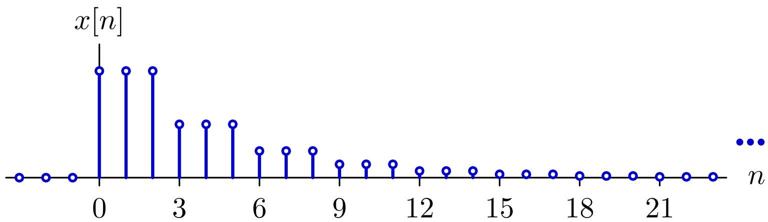
## Steps

**Part a.** Let $x[n]$ represent the following discrete-time signal

$$x[n] = \begin{cases} 0 & \text{for } n < 0 \\ a^0 & \text{for } n = 0, 1, 2 \\ a^1 & \text{for } n = 3, 4, 5 \\ a^2 & \text{for } n = 6, 7, 8 \\ \ldots \end{cases}$$

where $a$ is a real number between 0 and 1, as shown in the plot below.




Determine a closed form expression for $X(\Omega)$, which is the discrete-time Fourier transform of $x[n]$.

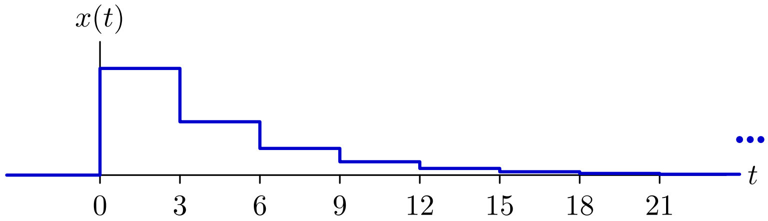
$$X(\Omega) = \boxed{\frac{1 + e^{-j\Omega} + e^{-j2\Omega}}{1 - ae^{-j3\Omega}}}$$

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \\ &= \sum_{m=0}^{\infty} a^m \left( e^{-j\Omega 3m} + e^{-j\Omega(3m+1)} + e^{-j\Omega(3m+2)} \right) \\ &= \sum_{m=0}^{\infty} a^m e^{-j\Omega 3m} \left(1 + e^{-j\Omega} + e^{-j2\Omega}\right) \\ &= \frac{1 + e^{-j\Omega} + e^{-j2\Omega}}{1 - ae^{-j3\Omega}} \end{aligned}$$

**Part b.** Let $x(t)$ represent the following continuous-time signal

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ a^0 & \text{for } 0 \leq t < 3 \\ a^1 & \text{for } 3 \leq t < 6 \\ a^2 & \text{for } 6 \leq t < 9 \\ \dots \end{cases}$$

where $a$ is a real number between 0 and 1, as shown in the plot below.




Determine a closed-form expression for $X(\omega)$, which is the continuous-time Fourier transform of $x(t)$.

$$X(\omega) = \boxed{\left(\frac{1}{j\omega}\right)\left(\frac{1-e^{-j3\omega}}{1-ae^{-j3\omega}}\right)}$$

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \\
&= \int_0^3 a^0 e^{-j\omega t}\,dt + \int_3^6 a^1 e^{-j\omega t}\,dt + \int_6^9 a^2 e^{-j\omega t}\,dt + \cdots \\
&= a^0\left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^3 + a^1\left[\frac{e^{-j\omega t}}{-j\omega}\right]_3^6 + a^2\left[\frac{e^{-j\omega t}}{-j\omega}\right]_6^9 + \cdots \\
&= -\frac{1}{j\omega}\left(a^0\left(e^{-j\omega 3} - e^{-j\omega 0}\right) + a^1 e^{-j\omega 3}\left(e^{-j\omega 3} - e^{-j\omega 0}\right) + a^2 e^{-j\omega 6}\left(e^{-j\omega 3} - e^{-j\omega 0}\right)\right) \\
&= \left(\frac{1-e^{-j3\omega}}{j\omega}\right)\left(\sum_{m=0}^{\infty}\left(ae^{-j3\omega}\right)^m\right) \\
&= \left(\frac{1-e^{-j3\omega}}{j\omega}\right)\left(\frac{1}{1-ae^{-j3\omega}}\right)
\end{aligned}$$