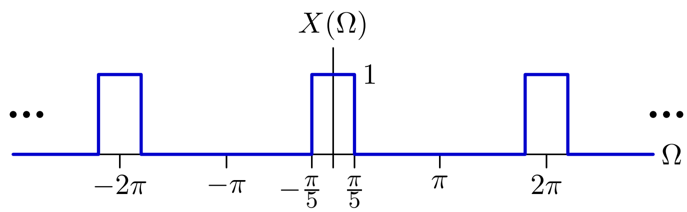
Slow Down

Let $x[n]$ represent a discrete time signal whose DTFT is given by

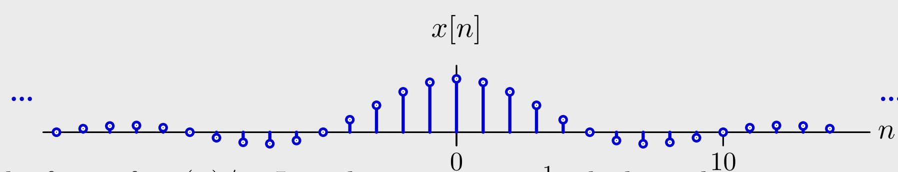
$$X(\Omega) = \begin{cases} 1 & \text{if } |\Omega| < \frac{\pi}{5} \\ 0 & \text{if } \frac{\pi}{5} < |\Omega| < \pi \end{cases}$$

and is periodic in Ω with period 2π as shown below.



Part a. Determine an expression for $x[n]$. Plot $x[n]$ and label the important features of your plot.

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}\Omega = \frac{1}{2\pi}\int_{-\pi/5}^{\pi/5} e^{j\Omega n}d\Omega = \frac{1}{2\pi}\left[\frac{e^{j\Omega n}}{jn}\right]_{-\pi/5}^{\pi/5} = \frac{\sin(\pi n/5)}{\pi n}$$

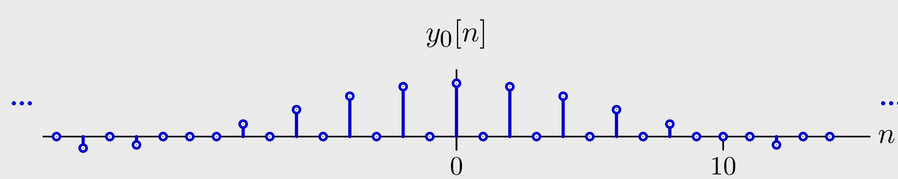


The function has the form of $\sin(n)/n$. Its value at $n = 0$ is $\frac{1}{5}$ which is what we expect since the area under $X(\Omega)$ for $-\pi < \Omega < \pi$ divided by 2π is $1/5$. The function $x[n] = 0$ at $n = \pm 5, \pm 10, \pm 15, \ldots$.

Part b. A new signal $y_0[n]$ is derived by stretching $x[n]$ as follows:

$$y_0[n] = \begin{cases} x\left[\frac{n}{2}\right] & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

Make a plot of $y_0[n]$ and label its key features.

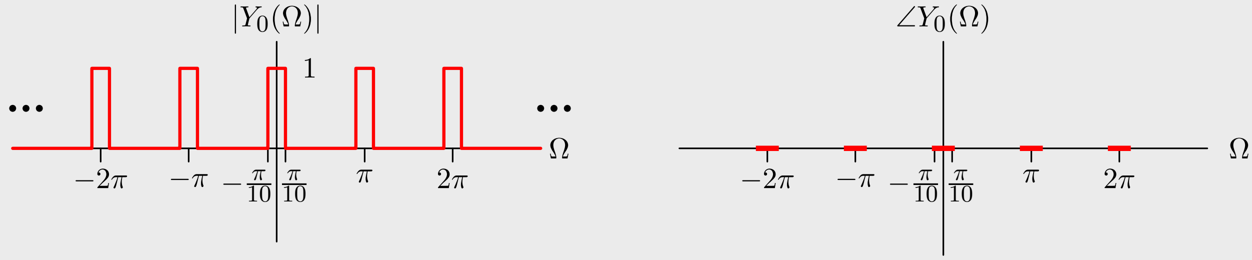


Part c. Determine an expression for $Y_0(\Omega)$ in terms of $X(\Omega)$. Sketch the magnitude and angle of $Y_0(\Omega)$ on the axes below. Label all important parameters of your plots.

$$Y_0(\Omega) = \sum_{n=-\infty}^{\infty} y_0[n]e^{-j\Omega n} = \sum_{n \text{ even}} x\left[n/2\right]e^{-j\Omega n}$$

Let $n = 2m$:

$$Y_0(\Omega) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\Omega 2m} = X(2\Omega)$$



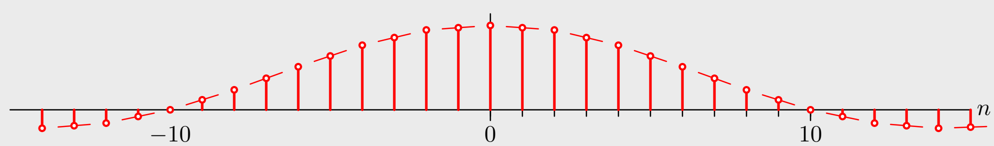
Briefly describe the important differences between $X(\Omega)$ and $Y_0(\Omega)$.

The passband of $Y_0(\Omega)$ is only half as wide as that of $X(\Omega)$. Also, unlike $X(\Omega)$, there are copies of the passband of $Y_0(\Omega)$ at Ω equal to odd multiples of π.

Part d. The $y_0[n]$ signal alternates between non-zero and zero values. To reduce the effect of the zero values, we define

$$y_1[n] = \frac{1}{2}y_0[n-1] + y_0[n] + \frac{1}{2}y_0[n+1]$$

Plot $y_1[n]$ and label the important features of your plot. Briefly describe the relation between $y_0[n]$ and $y_1[n]$.



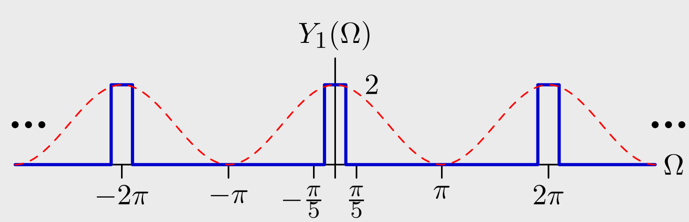
Briefly describe the relation between $y_0[n]$ and $y_1[n]$.

The zero-values that were inserted into $y_0[n]$ have been replaced with the average of the samples on either side of them, as in piecewise linear interpolation.

Part e. Determine an expression for $Y_1(\Omega)$ (the Fourier transform of $y_1[n]$) in terms of $Y_0(\Omega)$.

$$\begin{aligned} Y_1(\Omega) &= \sum_{n=-\infty}^{\infty} y_1[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}y_0[n-1] + y_0[n] + \frac{1}{2}y_0[n+1]\right)e^{-j\Omega n} \\ &= \frac{1}{2}e^{-j\Omega}Y_0(\Omega) + Y_0(\Omega) + \frac{1}{2}e^{j\Omega}Y_0(\Omega) = \Big(1+\cos(\Omega)\Big)Y_0(\Omega) \end{aligned}$$

Make a plot of $Y_1(\Omega)$.



Briefly describe the relation between $Y_0(\Omega)$ and $Y_1(\Omega)$.

The overall amplitude of $Y_1(\Omega)$ is twice that of $Y_0(\Omega)$. This results because the values of $y_0[n]$ are zero for odd values of n, while those for $y_1[n]$ are not. Components of $Y_1(\Omega)$ near $\Omega = \pi$ are greatly reduced in magnitude relative to those in $Y_0(\Omega)$.

The net effect of these changes is to generate a new signal $Y_1(\Omega)$ with half the bandwidth of $X(\Omega)$.