6.3000: Signal Processing

Discrete Fourier Transform 1

- Discrete Fourier Transform (DFT)
- Relations to Discrete-Time Fourier Series (DTFS)
- Relations to Discrete-Time Fourier Transform (DTFT)

March 18, 2025

Yet Another Fourier Representation

Why do we need another Fourier Representation?

Fourier series represent signals as sums of sinusoids. They provide insights that are not obvious from time representations, but Fourier series are only defined for periodic signals.

$$X[k] = \sum_{n=\langle N\rangle} x[n]e^{-j2\pi kn/N} \qquad \text{(summed over a period)}$$

Fourier transforms have no periodicity constaint:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \qquad \text{(summed over all samples } n\text{)}$$

but are functions of continuous domain (Ω).

$\rightarrow$ not convenient for numerical computations

Discrete Fourier Transform: discrete frequencies for aperiodic signals.

Discrete Fourier Transform

Definition and comparison to other Fourier representations.

	analysis	**synthesis**
DTFS:	$X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j \frac{2\pi k}{N} n}$	$x[n] = \sum_{k=\langle N \rangle} X[k] e^{j \frac{2\pi k}{N} n}$
DTFT:	$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$	$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$
DFT:	$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi k}{N} n}$	$x[n] = \sum_{k=\langle N \rangle} X[k] e^{j \frac{2\pi k}{N} n}$

Major differences:

DTFS: $x[n]$ is presumed to be periodic in N

DTFT: $x[n]$ is arbitrary

DFT: only a portion of an arbitrary $x[n]$ is considered

Relation Between DFT and DTFS

If we compute the DFT using a window length N that is equal to the period of a periodic signal, then the DFT and DTFS coefficients are equal.

Let $x_1[n] = \cos\frac{2\pi n}{64}$. Then if $N{=}64$, the DFT coefficients are

$$X_1[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} = \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k-63]$$

as plotted below.

The DFT coefficients are the same as the Fourier series coefficients.

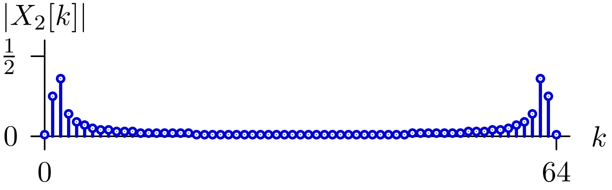
Relation Between DFT and DTFS

If a signal is not periodic in the DFT window length N, then there are no Fourier series coefficients to compare.

Let $x_2[n] = \cos\frac{3\pi n}{64}$. Then if N=64, the DFT coefficients are

$$X_2[k] = \frac{1}{N}\sum_{n=0}^{N-1} x_2[n]e^{-j\frac{2\pi}{N}kn}$$

are plotted below.

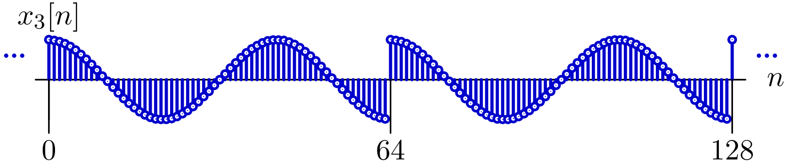


Even though $x_2[n]$ contains a single frequency $\Omega = 3\pi/64$, there are large coefficients at many different frequencies k.

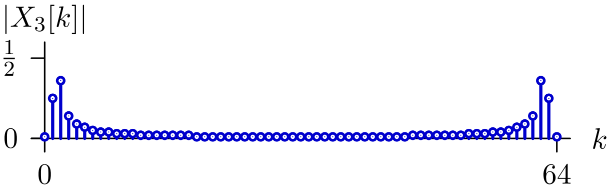
The reason is that $x_2[n]$ is not periodic in n with period $N = 64$.

Relation Between DFT and DTFS

Although $x_2[n] = \cos\frac{3\pi n}{64}$ is not periodic in $N{=}64$, we can define a signal $x_3[n]$ that is equal to $x_2[n]$ for $0 \le n < 64$ and that is periodic in $N{=}64$.



The DFT coefficients for this signal are the same as those for $x_2[n]$:



Furthermore, the DFT coefficients of $x_3[n]$ equal the DTFS coefficients of $x_3[n]$. The large number of non-zero coefficients are necessary to produce the step discontinuity at $n = 64$.

Two Ways to Think About the DFT

We just compared the DFT to the DTFS:

1. The DFT of a signal $x[n]$ is equal to the **DTFS** of a version of $x[n]$ that is periodically extended so that it is periodic in N.

$\rightarrow$ emphasizes the importance of **periodicity** in time.

2. We can also gain intuition for the DFT by comparing it to the DTFT.

Relation Between DFT and DTFT

The DFT can also be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $\frac{1}{N}$.

Let $x_w[n] = x[n] \times w[n]$ represent a **windowed** version of $x[n]$ where

$$w[n] = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Then the Fourier transform of $x_w[n]$ is

$$X_w(\Omega) = \sum_{n=-\infty}^{\infty} x_w[n] e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n]w[n] e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\Omega n}$$

Sample the resulting function of Ω at $\Omega = \frac{2\pi k}{N}$:

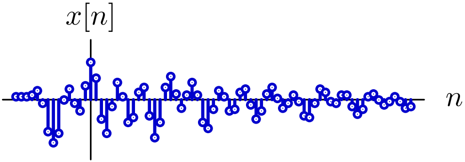
$$X_w\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2\pi k}{N}\right)n}$$

Divide both sides by N:

$$\frac{1}{N} X_w\left(\frac{2\pi k}{N}\right) = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} = X[k], \quad \text{which is the DFT of } x[n]$$

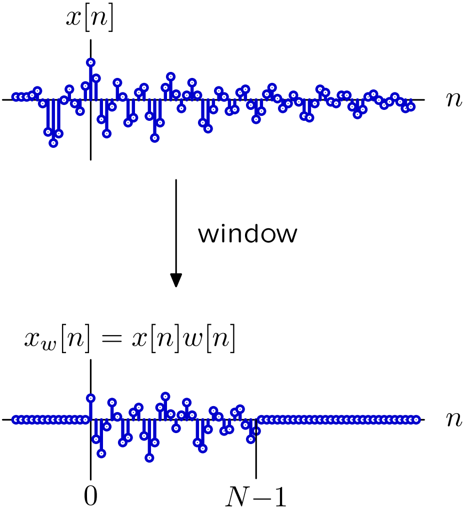
Relation Between DFT and DTFT

Graphical depiction of relation between DFT and DTFT.



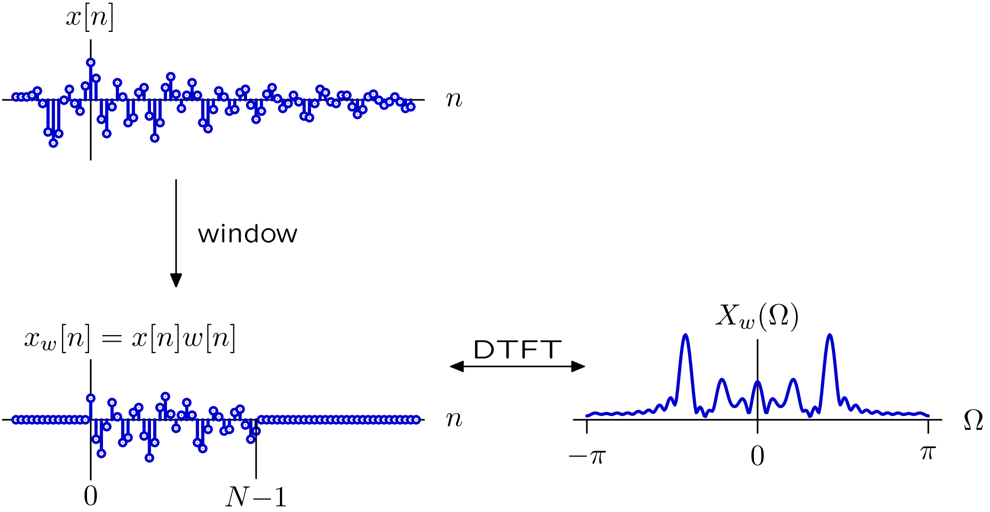
Relation Between DFT and DTFT

Graphical depiction of relation between DFT and DTFT.



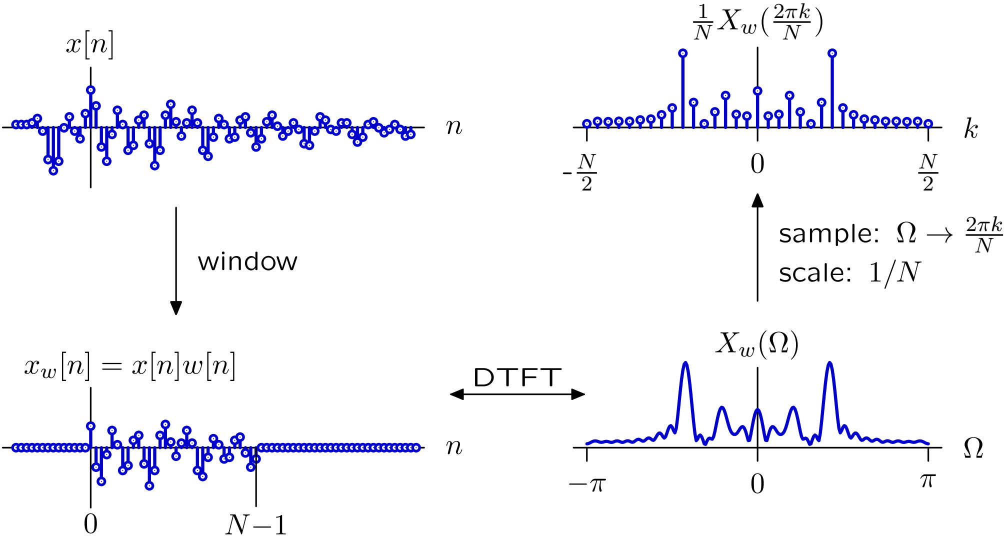
Relation Between DFT and DTFT

Graphical depiction of relation between DFT and DTFT.



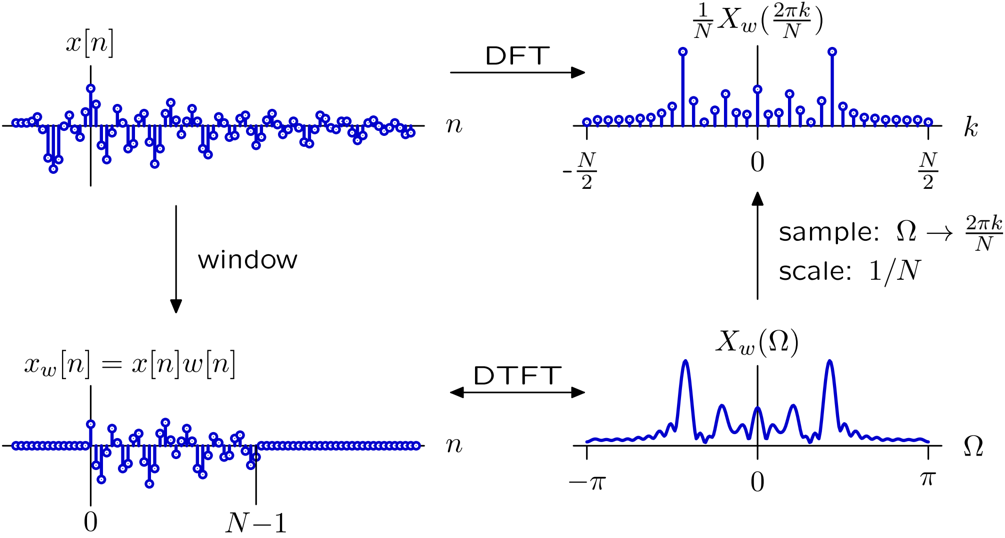
Relation Between DFT and DTFT

Graphical depiction of relation between DFT and DTFT.



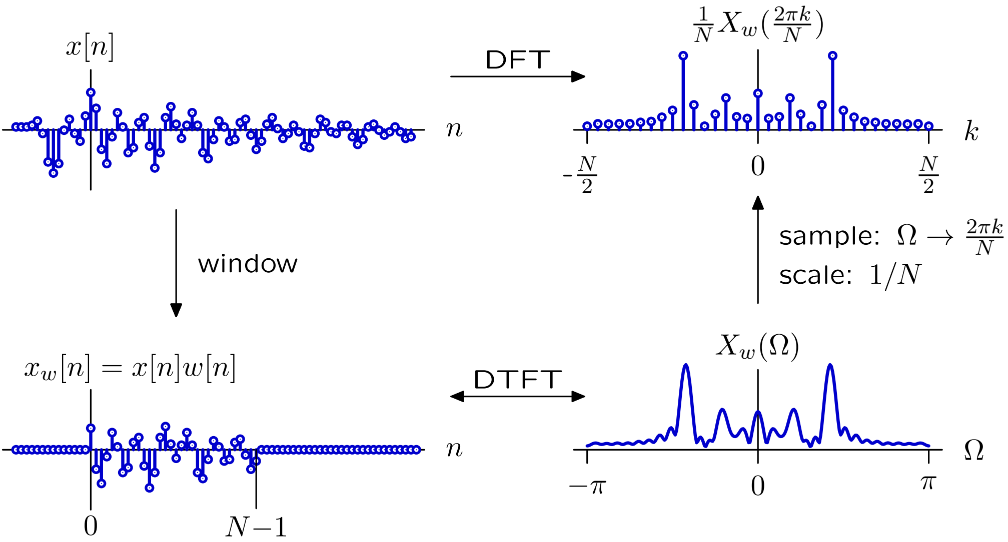
Relation Between DFT and DTFT

Graphical depiction of relation between DFT and DTFT.



Relation Between DFT and DTFT

Graphical depiction of relation between DFT and DTFT.



While sampling and scaling are important, it is the **windowing** that most affects frequency content.

Effect of Windowing on Fourier Representations

Determine effects of windowing on signals with a single frequency Ω_o.

Step 1: Find $X(\Omega)$, the DTFT of a complex exponential signal:
$x[n] = e^{j\Omega_o n}$

Step 2: Find $X_w(\Omega)$, the DTFT of a windowed version of $x[n]$:
$x_w[n] = x[n]w[n]$

Step 3: Compare $X_w(\Omega)$ to $X(\Omega)$.

Check Yourself

Step 1:

Let $x[n] = e^{j\Omega_o n}$

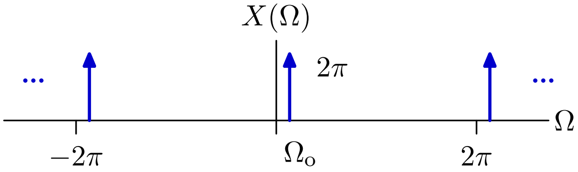
Find $X(\Omega)$, which is the DTFT of $x[n]$.

Effect of Windowing on Fourier Representations

Step 1: Find the DTFT of a complex exponential $x[n] = e^{j\Omega_o n}$.

The DTFT of a complex exponential is a train of impulses.

$$e^{j\Omega_o n} \stackrel{\text{DTFT}}{\Longrightarrow}$$



This is easy to verify using the DTFT synthesis equation.

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}\,d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\Omega-\Omega_o)e^{j\Omega n}\,d\Omega$$

$$= \int_{-\pi}^{\pi} \delta(\Omega-\Omega_o)e^{j\Omega_o n}\,d\Omega = e^{j\Omega_o n}\int_{-\pi}^{\pi} \delta(\Omega-\Omega_o)\,d\Omega$$

$$= e^{j\Omega_o n}$$

Check Yourself

Step 2:

Let $x_w[n]$ represent a windowed version of $x[n] = e^{j\Omega_o n}$.

$$x_w[n] = x[n]w[n] = e^{j\Omega_o n}w[n]$$

Find an expression for the Fourier transform $X_w(\Omega)$ in terms of the Fourier transform $W(\Omega)$ of $w[n]$.

Effect of Windowing on Fourier Representations

Step 2: Find the DTFT of $x_w[n]$, a windowed version of $x[n]$.

Let $x_w[n]$ represent a windowed version of $x[n] = e^{j\Omega_o n}$.

$$x_w[n] = x[n]w[n] = e^{j\Omega_\mathrm{o} n} w[n]$$

Then

$$\begin{aligned}
X_w(\Omega) &= \sum_{n=-\infty}^{\infty} e^{j\Omega_\mathrm{o} n} w[n] e^{-j\Omega n} && \text{DTFT analysis equation} \\
&= \sum_{n=-\infty}^{\infty} w[n] e^{-j(\Omega-\Omega_\mathrm{o})n} && \text{combine exponential terms} \\
&= W(\Omega - \Omega_\mathrm{o}) && \text{DTFT of } w[n]\text{, shifted in frequency}
\end{aligned}$$

The DTFT of a windowed version of a complex exponential signal is a shifted version of the DTFT of the window signal.

$$e^{j\Omega_\mathrm{o} n} w[n] \quad \overset{\text{DTFT}}{\Longrightarrow} \quad W(\Omega - \Omega_\mathrm{o})$$

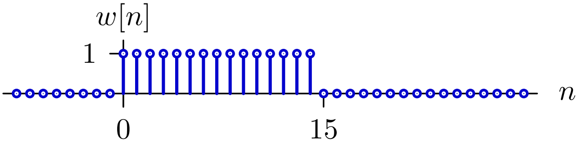
$\rightarrow$ Need to know $W(\Omega)$.

Effect of Windowing on Fourier Representations

Simplest window is rectangular, with width of N (length of DFT analysis)

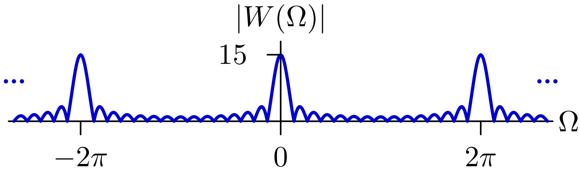
$$w[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

as shown below for $N = 15$.



The DTFT of $w[n]$ is

$$W(\Omega) = \sum_{n=-\infty}^{\infty} w[n] e^{-j\Omega n} = \sum_{n=0}^{N-1} e^{-j\Omega n} = \frac{1 - e^{-j\Omega N}}{1 - e^{-j\Omega}} = \frac{\sin \frac{N\Omega}{2}}{\sin \frac{\Omega}{2}} e^{-j\Omega \frac{N-1}{2}}$$

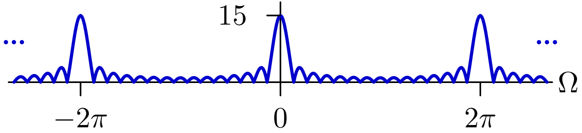


Effect of Windowing on Fourier Representations

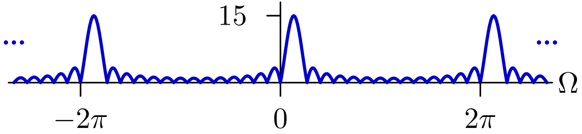
The DTFT of a windowed version of a complex exponential signal is a shifted version of the DTFT of the window signal.

$$x_w[n] = e^{j\Omega_o n} w[n] \quad \stackrel{\text{DTFT}}{\Longrightarrow} \quad X_w(\Omega) = W(\Omega - \Omega_o)$$

$|W(\Omega)|$



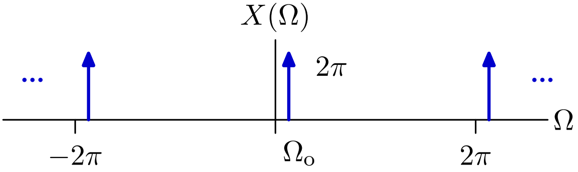
$|X_w(\Omega)| = |W(\Omega - \Omega_o)|$



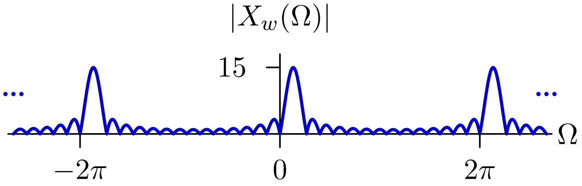
Effect of Windowing on Fourier Representations

Step 3: Compare $X_w(\Omega)$ to $X(\Omega)$.

$$x[n] = e^{j\Omega_o n} \quad \stackrel{\text{DTFT}}{\Longrightarrow}$$



$$x_w[n] = x[n]w[n] \quad \stackrel{\text{DTFT}}{\Longrightarrow}$$

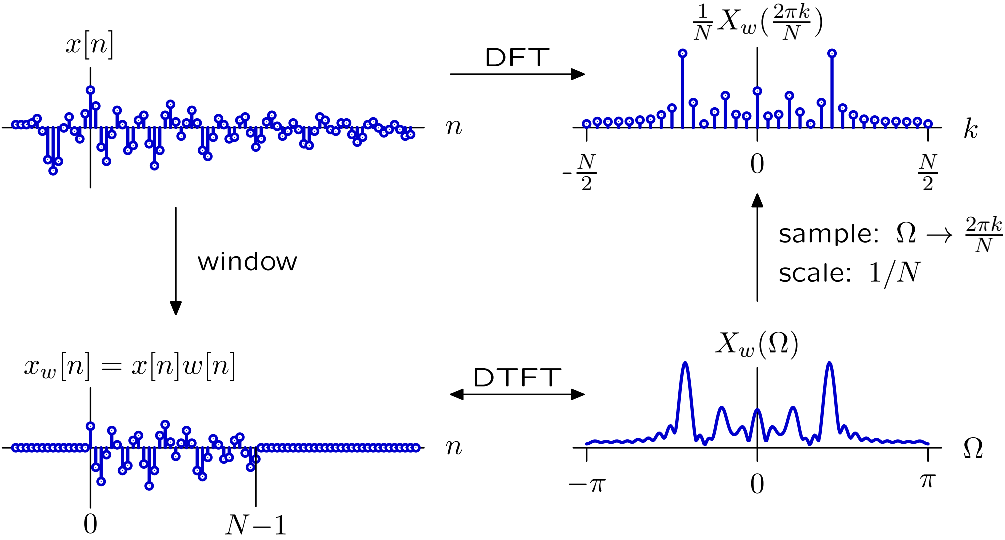


The frequency content of $X(\Omega)$ is at discrete frequencies $\Omega = \Omega_o + 2\pi m$.

The frequency content of $X_w(\Omega)$ is most dense at these same frequencies, but is spread out over almost all other frequencies as well.

Relation Between DFT and DTFT

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$.



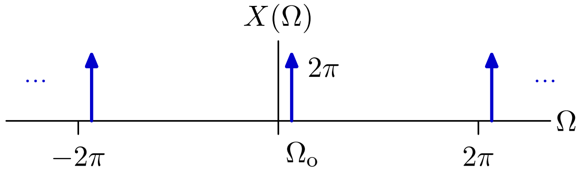
Next: apply these steps to a sinusoidal input.

Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$. Here $\Omega_o = \frac{2\pi}{15}$.

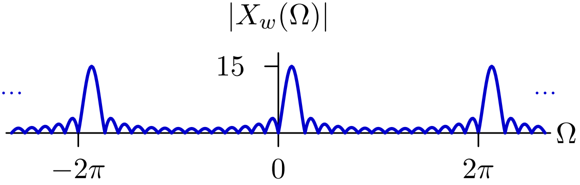
original signal

$$x[n] = e^{j\Omega_o n} \quad \overset{\text{DTFT}}{\Longrightarrow}$$



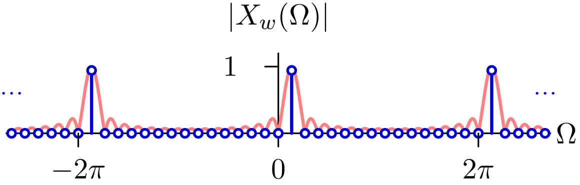
windowed

$$x_w[n] = x[n]w[n] \quad \overset{\text{DTFT}}{\Longrightarrow}$$



sampled and scaled

$$x_w[n] = x[n]w[n] \quad \overset{\text{DFT}}{\Longrightarrow}$$



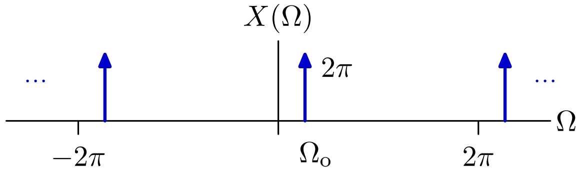
One sample is taken at the peak, and the others fall on zeros.

Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$. Here $\Omega_o = \frac{4\pi}{15}$.

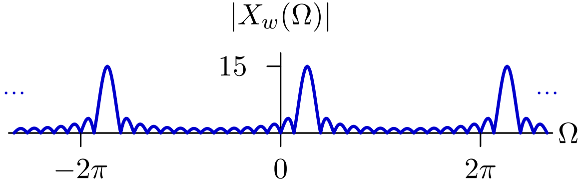
original signal

$$x[n] = e^{j\Omega_o n} \quad \overset{\text{DTFT}}{\Longrightarrow}$$



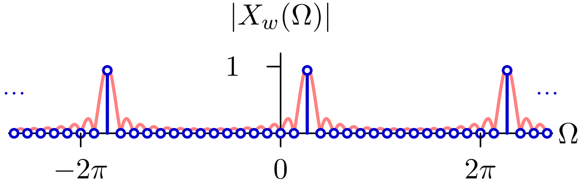
windowed

$$x_w[n] = x[n]w[n] \quad \overset{\text{DTFT}}{\Longrightarrow}$$



sampled and scaled

$$x_w[n] = x[n]w[n] \quad \overset{\text{DFT}}{\Longrightarrow}$$



One sample is taken at the peak, and the others fall on zeros.

Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$. Here $\Omega_o = \frac{6\pi}{15}$.

original signal

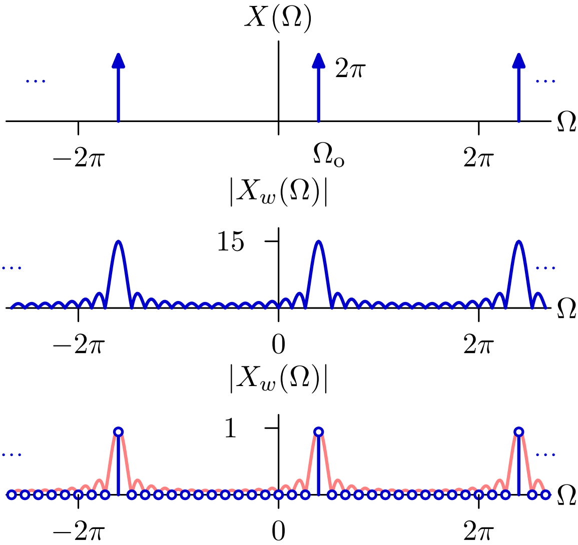
$$x[n] = e^{j\Omega_o n} \stackrel{\text{DTFT}}{\Longrightarrow}$$

windowed

$$x_w[n] = x[n]w[n] \stackrel{\text{DTFT}}{\Longrightarrow}$$

sampled and scaled

$$x_w[n] = x[n]w[n] \stackrel{\text{DFT}}{\Longrightarrow}$$



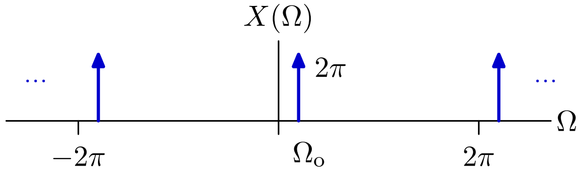
One sample is taken at the peak, and the others fall on zeros.

Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$. Here $\Omega_o = \frac{3\pi}{15}$.

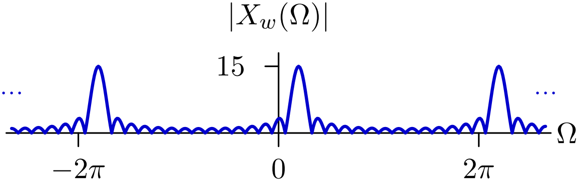
original signal

$$x[n] = e^{j\Omega_o n} \overset{\text{DTFT}}{\Longrightarrow}$$



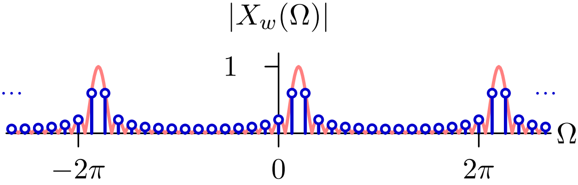
windowed

$$x_w[n] = x[n]w[n] \overset{\text{DTFT}}{\Longrightarrow}$$



sampled and scaled

$$x_w[n] = x[n]w[n] \overset{\text{DFT}}{\Longrightarrow}$$



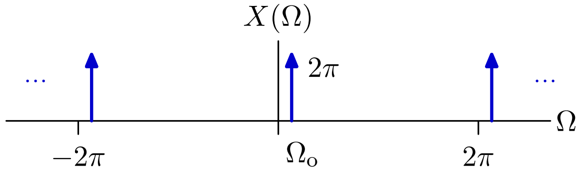
Now none of the samples fall on zeros.

Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$. Here $\Omega_o = \frac{2\pi}{15}$.

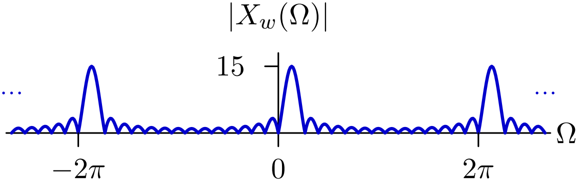
original signal

$$x[n] = e^{j\Omega_o n} \stackrel{\text{DTFT}}{\Longrightarrow}$$



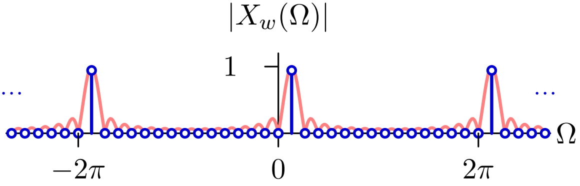
windowed

$$x_w[n] = x[n]w[n] \stackrel{\text{DTFT}}{\Longrightarrow}$$



sampled and scaled

$$x_w[n] = x[n]w[n] \stackrel{\text{DFT}}{\Longrightarrow}$$



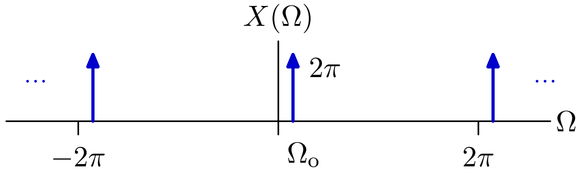
Generally, the relation between the samples is complicated.

Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$. Here $\Omega_o = \frac{2.2\pi}{15}$.

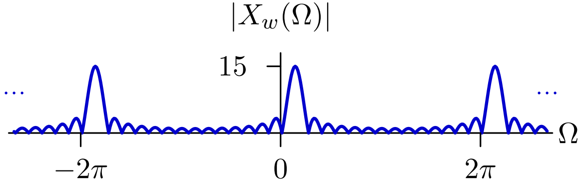
original signal

$$x[n] = e^{j\Omega_o n} \quad \overset{\text{DTFT}}{\Longrightarrow}$$



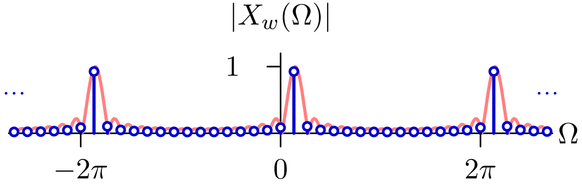
windowed

$$x_w[n] = x[n]w[n] \quad \overset{\text{DTFT}}{\Longrightarrow}$$



sampled and scaled

$$x_w[n] = x[n]w[n] \quad \overset{\text{DFT}}{\Longrightarrow}$$



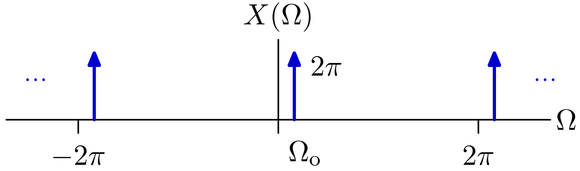
Generally, the relation between the samples is complicated.

Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$. Here $\Omega_o = \frac{2.4\pi}{15}$.

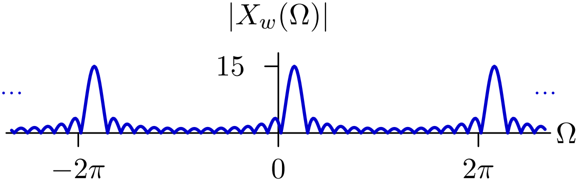
original signal

$$x[n] = e^{j\Omega_o n} \quad \overset{\text{DTFT}}{\Longrightarrow}$$



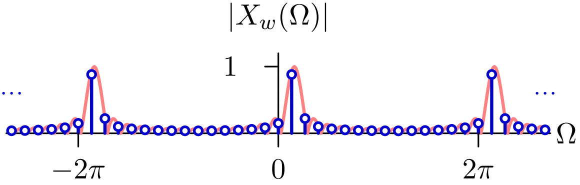
windowed

$$x_w[n] = x[n]w[n] \quad \overset{\text{DTFT}}{\Longrightarrow}$$



sampled and scaled

$$x_w[n] = x[n]w[n] \quad \overset{\text{DFT}}{\Longrightarrow}$$



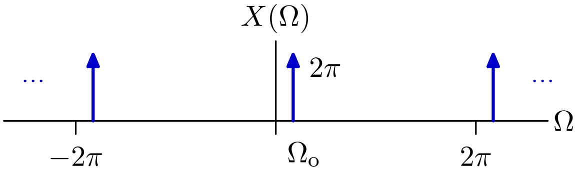
Generally, the relation between the samples is complicated.

Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$. Here $\Omega_o = \frac{2.6\pi}{15}$.

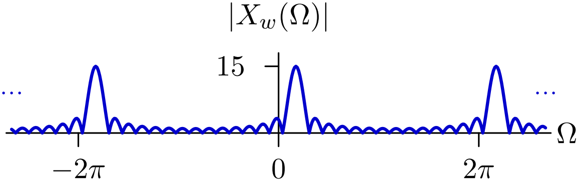
original signal

$$x[n] = e^{j\Omega_o n} \quad \overset{\text{DTFT}}{\Longrightarrow}$$



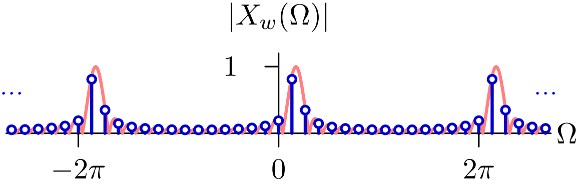
windowed

$$x_w[n] = x[n]w[n] \quad \overset{\text{DTFT}}{\Longrightarrow}$$



sampled and scaled

$$x_w[n] = x[n]w[n] \quad \overset{\text{DFT}}{\Longrightarrow}$$



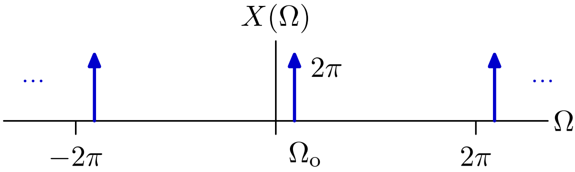
Generally, the relation between the samples is complicated.

Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$. Here $\Omega_o = \frac{2.8\pi}{15}$.

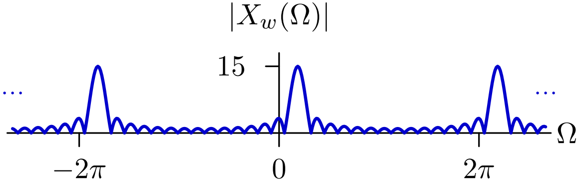
original signal

$$x[n] = e^{j\Omega_o n} \quad \overset{\text{DTFT}}{\Longrightarrow}$$



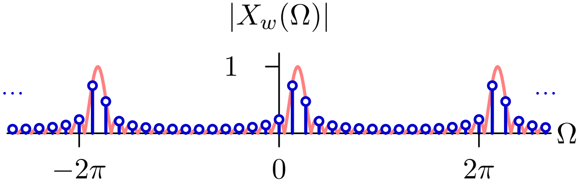
windowed

$$x_w[n] = x[n]w[n] \quad \overset{\text{DTFT}}{\Longrightarrow}$$



sampled and scaled

$$x_w[n] = x[n]w[n] \quad \overset{\text{DFT}}{\Longrightarrow}$$



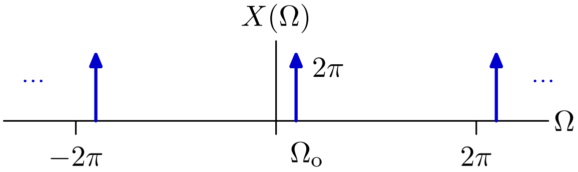
Generally, the relation between the samples is complicated.

Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$. Here $\Omega_o = \frac{3\pi}{15}$.

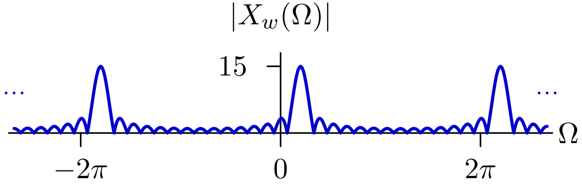
original signal

$$x[n] = e^{j\Omega_o n} \overset{\text{DTFT}}{\Longrightarrow}$$



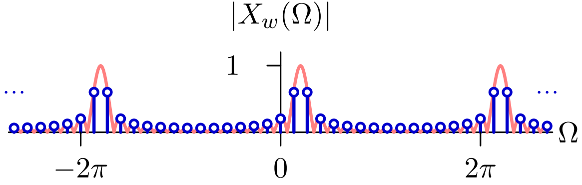
windowed

$$x_w[n] = x[n]w[n] \overset{\text{DTFT}}{\Longrightarrow}$$



sampled and scaled

$$x_w[n] = x[n]w[n] \overset{\text{DFT}}{\Longrightarrow}$$



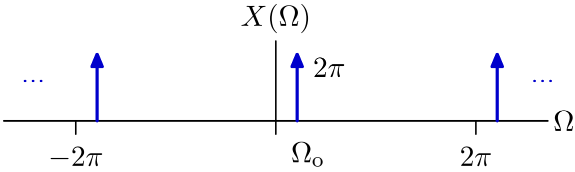
Generally, the relation between the samples is complicated.

Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$. Here $\Omega_o = \frac{3.2\pi}{15}$.

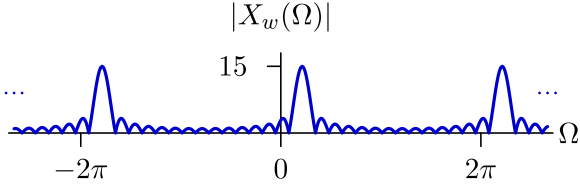
original signal

$$x[n] = e^{j\Omega_o n} \quad \overset{\text{DTFT}}{\Longrightarrow}$$



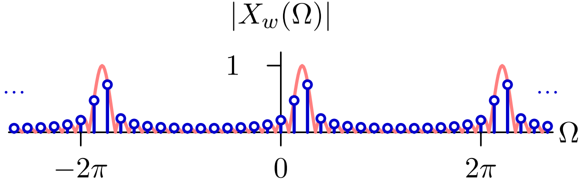
windowed

$$x_w[n] = x[n]w[n] \quad \overset{\text{DTFT}}{\Longrightarrow}$$



sampled and scaled

$$x_w[n] = x[n]w[n] \quad \overset{\text{DFT}}{\Longrightarrow}$$



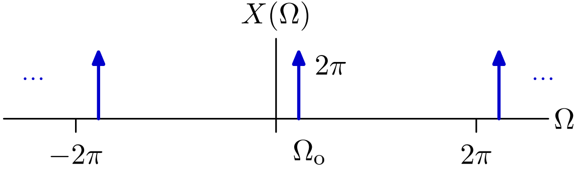
Generally, the relation between the samples is complicated.

Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$. Here $\Omega_o = \frac{3.4\pi}{15}$.

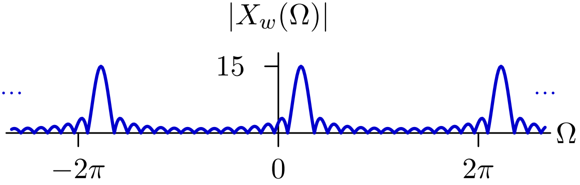
original signal

$$x[n] = e^{j\Omega_o n} \quad \overset{\text{DTFT}}{\Longrightarrow}$$



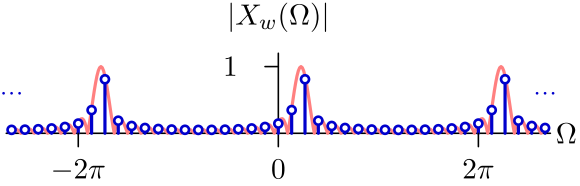
windowed

$$x_w[n] = x[n]w[n] \quad \overset{\text{DTFT}}{\Longrightarrow}$$



sampled and scaled

$$x_w[n] = x[n]w[n] \quad \overset{\text{DFT}}{\Longrightarrow}$$



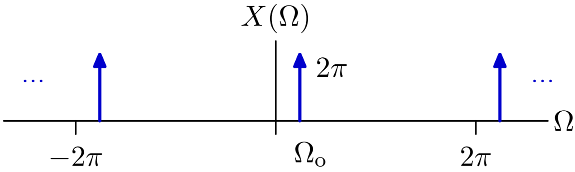
Generally, the relation between the samples is complicated.

Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$. Here $\Omega_o = \frac{3.6\pi}{15}$.

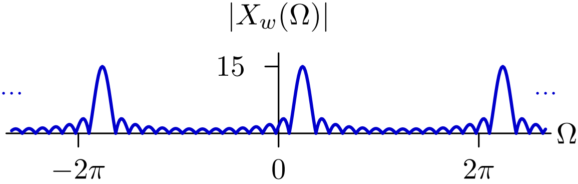
original signal

$$x[n] = e^{j\Omega_o n} \quad \overset{\text{DTFT}}{\Longrightarrow}$$



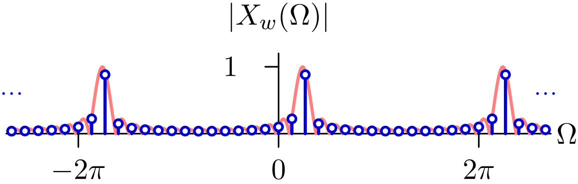
windowed

$$x_w[n] = x[n]w[n] \quad \overset{\text{DTFT}}{\Longrightarrow}$$



sampled and scaled

$$x_w[n] = x[n]w[n] \quad \overset{\text{DFT}}{\Longrightarrow}$$



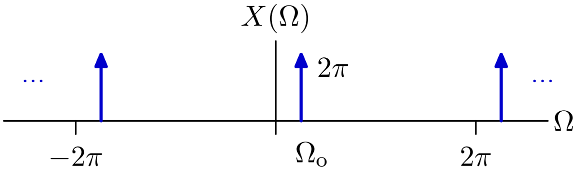
Generally, the relation between the samples is complicated.

Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$. Here $\Omega_o = \frac{3.8\pi}{15}$.

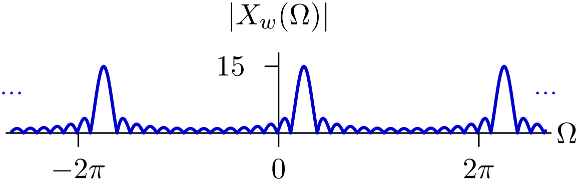
original signal

$$x[n] = e^{j\Omega_o n} \quad \overset{\text{DTFT}}{\Longrightarrow}$$



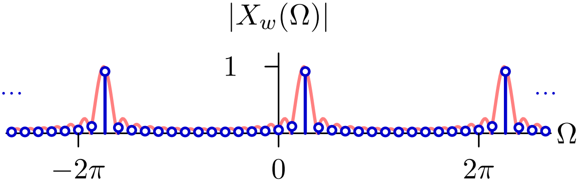
windowed

$$x_w[n] = x[n]w[n] \quad \overset{\text{DTFT}}{\Longrightarrow}$$



sampled and scaled

$$x_w[n] = x[n]w[n] \quad \overset{\text{DFT}}{\Longrightarrow}$$



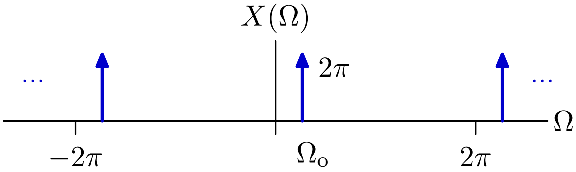
Generally, the relation between the samples is complicated.

Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $1/N$. Here $\Omega_o = \frac{4\pi}{15}$.

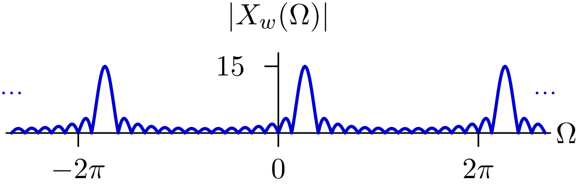
original signal

$$x[n] = e^{j\Omega_o n} \stackrel{\text{DTFT}}{\Longrightarrow}$$



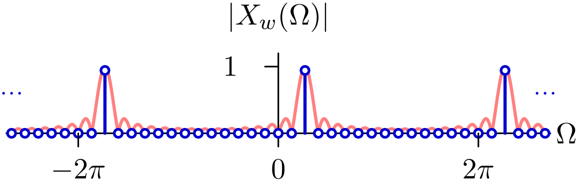
windowed

$$x_w[n] = x[n]w[n] \stackrel{\text{DTFT}}{\Longrightarrow}$$



sampled and scaled

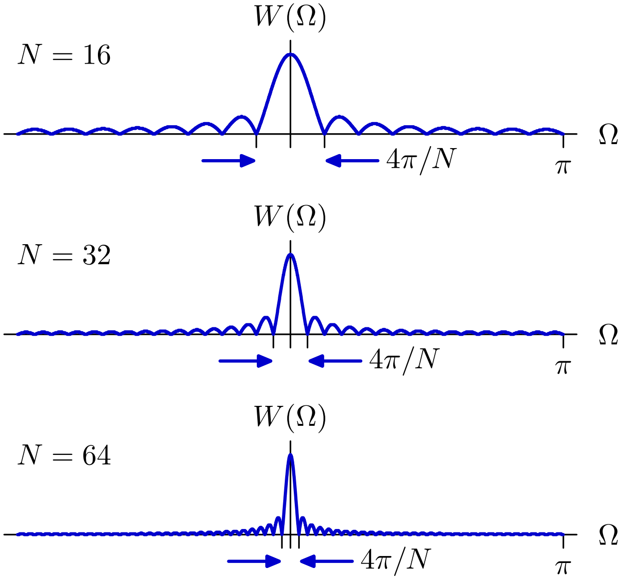
$$x_w[n] = x[n]w[n] \stackrel{\text{DFT}}{\Longrightarrow}$$



Generally, the relation between the samples is complicated.

Spectral Blurring introduces a Time/Frequency Tradeoff

Longer windows provide finer frequency resolution.

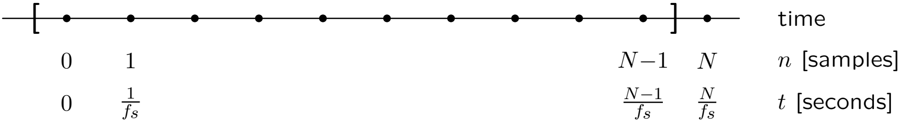


The width of the central lobe is inversely related to window length.

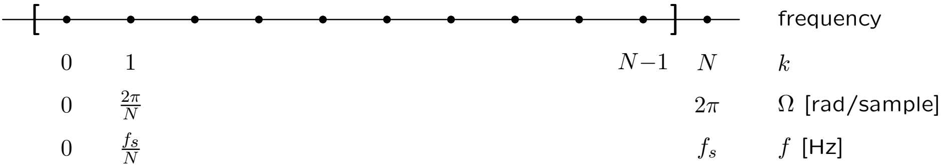
Frequency Resolution

The DFT analysis period N determines both the window in time that is analyzed and the frequency resolution of the result.

The time window is divided into N samples numbered $n = 0$ to $N-1$.



Discrete frequencies are similarly numbered as $k = 0$ to $N-1$.



As the analysis length N increases, both temporal duration and spectral resolution increase.

Summary

Today we introduced a new Fourier representation for DT signals: the Discrete Fourier Transform (DFT).

The DFT has a number of features that make it particularly convenient.

- It is not limited to periodic signals.
- It has discrete domain (k instead of Ω) and finite length: convenient for numerical computation.

The finite analysis window of the DFT can smear the resulting spectral representation.

- The DFT is equivalent to the DTFS of a periodically extended version of the input signal. The smear results because of discontinuities introduced by periodic extension.
- The DFT is equivalent to the DTFT of a windowed version of the input signal that is then sampled and scaled in amplitude. The windowing smears the spectral representation because of discontinuities introduced by the windowing.

Two Ways to Think About the DFT

Compare to DTFS:

1. The DFT of a signal $x[n]$ is equal to the **DTFS** of a version of $x[n]$ that is periodically extended so that it is periodic in N.
$\rightarrow$ emphasizes the importance of **periodicity** in time.

Compare to DTFT:

2. The DFT is equal to samples of the **DTFT** of a windowed version of the original signal.
$\rightarrow$ emphasizes the importance of **spectral smear** in frequency.

The DTFS and DTFT offer different and complementary

- **rules** for constructing all of the components of the DFT, and
- **intuition** for understanding the origin of "extra" components of DFT.

Question of the Day

Find an expression for the 10-point DFT (i.e., $N = 10$) of the following signal:

$$x[n] = (-1)^n$$