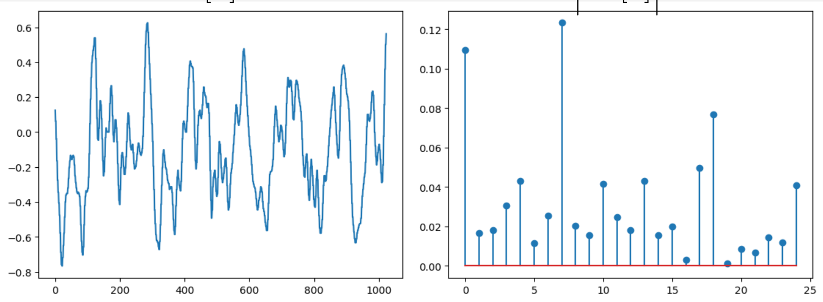
$$\frac{f_o}{f_s} = \frac{k_o}{N} \rightarrow f_o = \frac{k_o}{N} f_s = \frac{7}{1024} \times 44100 \approx 301.46\,\text{Hz}$$

This frequency is between D4 ($293.66\,\text{Hz}$) and E-flat-4 ($311.13\,\text{Hz}$).

Check Yourself

The magnitude of the DFT is largest at $k = 7$ ($f_s = 44100\,\text{Hz}$).

$x_1[n]$ $\quad$ $\left|X_1[k]\right|$

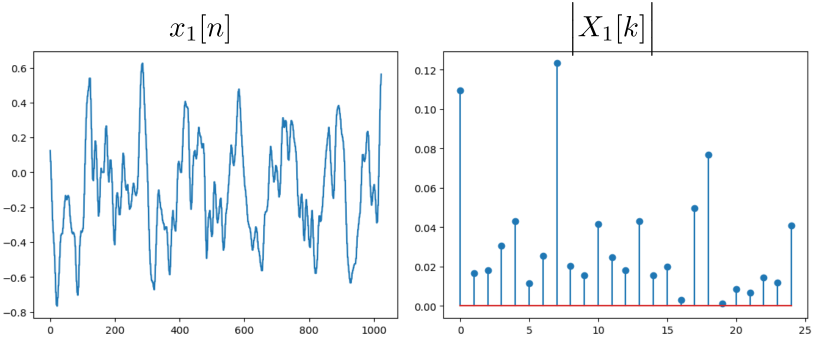


What is the corresponding frequency in Hz? 2

1. 293.66 Hz
2. 301.46 Hz
3. 146.83 Hz
4. 150.73 Hz
5. None of the above

Frequency Resolution

Information about pitch is at low frequencies. Zoom in on $k = 0$ to 24.



The DFT provides integer resolution in k. Therefore, the peak at $k = 7$ could be off by as much as $\pm\frac{1}{2}$.

$$\Delta f = \frac{\Delta k}{N} f_s = \frac{1/2}{1024} \times 44100 \approx 21.5\,\text{Hz}$$

Thus the frequency of the biggest peak is $280 < f_o < 323$, easily including both D ($293.66\,$Hz) and E-flat ($311.13\,$Hz).

Improving Frequency Resolution

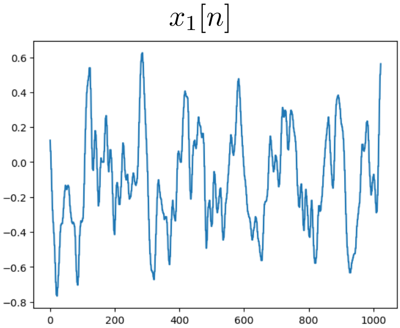
We can increase N to increase the number of analyzed frequencies.

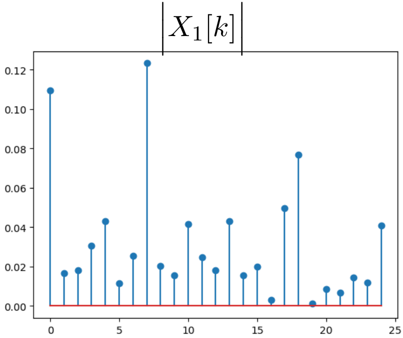
Two methods to increase N:

- zero-padding (add zeros to increase length of input)
- increase sample size

Zero Padding

Original (N=1024).

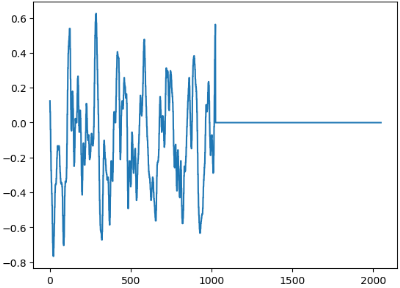




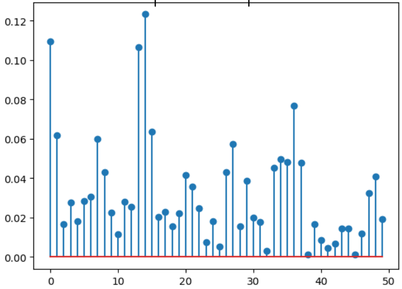
Zero Padding

What happens if we increase the length of the signal by adding zeros?

$x_2[n]$



$\left|2X_2[k]\right|$



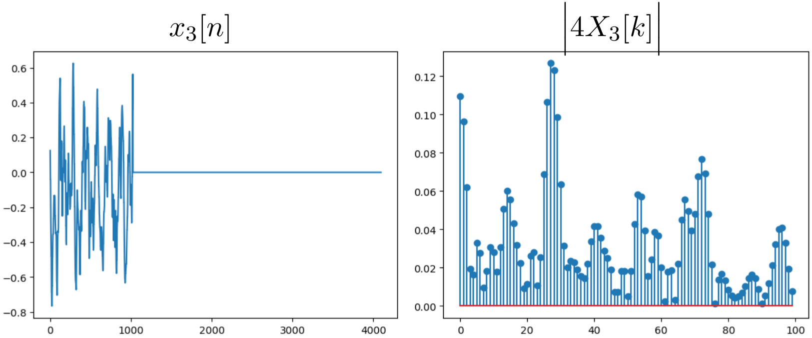
$$X_2[k]=\frac{1}{2N}\sum_{n=0}^{2N-1}x_2[n]e^{-j\frac{2\pi k}{2N}n}=\frac{1}{2N}\sum_{n=0}^{N-1}x_1[n]e^{-j\frac{2\pi k}{2N}n}=\frac{1}{2N}\sum_{n=0}^{N-1}x_1[n]e^{-j\frac{2\pi}{N}\left(\frac{k}{2}\right)n}$$

$$=\begin{cases}\frac{1}{2}X_1[k/2] & \text{if } k \text{ is even}\\ \text{new information} & \text{if } k \text{ is odd}\end{cases}$$

Lengthening $x_1[n]$ with zeros stretches the DFT by inserting new coefficients of X_2 between adjacent coefficients of X_1.

Zero Padding

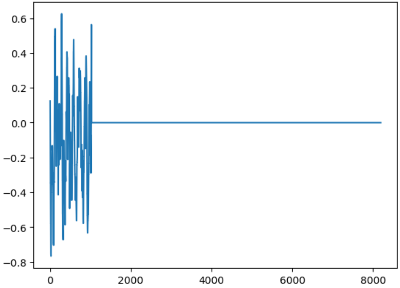
Lengthen by a factor of 4 (N=4096).



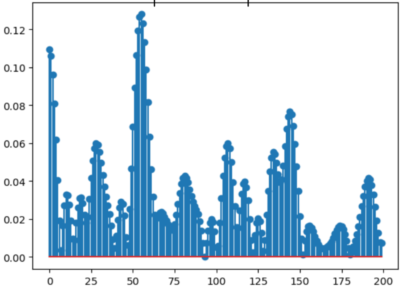
Zero Padding

Lengthen by a factor of 8 (N=8192).

$x_4[n]$



$\left|8X_4[k]\right|$

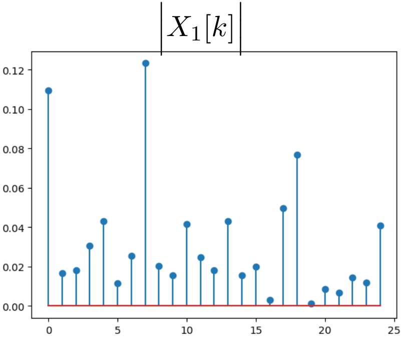


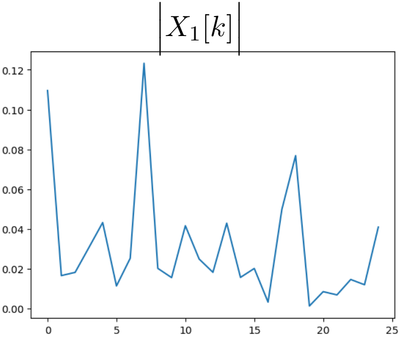
Zero Padding

The stem plots can be distracting when they are close together. (They also take a long time to compute!) Replot using lines (but remember that the signals are DT).

Zero Padding

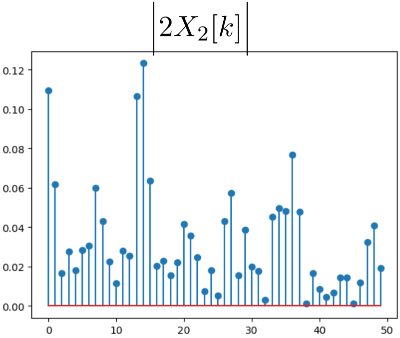
Original (N=1024).

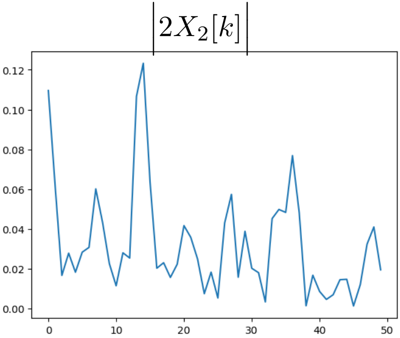




Zero Padding

Lengthen by a factor of 2 (N=2048).





Zero Padding

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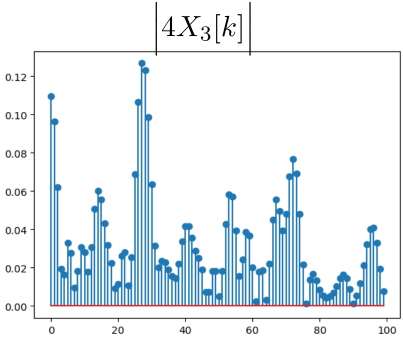
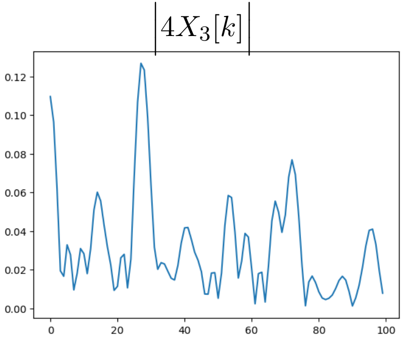
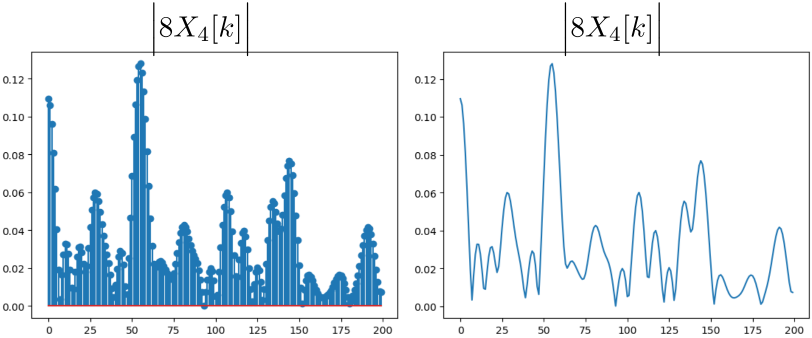




Zero Padding

Lengthen by a factor of 8 (N=8192).



Peak is now at $k = 55$.

$$f_o = \frac{k_o}{N} f_s = \frac{55}{8 \times 1024} 44100 \approx 296\,\text{Hz}$$

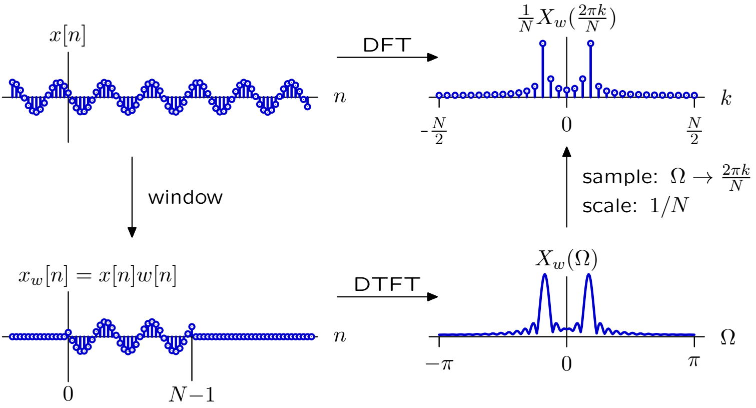
compared to our previous estimate of $301.46\,\text{Hz}$.

More importantly, frequencies are sampled more densely:

$$\Delta f = \frac{\Delta k}{N} f_s = \frac{1/2}{8 \times 1024} \times 44100 \approx 2.7\,\text{Hz}$$

But we still cannot tell if the note was D (293.66 Hz) or E-flat (311.13 Hz).

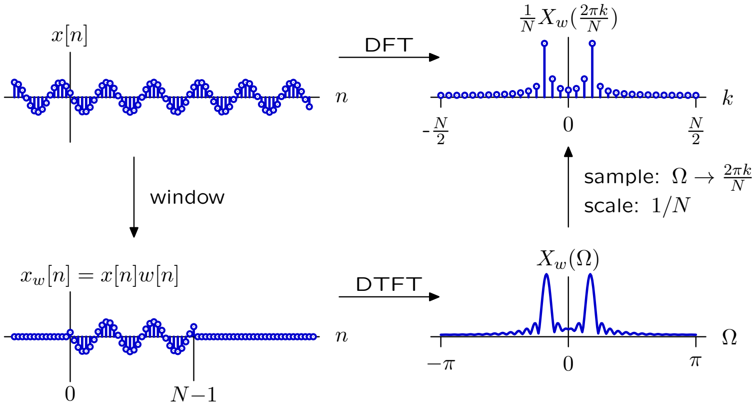
Check Yourself



Which of the following is/are true?

1. Zero-padding has no effect on the DTFT of $x_w[n]$.
2. Zero-padding decreases spectral smear in the DTFT.
3. Zero-padding has no effect on the sampled version $X_w(\Omega)$.
4. Zero-padding decreases the frequency interval (Hz) of DFT samples.

Check Yourself



Which of the following is/are true?

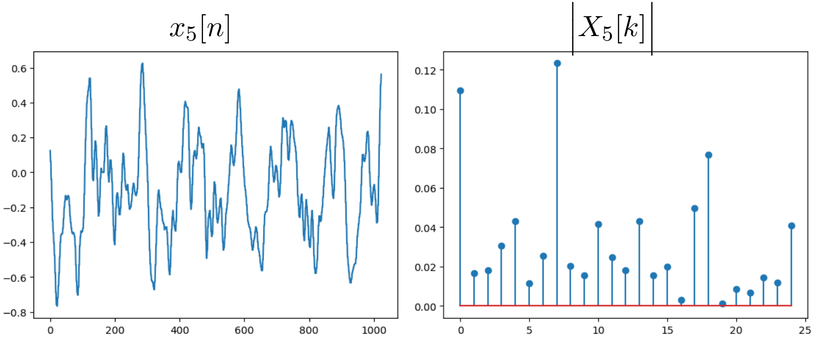
1. Zero-padding has no effect on the DTFT of $x_w[n]$. $\surd$
2. Zero-padding decreases spectral smear in the DTFT of $x_w[n]$. ✗
3. Zero-padding has no effect on the sampled version of $X_w(\Omega)$. ✗
4. Zero-padding decreases the frequency interval (Hz) of DFT samples. $\surd$

More Data

In order to increase **frequency resolution**, we need to include more data.

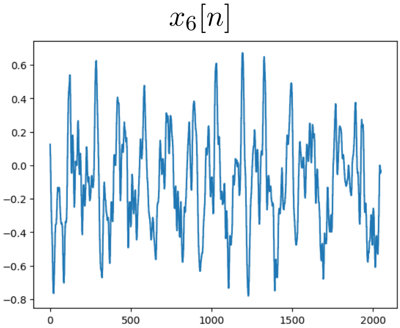
More Data

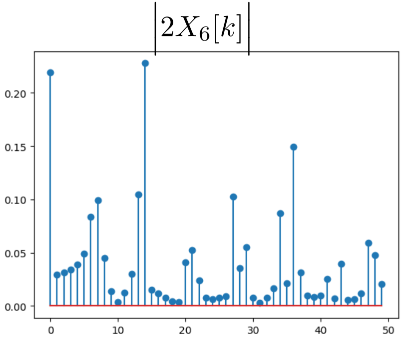
Original (N=1024).



More Data

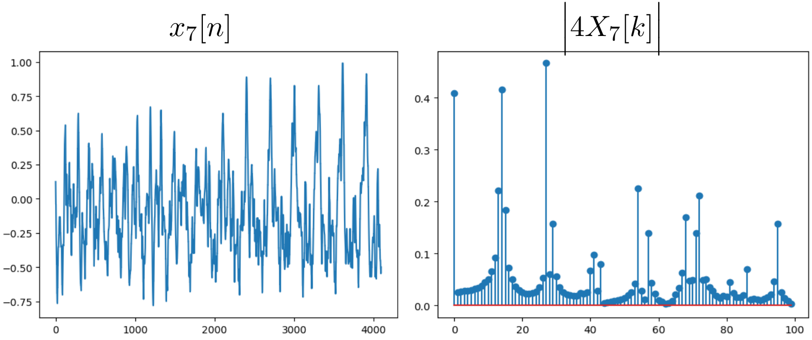
Lengthen by a factor of 2 (N=2048).





More Data

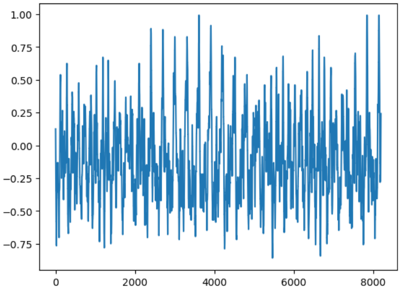
Lengthen by a factor of 4 (N=4096).



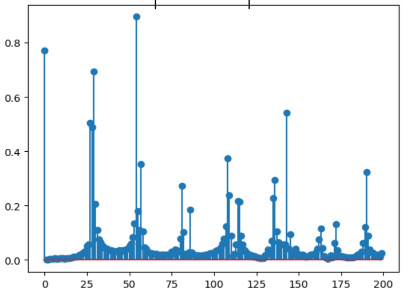
More Data

Lengthen by a factor of 8 (N=8192).

$x_8[n]$



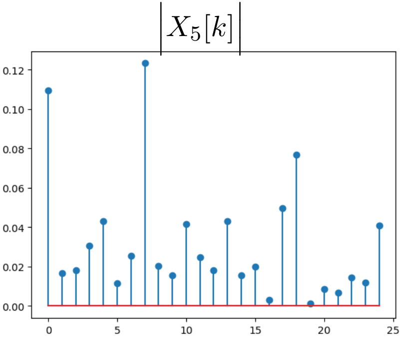
$\left|8X_8[k]\right|$

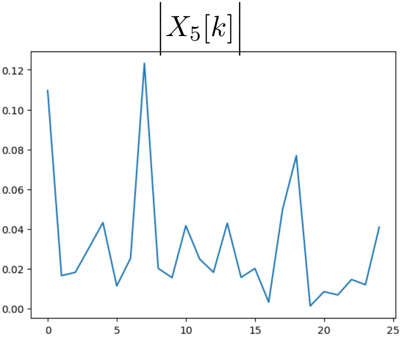


Switching again to line plots ...

More Data

Original (N=1024).





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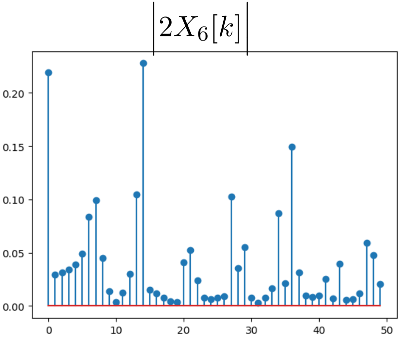

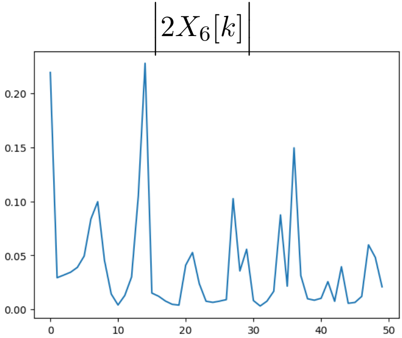

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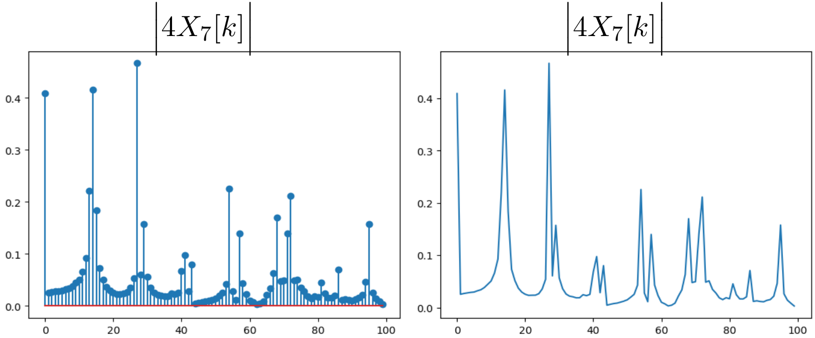
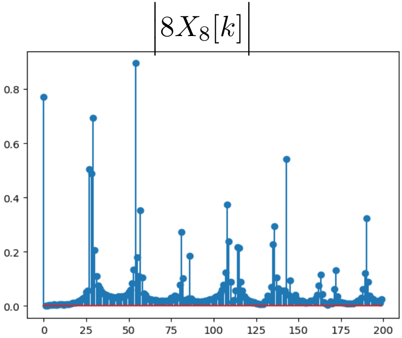
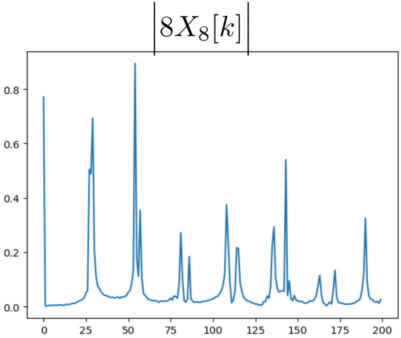


More Data

Lengthen by a factor of 8 (N=8192).

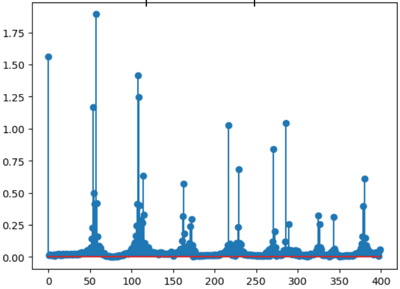




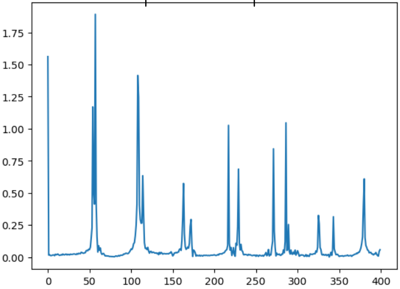
More Data

Lengthen by a factor of 16 (N=16,384).

$\left|16X_9[k]\right|$

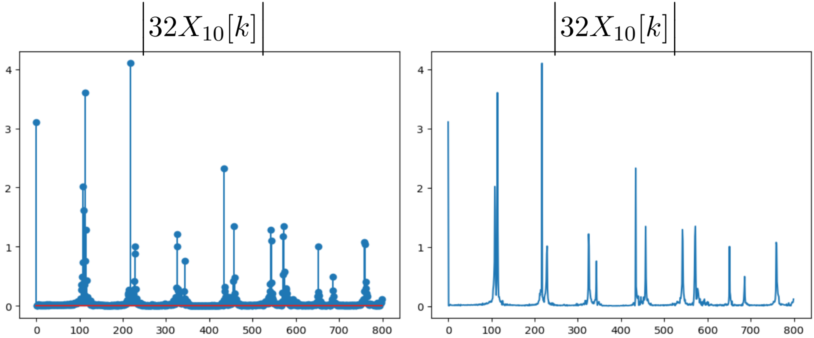


$\left|16X_9[k]\right|$



More Data

Lengthen by a factor of 32 (N=32,768).

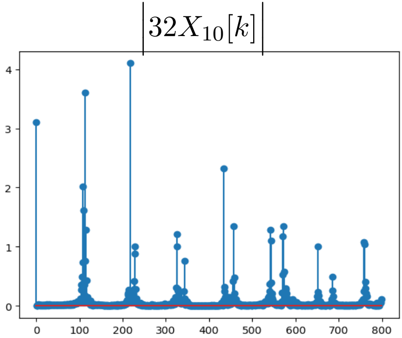


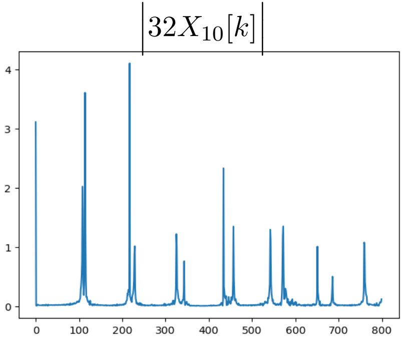
Clear peaks at $k = 217$ and $k = 228$ ($f = 292.04$ Hz and $f = 306.85$ Hz).
$\rightarrow$ close to D (293.66 Hz) and E-flat (311.13 Hz): both notes are present!

Anything else?

More Data

Lengthen by a factor of 32 (N=32,768).





Clear peaks at $k = 217$ and $k = 228$ ($f = 292.04$ Hz and $f = 306.85$ Hz).
$\rightarrow$ close to D (293.66 Hz) and E-flat (311.13 Hz): both notes are present!

Notice that these are the second harmonics of lower frequencies.
$\rightarrow$ an octave lower than was suggested by the analysis with $N = 1024$.

The fundamental components were not clearly resolved with $N = 1024$ but are clear with $N = 32{,}768$.

Summary: Frequency Resolution

Increasing the length of the analysis by zero padding increases the **number of frequency points** (because sampling is more dense) but does not increase frequency **resolution** (because windowing is unchanged).

To increase frequency resolution we must increase the number of data that are analyzed.

Implementing Convolution with DFT

In addition to being useful for characterizing the frequency content of a signal, the DFT can also be used to implement convolution.

Recall the convolution result for the DTFT.

If

$$f_a[n] \quad \overset{\text{DTFT}}{\Longrightarrow} \quad F_a(\Omega)$$

and

$$f_b[n] \quad \overset{\text{DTFT}}{\Longrightarrow} \quad F_b(\Omega)$$

then

$$(f_a * f_b)[n] \quad \overset{\text{DTFT}}{\Longrightarrow} \quad F_a(\Omega)F_b(\Omega)$$

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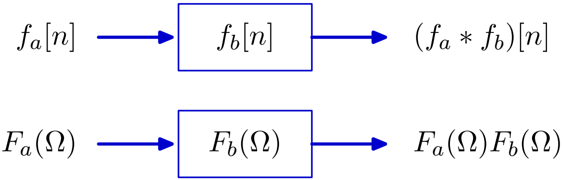
This property is the basis of the **filtering** view of a system:



Regular Convolution

Why does multiplication in frequency correspond to convolution in time?

Let $F(\Omega) = F_a(\Omega) \times F_b(\Omega)$. Find $f[n]$.

$$
\begin{aligned}
f[n] &= \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega n}\,d\Omega \\
&= \frac{1}{2\pi}\int_{2\pi} F_a(\Omega)F_b(\Omega)e^{j\Omega n}\,d\Omega \\
&= \frac{1}{2\pi}\int_{2\pi} F_a(\Omega)\Bigg(\sum_{m=-\infty}^{\infty} f_b[m]e^{-j\Omega m}\Bigg)e^{j\Omega n}\,d\Omega \\
&= \sum_{m=-\infty}^{\infty} f_b[m]\underbrace{\frac{1}{2\pi}\int_{2\pi} F_a(\Omega)e^{j\Omega(n-m)}\,d\Omega}_{f_a[n-m]} \\
&= \sum_{m=-\infty}^{\infty} f_b[m]f_a[n-m] \equiv (f_a * f_b)[n]
\end{aligned}
$$

Multiplying in frequency is equivalent to convolving in time.

Implementing Convolution with DFT

The argument for the DFT is similar to the one for the DTFT.

Let $F[k] = F_a[k] \times F_b[k]$. Find $f[n]$.

$$f[n] = \sum_{k=0}^{N-1} F[k] e^{j\frac{2\pi k}{N}n} = \sum_{k=0}^{N-1} F_a[k] F_b[k] e^{j\frac{2\pi k}{N}n}$$

$$= \sum_{k=0}^{N-1} F_a[k] \left(\frac{1}{N} \sum_{m=0}^{N-1} f_b[m] e^{-j\frac{2\pi k}{N}m} \right) e^{j\frac{2\pi k}{N}n}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] \underbrace{\left(\sum_{k=0}^{N-1} F_a[k] e^{j\frac{2\pi k}{N}(n-m)} \right)}_{f_a[n-m]?}$$

No! $f_a[n]$ is only defined for $0 \le n < N$, but $n-m$ can fall outside that range.

Implementing Convolution with DFT

The argument for the DFT is similar to the one for the DTFT.

Let $F[k] = F_a[k] \times F_b[k]$. Find $f[n]$.

$$f[n] = \sum_{k=0}^{N-1} F[k] e^{j\frac{2\pi k}{N}n} = \sum_{k=0}^{N-1} F_a[k] F_b[k] e^{j\frac{2\pi k}{N}n}$$

$$= \sum_{k=0}^{N-1} F_a[k] \left(\frac{1}{N} \sum_{m=0}^{N-1} f_b[m] e^{-j\frac{2\pi k}{N}m} \right) e^{j\frac{2\pi k}{N}n}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] \underbrace{\left(\sum_{k=0}^{N-1} F_a[k] e^{j\frac{2\pi k}{N}(n-m)} \right)}_{f_a[(n-m) \bmod N]}$$

Since k is integer, the complex exponential is periodic in $n-m$, period N. Therefore the parenthesized expression is $f_a[(n-m) \bmod N]$, and

$$f[n] = \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] f_a[(n-m) \bmod N] \equiv \frac{1}{N} (f_a \circledast f_b)[n]$$

Implementing Convolution with DFT

Multiplfying DFTs is equivalent to **circular** convolution in time.

If

$$f_a[n] \stackrel{\text{DFT}}{\Longrightarrow} F_a[k]$$

and

$$f_b[n] \stackrel{\text{DFT}}{\Longrightarrow} F_b[k]$$

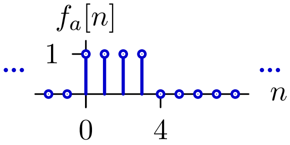
then

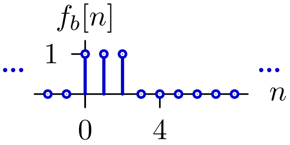
$$\frac{1}{N}(f_a \circledast f_b)[n] \stackrel{\text{DFT}}{\Longrightarrow} F_a[k]F_b[k]$$

where

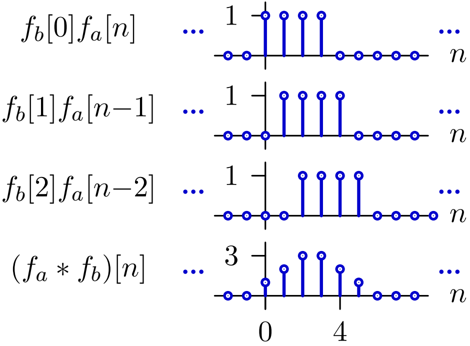
$$(f_a \circledast f_b)[n] = \sum_{m=0}^{N-1} f_b[m] f_a[(n-m) \bmod N]$$

Superposition View of Conventional Convolution

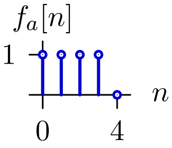


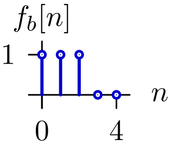


$$(f_a * f_b)[n] = \sum_{m=-\infty}^{\infty} f_b[m] f_a[n-m] = f_b[0] f_a[n] + f_b[1] f_a[n-1] + f_b[2] f_a[n-2]$$



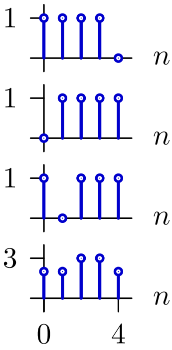
Superposition View of Circular Convolution (N=5)





$$\left(f_a \circledast f_b\right)[n] = \sum_{m=0}^{N-1} f_b[m] f_a[(n-m) \bmod N]$$

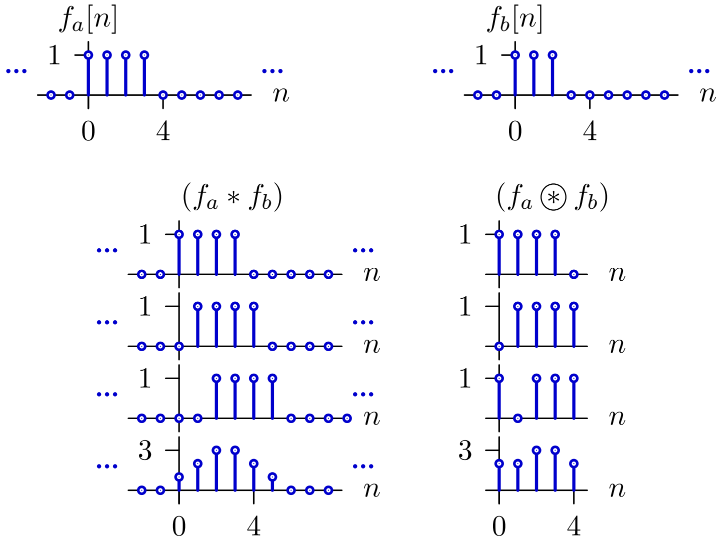
$$= f_b[0] f_a[n \bmod N] + f_b[1] f_a[(n-1) \bmod N] + f_b[2] f_a[(n-2) \bmod N]$$



$f_b[0] f_a[n \bmod N]$	$f_a[0]$	$f_a[1]$	$f_a[2]$	$f_a[3]$	$f_a[4]$
$f_b[1] f_a[(n-1) \bmod N]$	$f_a[4]$	$f_a[0]$	$f_a[1]$	$f_a[2]$	$f_a[3]$
$f_b[2] f_a[(n-2) \bmod N]$	$f_a[3]$	$f_a[4]$	$f_a[0]$	$f_a[1]$	$f_a[2]$
$(f_a \circledast f_b)[n]$					

Side By Side

The parts of the conventional convolution that would fall outside the DFT window "alias" to points inside the DFT window.



Summary

Today we discussed two critical issues in using the DFT.

- Frequency resolution – how the length of a signal determines the ability to discriminate frequencies using the DFT.
- Circular Convolution – how the DFT can be used to carry out time domain operations.

Question of the Day

Determine the value of the following expression:

$$\left(\delta[n-2] \circledast \left(u[n-1] - u[n-4]\right)\right)[2]$$

where the circular convolution has length $N=5$.