- **Complexity**

Answer the following questions without the use of a calculator or computer. Briefly explain your answers.

Complex numbers

a. Determine the real and imaginary parts of $1/(e^{j3\pi/4}+\frac{1}{\sqrt{2}}e^{j\pi/2})$.

We can simplify the denominator by converting each of the complex exponentials to cartesian form:

$$\frac{1}{e^{j3\pi/4}+\frac{1}{\sqrt{2}}e^{j\pi/2}} = \frac{1}{-\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}}+\frac{j}{\sqrt{2}}} = \frac{1}{\frac{-1+2j}{\sqrt{2}}} = \frac{\sqrt{2}}{-1+2j}\,.$$

Then multiply by $\frac{-1-2j}{-1-2j}$ to make the denominator real:

$$\left(\frac{\sqrt{2}}{-1+2j}\right)\left(\frac{-1-2j}{-1-2j}\right) = -\frac{\sqrt{2}}{5}(1+2j)$$

Thus the real part is $-\frac{\sqrt{2}}{5}$ and the imaginary part is $-\frac{2\sqrt{2}}{5}$.

b. Determine the real and imaginary parts of $(1-j\sqrt{3})^{12}$.

Raising to a power is easier when a complex number is in polar form than it is when it is in cartesian form. Converting $1-j\sqrt{3}$ to polar form yields

$$1-j\sqrt{3} = 2e^{-j\pi/3}\,.$$

Raising this number to the twelfth power yields

$$(2e^{-j\pi/3})^{12} = 2^{12}e^{-j12\pi/3} = 2^{12}e^{-j4\pi} = 2^{12} = 4096\,.$$

Thus the real part is 4096 and the imaginary part is zero.

c. Determine the real and imaginary parts of j^j.

Replace the j in the base with the equivalent expression $e^{j\pi/2}$ to get

$$j^j = (e^{j\pi/2})^j\,.$$

Multiplying two exponential functions is equivalent to adding their exponents. Carrying out this operation yields

$$j^j = (e^{j\pi/2})^j = e^{-\pi/2}\,.$$

Thus j^j is real valued. The real part is $e^{-\pi/2}$ and the imaginary part is 0.

Extra credit: The expression $e^{j\pi/2}$ is just one of infinitely many expressions equivalent to j. A more comprehensive list includes $e^{j(2\pi n-\pi/2)}$ for any integer n. Thus j^j can take on multiple values: $e^{-(2\pi n-\pi/2)}$ for any integer n. All of these values are real, so the imaginary part of j^j is always 0, but the real part can have any of the values given by $e^{-(2\pi n-\pi/2)}$ for any integer n.

Complex exponentials

d. Determine the real and imaginary parts of $(1+j)e^{j\omega t}$.

$$(1+j)e^{j\omega t} = (1+j)(\cos(\omega t) + j\sin(\omega t)) = \cos(\omega t) - \sin(\omega t) + j(\cos(\omega t) + \sin(\omega t))$$

The real part is $\cos(\omega t) - \sin(\omega t)$ and the imaginary part is $\cos(\omega t) + \sin(\omega t)$.

e. Determine the real and imaginary parts of $e^{j\pi/2}e^{j\omega t}$.

$$e^{j\pi/2}e^{j\omega t} = je^{j\omega t} = j\cos(\omega t) - \sin(\omega t)$$

The real part is $-\sin(\omega t)$ and the imaginary part is $\cos(\omega t)$.

f. Determine the real and imaginary parts of $(\cos(\omega t)+j\sin(\omega t))^n$ where n is an integer.

$$(\cos(\omega t)+j\sin(\omega t))^n = (e^{j\omega t})^n = e^{jn\omega t} = \cos(n\omega t)+j\sin(n\omega t)\,.$$

Therefore the real part is $\cos(n\omega t)$ and the imaginary part is $\sin(n\omega t)$.

g. Find a complex number c so that $\mathrm{Re}\left(ce^{j\omega t}\right) = A\cos(\omega t) + B\sin(\omega t)$ for all real numbers ω and t.

Let $c = a+jb$. Then

$$\mathrm{Re}\left(ce^{j\omega t}\right) = \mathrm{Re}\left((a+jb)e^{j\omega t}\right) = a\cos(\omega t) - b\sin(\omega t) = A\cos(\omega t)+B\sin(\omega t)\,.$$

Therefore $a = A$, $b = -B$, and $c = A-jB$.