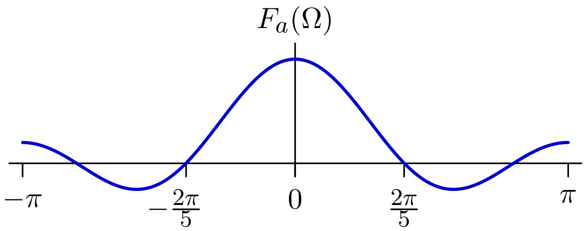
Interpreting Fourier Transforms

Let $F_a(\Omega)$ represent the Discrete-Time Fourier Transform of a discrete-time signal $f_a[n]$, which is a pulse of unknown width.

$$f_a[n] = \begin{cases} 1 & -W \le n \le W \\ 0 & \text{otherwise} \end{cases} \quad \overset{\text{DTFT}}{\Longrightarrow}$$



Let $F_b[k]$ represent a sampled version of $F_a(\Omega)$,

$$F_b[k] = F_a\left(\frac{2\pi k}{9}\right)$$

where k is an integer.

Part 1. Use an inverse Discrete Fourier Transform with length $N = 9$ to find $f_b[n]$:

$$f_b[n] \quad \underset{N=9}{\overset{\text{DFT}}{\Longleftrightarrow}} \quad F_b[k]$$

Enter your answer in the boxes below.

$f_b[0]$:	9	$f_b[1]$:	9	$f_b[2]$:	9
$f_b[3]$:	0	$f_b[4]$:	0	$f_b[5]$:	0
$f_b[6]$:	0	$f_b[7]$:	9	$f_b[8]$:	9

Start by finding W. We have an expression for $f_a[n]$ with W as a parameter. We can take the DTFT of this expression and then find the value of W that makes our analytic expression for the DTFT match the given plot of $F_a(\Omega)$.

While it is straightforward to find the DTFT of $f_a[n]$ by using the DTFT analysis equation, it's a bit easier to find the DTFT of a shifted version, $f_a[n-W]$, and then multiply the result by a phase factor $e^{j\Omega W}$ to compensate for the time shift, as follows.

$$F_a(\Omega) = e^{j\Omega W}\left(\sum_{n=0}^{2W} e^{-j\Omega n}\right) = \frac{e^{j\Omega(W+\frac{1}{2})}}{e^{j\Omega/2}} \times \frac{1-e^{-j\Omega(2W+1)}}{1-e^{-j\Omega}} = \frac{e^{j\Omega(W+\frac{1}{2})}-e^{-j\Omega(W+\frac{1}{2})}}{e^{j\Omega/2}-e^{-j\Omega/2}} = \frac{\sin(\Omega(W+\frac{1}{2}))}{\sin(\Omega/2)}$$

The zero crossing of this expression that is closest to $\Omega = 0$ is at $\Omega = \frac{\pi}{W+\frac{1}{2}}$. The corresponding zero crossing in the plot is at $2\pi/5$. Equating those yields $W = 2$, and $f_a[n]$ is fully determined as

$$f_a[n] = \begin{cases} 1 & \text{if } -2 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

We can compute $f_b[n]$ from $F_b[k]$ using the inverse DFT relation with length $N = 9$

$$f_b[n] = \sum_{k=0}^{8} F_b[k] e^{j2\pi kn/9}$$

and then relate this back to $f_a[n]$ through the transform relation $F_b[k] = F_a(2\pi k/9)$:

$$f_b[n] = \sum_{k=0}^{8} F_a(2\pi k/9) e^{j2\pi kn/9}\,.$$

Substituting the definition of the DTFT for F_a, we can rewrite this expression as

$$f_b[n] = \sum_{k=0}^{8}\sum_{m=-\infty}^{\infty} f_a[m] e^{-j2\pi km/9} e^{j2\pi kn/9}\,.$$

Swapping the order of summation allows us to factor the $f_a[m]$ term out of the sum on k:

$$f_b[n] = \sum_{m=-\infty}^{\infty} f_a[m] \sum_{k=0}^{8} e^{j2\pi k(n-m)/9} = \sum_{m=-\infty}^{\infty} f_a[m] 9\delta[(n-m) \bmod 9]$$

The delta function in the previous expression is zero unless m is equal to the sum of n plus an integer multiple of 9 (i.e., $9l$ where l is an integer):

$$f_b[n] = \sum_{l=-\infty}^{\infty} f_a[n-9l]\,.$$

Since $f_a[n]$ is non-zero for just 5 discrete times, the sum on l reduces to the following 5 cases:

$$\begin{aligned} f_b[-2] &= 9f_a[7] \\ f_b[-1] &= 9f_a[8] \\ f_b[0] &= 9f_a[0] \\ f_b[1] &= 9f_a[1] \\ f_b[2] &= 9f_a[2] \end{aligned}$$

The values of $f_b[n]$ at $n = \pm 3$ and $n = \pm 4$ are 0.

Part 2. Let $F_c[k]$ represent the DFT that results when each element of $F_b[k]$ is squared:

$$F_c[k] = F_b^2[k]\,.$$

Use an inverse Discrete Fourier Transform with length $N = 9$ to find $f_c[n]$:

$$f_c[n] \quad \underset{N=9}{\overset{\text{DFT}}{\Longleftrightarrow}} \quad F_c[k]$$

Enter your answer in the boxes below.

$f_c[0]$:	45	$f_c[1]$:	36	$f_c[2]$:	27
$f_c[3]$:	18	$f_c[4]$:	9	$f_c[5]$:	9
$f_c[6]$:	18	$f_c[7]$:	27	$f_c[8]$:	36

Since $F_c[k] = F_b^2[k]$, we must have $f_c[n] = \frac{1}{N}\left(f_b \circledast f_b\right)[n]$.

We can compute the circular convolution using superposition, etc, and then divide by $N = 9$ to find that

$$f_c[n] = \begin{cases} 45 & \text{if } n = 0 \\ 36 & \text{if } n = 1 \text{ or } n = 8 \\ 27 & \text{if } n = 2 \text{ or } n = 7 \\ 18 & \text{if } n = 3 \text{ or } n = 6 \\ 9 & \text{if } n = 4 \text{ or } n = 5 \end{cases}$$