Trigonometric Expansions

In this problem, we compare two methods for expanding a function $f(\theta)$ as a series of the form

$$f(\theta) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\theta) + \sum_{k=1}^{\infty} d_k \sin(k\theta)\,.$$

Part a. Use the trigonometric identities provided on the last page of this homework assignment plus the rules of ordinary algebra to determine the values of the non-zero coefficients c_k and d_k needed to expand the function

$$f_1(\theta) = \cos^5(\theta)\,.$$

$$\begin{aligned}
f_1(\theta) = \cos^5(\theta) &= \cos^2(\theta)\cos^2(\theta)\cos(\theta) \\
&= \left(\frac{1+\cos(2\theta)}{2}\right) \times \left(\frac{1+\cos(2\theta)}{2}\right) \times \cos(\theta) \\
&= \left(\frac{1+2\cos(2\theta)+\cos^2(2\theta)}{4}\right) \times \cos(\theta) \\
&= \left(\frac{1+2\cos(2\theta)+\frac{1+\cos(4\theta)}{2}}{4}\right) \times \cos(\theta) \\
&= \left(\frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)\right)\cos(\theta) \\
&= \frac{3}{8}\cos(\theta) + \frac{1}{2}\cos(2\theta)\cos(\theta) + \frac{1}{8}\cos(4\theta)\cos(\theta) \\
&= \frac{3}{8}\cos(\theta) + \frac{1}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta) + \frac{1}{16}\cos(3\theta) + \frac{1}{16}\cos(5\theta) \\
&= \frac{5}{8}\cos(\theta) + \frac{5}{16}\cos(3\theta) + \frac{1}{16}\cos(5\theta)
\end{aligned}$$

The coefficient $c_1 = 5/8$, $c_3 = 5/16$, and $c_5 = 1/16$. The other coefficients are zero.

Part b. An alternative to trigonometric identities is to use complex exponentials. Determine the non-zero coefficients c_k and d_k as in the previous part – but this time use Euler's formula and complex numbers, but no trigonometric identifies.

We can use Euler's formula to rewrite the target function as

$$f_1(\theta) = \left(\frac{e^{j\theta}+e^{-j\theta}}{2}\right)^5.$$

Then we can expand the fifth power by repeated multiplication (or by using the binomial equation) to get

$$f_1(\theta) = \frac{1}{32}\left(e^{j5\theta} + 5e^{j3\theta} + 10e^{j\theta} + 10e^{-j\theta} + 5e^{-3j\theta} + e^{-5j\theta}\right).$$

Finally, we can pair complex exponentials with their negative partners and use Euler's formula to obtain

$$f_1(\theta) = \frac{5}{8}\cos(\theta) + \frac{5}{16}\cos(3\theta) + \frac{1}{16}\cos(5\theta)\,.$$

As before, the coefficient $c_1 = 5/8$, $c_3 = 5/16$, and $c_5 = 1/16$. The others are zero.

Part c. Use the trigonometric identities provided on the last page of this homework assignment plus the rules of ordinary algebra to determine the values of the non-zero coeeficients c_k and d_k needed to expand the function

$$f_2(\theta) = \sin^5(\theta)\,.$$

$$\begin{aligned}
f_2(\theta) = \sin^5(\theta) &= \sin^2(\theta)\sin^2(\theta)\sin(\theta) \\
&= \left(\frac{1-\cos(2\theta)}{2}\right) \times \left(\frac{1-\cos(2\theta)}{2}\right) \times \sin(\theta) \\
&= \left(\frac{1-2\cos(2\theta)+\cos^2(2\theta)}{4}\right) \times \sin(\theta) \\
&= \left(\frac{1-2\cos(2\theta)+\frac{1+\cos(4\theta)}{2}}{4}\right) \times \sin(\theta) \\
&= \left(\frac{3}{8} - \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)\right)\sin(\theta) \\
&= \frac{3}{8}\sin(\theta) - \frac{1}{2}\cos(2\theta)\sin(\theta) + \frac{1}{8}\cos(4\theta)\sin(\theta) \\
&= \frac{3}{8}\sin(\theta) - \frac{1}{4}\sin(3\theta) + \frac{1}{4}\sin(\theta) + \frac{1}{16}\sin(5\theta) - \frac{1}{16}\sin(3\theta) \\
&= \frac{5}{8}\sin(\theta) - \frac{5}{16}\sin(3\theta) + \frac{1}{16}\sin(5\theta)
\end{aligned}$$

The coefficient $d_1 = 5/8$, $d_3 = -5/16$, and $d_5 = 1/16$. The other coefficients are zero.

Part d. Determine the non-zero coefficients c_k and d_k as in the previous part – but this time use Euler's formula and complex numbers, but no trigonometric identities.

Use Euler's formula to rewrite the target function as

$$f_2(\theta) = \left(\frac{e^{j\theta} - e^{-j\theta}}{2j}\right)^5 .$$

Then expand the fifth power by repeated multiplication (or by using the binomial formula) to obtain

$$f_2(\theta) = \frac{1}{32j}\left(e^{j5\theta} - 5e^{j3\theta} + 10e^{j\theta} - 10e^{-j\theta} + 5e^{-3j\theta} - e^{-5j\theta}\right) .$$

Then pair the complex exponentials with their negative partners and use Euler's formula to obtain

$$f_2(\theta) = \frac{5}{8}\sin(\theta) - \frac{5}{16}\sin(3\theta) + \frac{1}{16}\sin(5\theta)\,.$$

As before, the coefficient $d_1 = 5/8$, $d_3 = -5/16$, and $d_5 = 1/16$. The others are zero.

Part e. List the mathematical relations that you used in each of the previous parts. Briefly describe the pros and cons of using trigonometric identities versus Euler's formula.

The solution to part a used the product of cosines rule repeatedly. The solution to part c used the product of sines and the product of sines with cosines. Thus these solutions are similar, but used different trig identities.

The solution to parts b and d used Euler's formula to convert the original trig functions tocomplex exponentials. The resulting expression was expanded with the binomial theorem (Pascal's triangle). And that result was converted back to trig form using Euler's formula.

The point of this problem is that a single equation – namely Euler's formula – substitutes for any number of trigonometric identities. Rather than remembering and learning to use the many trig identities that exist, we can remember and learn to use a single equation (Euler's formula) if we use complex numbers.